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TERNARY HALFGROUPOIDS AND COORDINATIZATION

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(Preliminary communication)

§ 1. Definition 1.1. A ternary halfgroupoid is a couple (S, τ) where S is a set with $\mathit{card}\ S \geq 2$ and τ is a mapping of some nonempty set $\mathcal{D}omain\ \tau \subseteq S \times S \times S$ into S. If $\mathcal{D}omain\ \tau = S \times S \times S$ we get a ternary groupoid.

Definition 1.1.a. Let $T = (S, \tau)$ and $T' = (S', \tau')$ be ternary halfgroupoids. An isotopism $\sigma : T \longrightarrow T'$ is a quadruple $(\sigma_1, \sigma_2, \sigma_3, \sigma_4)$ such that $\sigma_i : S \longrightarrow S'$ $(i = 1, 2, 3, 4)$ is a bijection, $\{(a^{\sigma_1}, a^{\sigma_2}, a^{\sigma_3}) \mid (a, b, c) \in \mathcal{D}omain\ \tau\} = \mathcal{D}omain\ \tau'$ and $\tau'(a^{\sigma_1}, b^{\sigma_2}, c^{\sigma_3}) = (\tau(a, b, c))^{\sigma_4}$ for all $(a, b, c) \in \mathcal{D}omain\ \tau$. For $T = T'$ we get an autotopism. For $\sigma_1 = \sigma_2 = \sigma_3 = \sigma_4$ we obtain an isomorphism which becomes an automorphism if $T = T'$.

Definition 1.2. A g.p. presystem [1] is a quadruple $(\mathcal{P}, \mathcal{L}, I, /\!/)$ where (i) $\mathcal{P}$ and $\mathcal{L}$ are nonempty sets of elements called the points and the lines respectively, (ii) I is a binary relation between $\mathcal{P}$

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[1] g.p. = with generalized parallelity

and $\mathcal{L}$ such that for each $p \in \mathcal{P}$ ($\ell \in \mathcal{L}$) there exists a line ℓ (a point p) with $p \mathrm{I} \ell$ and (iii) $//$ is a decomposition of $\mathcal{L}$ with members $L \subseteq \mathcal{L}$ such that, for each $p \in \mathcal{P}$ there is at most one line $\ell \in L$ with $p \mathrm{I} \ell$.

Definition 1.2. a. Let $P = (\mathcal{P}, \mathcal{L}, \mathrm{I}, //)$ and $P' = (\mathcal{P}', \mathcal{L}', \mathrm{I}', //')$ be g.p. presystems.

An **isomorphism** $\rho : P \to P'$ is a couple (ρ_1, ρ_2) of bijections $\rho_1 : \mathcal{P} \to \mathcal{P}'$, $\rho_2 : \mathcal{L} \to \mathcal{L}'$ satisfying the following two properties: (i) $p \mathrm{I} \ell \iff p^{\rho_1} \mathrm{I} \ell^{\rho_2}$ and (ii) ℓ^{ρ_2}, m^{ρ_2} belong to a common member of $//'$ if ℓ, m belong to a common member of $//$ If $P = P'$, we get an **automorphism**.

Definition 1.3. A **g.p. system** is a triple $(\mathcal{P}, \mathcal{L}, //)$ where $\mathcal{P}$ is a nonempty set of elements called the **points**, $\mathcal{L}$ is a nonempty set of distinguished nonempty subsets of $\mathcal{P}$ called the **lines** and $// = (L_\iota)_{\iota \in \mathcal{D}omain\, //}$ is a family of nonempty subsets in $\mathcal{L}$ such that $\bigcup_{\iota \in \mathcal{D}omain\, //} L_\iota = \mathcal{L}$ and each member of $//$ is a decomposition in $\mathcal{P}$. If $L_\alpha \cap L_\beta = \emptyset$ whenever $\alpha \neq \beta$ we get a **parallel system**.

Definition 1.3. a. Let $P = (\mathcal{P}, \mathcal{L}, //)$ and $P' = (\mathcal{P}', \mathcal{L}', //')$ be g.p. systems. An **isomorphism** $\rho : P \to P'$ is a bijection $\rho : \mathcal{P} \to \mathcal{P}'$ having the following properties: (i) if $\ell \in \mathcal{L}$ then $\ell^\rho \in \mathcal{L}'$ and if $\ell' \in \mathcal{L}'$ then there is a line $\ell \in \mathcal{L}$ with $\ell^\rho = \ell'$; (ii) ℓ^ρ, m^ρ belong to a common member of $//'$ if ℓ, m belong to a common member of $//$.

If $P = P'$ we get an <u>automorphism</u>.

<u>Construction 1.1.</u> Let $\mathrm{T} = (S, \tau)$ be a ternary halfgroupoid. First we introduce some denotations: $\mathcal{D}omain_{i,j}\ \tau$ ($\mathcal{D}omain_{k}\ \tau$) is the projection of $\mathcal{D}omain\ \tau$ obtained by the omission of the components with prescribed indices $i, j = 1, 2, 3$ or $k = 1, 2, 3$ respectively. $\mathrm{Image}_u\ \tau$ is the set of all $\tau(x, y, u)$ such that $(x, y, u) \in \mathcal{D}omain\ \tau$ with a fixed $u \in \mathcal{D}omain_3\ \tau$. Λ_τ is the set of all $(u, v) \in S \times S$ with $u \in \mathcal{D}omain_3\ \tau$ and $v \in \mathcal{I}mage_u\ \tau$. Now put $\mathcal{P} = \mathcal{D}omain_{12}\ \tau$, $\mathcal{L} = \Lambda_\tau$, and define $I \subseteq \mathcal{P} \times \mathcal{L}$ by $(x, y)\, I\, (u, v) \Leftrightarrow \tau(x, u, u) = v$ for all admissible $(x, y, u) \in \mathcal{D}omain\ \tau$, $v \in \mathcal{I}mage_u\ \tau$. Further, set $L_u = \{(u, v) \in \Lambda_\tau \mid v \in \mathcal{I}mage_u\ \tau\}$ for every $u \in \mathcal{D}omain_3\ \tau$ and $/\!/ = \{L_u \mid u \in \mathcal{D}omain_3\ \tau\}$. Then $(\mathcal{P}, \mathcal{L}, I, /\!/)$ is a g.p. presystem which is canonically determined by T and will be denoted by $\overleftarrow{\mathbb{P}}(\mathrm{T})$.

<u>Construction 1.2.</u> Let a ternary halfgroupoid $\mathrm{T} = (S, \tau)$ be given. Put $\mathcal{P} = \mathcal{D}omain_{1,2}\ \tau$,

$\ell_{u,v} = \{(x, y) \in \mathcal{D}omain_{1,2}\ \tau \mid \tau(x, y, u) = v\}$

for each $(u, v) \in \Lambda_\tau$, $\mathcal{L} = \{\ell_{u,v} \mid (u, v) \in \Lambda_\tau\}$, $L_u = \{\ell_{u,v} \mid v \in \mathcal{I}mage_u\ \tau\}$ for each $u \in \mathcal{D}omain_3\ \tau$, $/\!/ = (L_u \mid u \in \mathcal{D}omain_3\ \tau$. Then $(\mathcal{P}, \mathcal{L}, /\!/)$ is a g.p. system which is canonically determined by T. This g.p. system shall be denoted by $\overleftarrow{\mathbb{P}}(\mathrm{T})$.

Construction 1.3. Let a g.p. presystem $P=(\mathcal{P},\mathcal{L},\mathrm{I},/\!/)$ be given where $P \subseteq S \times S$ for a sufficiently large set S. Then we can choose injections $\alpha : /\!/ \to S$ and $\beta_L : L \to S$ (for $L \in /\!/$) and define τ by $\tau(x,y,u)=v \iff (x,y)\,\mathrm{I}\,\beta^{-1}_{\alpha^{-1}(u)}(v)$ for all admissible $(x,y)\in\mathcal{P}$, $u\in\alpha(/\!/)$ and $v\in\beta_{\alpha^{-1}(u)}(\alpha^{-1}(u))$. This τ is well-defined on a certain subset of $S\times \times S\times S$ so that a ternary halfgroupoid (S,τ) is obtained. It is canonically determined by P, α and $(\beta_L)_{L\in/\!/}$, and it will be denoted by $\mathbb{T}(P,\alpha,(\beta_L)_{L\in/\!/})$.

Construction 1.4. Let a g.p. system $P=(\mathcal{P},\mathcal{L},/\!/)$ be given with $\mathcal{P}\subseteq S\times S$, S being a sufficiently large set. Then we can choose injections $\alpha : \mathcal{D}omain\ /\!/ \to S$ and $\beta_\iota : L_\iota \to S$ (for $\iota \in \mathcal{D}omain\ /\!/$) and define τ by $\tau(x,y,u)=v \iff (x,y)\in\beta^{-1}_{\alpha^{-1}(u)}(v)$ for all admissible $(x,y)\in\mathcal{P}$, $u\in\alpha(/\!/)$, $v\in\beta_{\alpha^{-1}(u)}(\alpha^{-1}(u))$. We obtain, as in Construction 1.3, a ternary halfgroupoid (S,τ) which is canonically determined by P, α, $(\beta_\iota)_{\iota\in\mathcal{D}omain\ /\!/}$, and which will be denoted by $\mathbb{T}(P,\alpha,(\beta_\iota)_{\iota\in\mathcal{D}omain\ /\!/})$.

Construction 1.5. Let $P=(\mathcal{P},\mathcal{L},\mathrm{I},/\!/)$ be a g.p. presystem. Put $\bar{\ell}=\{p\in\mathcal{P}\mid p\,\mathrm{I}\,\ell\}$ for each $\ell\in\mathcal{L}$. Define $\bar{\mathcal{L}}$ as the set $\{\bar{\ell}\mid \ell\in\mathcal{L}\}$. Further choose a bijection $\alpha : J \to /\!/$ where J is a convenient index set. Now let $\overline{/\!/}$ denote the family $(\overline{\alpha(\iota)})_{\iota\in J}$ where $\overline{\alpha(\iota)}=\{\bar{\ell}\mid \ell\in\alpha(\iota)\}$ for all $\iota\in J$. Then $(\mathcal{P},\bar{\mathcal{L}},\overline{/\!/})$ is a g.p. system which is canonically determined by P and α. This g.p. system will be denoted by $\widehat{\mathbb{P}}(P)$.

<u>Construction 1.6</u>. Let $T = (S, \tau)$ be a ternary halfgroupoid satisfying the middle cancellation law: if $\tau(x, y_1, u) = \tau(x, y_2, u)$ for (x, y_1, u), $(x, y_2, u) \in \mathcal{D}omain\ \tau$ then $y_1 = y_2$. Define $\tau^{\bullet}$ by $\tau^{\bullet}(x, u, v) = y \Leftrightarrow \tau(x, y, u) = v$ for all $(x, y, u) \in \mathcal{D}omain\ \tau$. Then $\tau^{\bullet}$ is well-defined on some uniquely determined subset of $S \times S \times S$ and $T^{\bullet} = (S, \tau^{\bullet})$ is a ternary halfgroupoid satisfying the right cancellation law: if $\tau^{\bullet}(x, u, v_1) = \tau^{\bullet}(x, u, v_2)$ for $(x, u, v_1), (x, u, v_2) \in \mathcal{D}omain\ \tau^{\bullet}$ then $v_1 = v_2$. Conversely, if $T = (S, \tau)$ is a ternary halfgroupoid satisfying the right cancellation law, we may define $\hat{\tau}$ by $\hat{\tau}(x, y, u) = v \Leftrightarrow \tau(x, u, v) = y$ for all $(x, u, v) \in \mathcal{D}omain\ \tau$. Such $\hat{\tau}$ is well-defined on some uniquely determined subset of $S \times S \times S$ and the obtained ternary halfgroupoid $\hat{T} = (S, \hat{\tau})$ satisfies the middle cancellation law.

<u>Remarks</u>. If $P = (\mathcal{P}, \mathcal{L}, I, /\!/)$ is a g.p. system then $\breve{\mathbb{P}}(\mathbb{T}(P, \alpha, (\beta_L)_{L \in /\!/})$ is isomorphic to P. If $P = (\mathcal{P}, \mathcal{L}, /\!/)$ is a g.p. system then $\overline{\mathbb{P}}(\mathbb{T}(P, \alpha, (\beta_\iota)_{\iota \in \mathcal{D}omain /\!/})) = P$. If P and P' are isomorphic g.p. presystems then also $\widehat{\mathbb{P}}(P)$, $\widehat{\mathbb{P}}(P')$ are isomorphic. If $T = (S, \tau)$ is a ternary halfgroupoid satisfying the middle cancellation law then define τ^* by $\tau^*(u, v, x) = y \Leftrightarrow \tau^{\bullet}(x, u, v) = v$

for all $(x, u, v) \in \mathit{Domain}\ \tau^{\bullet}$. The obtained halfgroupoid $T^* = (S, \tau^*)$ is said to be <u>dual</u> to T (and also $\overleftrightarrow{\mathbb{P}}(T), \overleftrightarrow{\mathbb{P}}(T^*)$ or $\overline{\mathbb{P}}(T), \overline{\mathbb{P}}(T^*)$ respectively can be said to be mutually <u>dual</u>). Clearly $(T^*)^* = T$.

§ 2. <u>Proposition 2.1.</u> Let σ be an autotopism of a given ternary halfgroupoid $T = (S, \tau)$. Then the mappings $(x, y) \to (x^{\sigma_1}, y^{\sigma_2})$ for $(x, y) \in \mathit{Domain}_{1,2}\ \tau$ and $(u, v) \to (u^{\sigma_3}, v^{\sigma_4})$ for $(u, v) \in \Lambda_\tau$ define an automorphism of $\overleftrightarrow{\mathbb{P}}(T)$.

<u>Proposition 2.2.</u> Let a g.p. presystem $P = (\mathcal{P}, \mathcal{L}, I, /\!/)$ be given where $\mathcal{P} = S_1 \times S_2$ for some sets S_1 and S_2 with $\mathit{card}\ S_1 \geq 2$, $\mathit{card}\ S_2 \geq 2$. Let S_3 and S_4 be arbitrary sets such that there is a bijection $\alpha : /\!/ \to S_3$ and there are injections $\beta_L : L \to S_4$ (for $L \in /\!/$) with $\bigcup_{L \in /\!/} \beta_L(L) = S$ and with $\beta_L(L) \cap \beta_M(M) = \emptyset$ whenever L, M are distinct members of $/\!/$. Then each coordinate automorphism[2] $\rho : P \to P$ induces an autotopism of $\mathbb{T}(P, \alpha, (\beta_L)_{L \in /\!/})$. If, moreover, $X \in /\!/$ with $\beta_X(x(b)) = b$ for $b \in S_2$ then $\sigma_4 \mid S_2 = \sigma_2$ and $0^{\sigma_3} = 0$ for $0 = \alpha(X)$.

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2 i.e., an automorphism of P preserving X as well as Y where (and also in the following)

$$X = \{\{(x, y) \in S_1 \times S_2 \mid y = b\} \mid b \in S_2\},\ Y = \{\{(x, y) \in S_1 \times S_2 \mid x = a\} \mid a \in S_1\}.$$

Proposition 2.3. Let $P = (\mathcal{P}, \mathcal{L}, /\!/)$ be a paralell system with $/\!/ = (L_\iota)_{\iota \in S}$ and with $\mathcal{P} = S \times S$ for a certain set S, $card\ S \geq 2$. Let $X = L_0$ for some element $0 \in S$ and $card\,(\mathcal{Y}(0) \cap \ell) = 1$ for each $\ell \in \mathcal{L}$. Then there is a $T = \mathbb{T}(P$, identity, $(\beta_\iota)_{\iota \in S})$ such that every coordinate automorphism $\varphi : P \rightarrow P$ induces an autotopism σ of T with $0^{\sigma_3} = 0$ and $\sigma_2 = \sigma_4$. Conversely, each autotopism σ of T with $0^{\sigma_3} = 0$ induces a coordinate automorphism of P.

Proposition 2.4. Let $P = (\mathcal{P}, \mathcal{L}, /\!/)$ be a parallel system with $/\!/ = (L_\iota)_{\iota \in S}$ and with (i) $\mathcal{P} = S \times S$ where S is a set, $card\ S \geq 2$, (ii) $X = L_0$ for some element $0 \in S$, (iii) $card\,(\mathcal{Y}(0) \cap \ell) = 1$ for all $\ell \in \mathcal{L}$, (iv) $d = \{(x, y) \in S \times S \mid x = y\} \in L_1$ for some element $1 \in S$ and (v) each point of $\mathcal{Y}(1)$ is contained in a unique line through $(0, 0)$ and each line through $(0, 0)$ intersects $\mathcal{Y}(1)$ in exactly one point. Then there is a $T = \mathbb{T}(\mathcal{P}, \alpha, (\beta_\iota)_{\iota \in S})$ such that every coordinate automorphism of P fixing $(0, 0)$ and $(1, 1)$ induces an automorphism of T fixing 0 (and 1). Conversely, every automorphism of T preserving 0 induces a coordinate automorphism of P fixing $(0, 0)$ and $(1, 1)$.

§ 3. **Definition 3.1.** A parallel system $P = (\mathcal{P}, \mathcal{L}, /\!/)$ is said to be **natural** if

(a) $\mathcal{P} = S \times S$ for a set S, $card\ S \geq 2$,

(b) $Domain\ /\!/ = S$, i.e. $/\!/ = (L_\iota)_{\iota \in S}$,

(c) $X = L_0$ for some element $0 \in S$,

(d) $card(x(a) \cap \ell) = card(y(a) \cap \ell) = 1$ for all $a \in S$ and $\ell \in \mathcal{L} \setminus (X \cup Y)$ and

(e) $d = \{(x, y) \in S \times S \mid x \in y\} \in \mathcal{L}$.

Definition 3.2. A ternary groupoid $T = (S, \tau)$ is said to be natural if (1^{τ}) for $u_1, u_2, v \in S$ with $u_1 \neq u_2$ there exist $x, y_1, y_2 \in S$ with $y_1 \neq y_2$ such that $\tau(x, y_1, u_1) \neq \tau(x, y_2, u_2)$, ($2^{\tau}$) the equation $\tau(x, y, u) = v$ has a unique solution $x \in S$ ($y \in S$) for any given $y, u, v \in S$ with $u \neq 0$ ($x, u, v \in S$), (3^{τ}) there is an element $0 \in S$ with $\tau(a, b, 0) = \tau(0, b, a) = b$ for all a, $b \in S$ and (4^{τ}) there is an element $1 \in S$ such that $\tau(a, a, 1) = 0$ for all $a \in S$.

Proposition 3.1. If $T = (S, \tau)$ is a natural ternary groupoid then (A) $0 \neq 1$, (B) from $\tau(x, y, u_1) = v_1 \Leftrightarrow \tau(x, y, u_2) = v_2$ for fixed $(u_1, v_1), (u_2, v_2) \in S \times S$ it follows $(u_1, v_1) = (u_2, v_2)$ and (C) $T^{\bullet}$ is characterized by the following conditions:

($5^{\tau^{\bullet}}$) for $u_1, u_2, v \in S$ with $u_1 \neq u_2$ there is an $x \in S$ such that $\tau^{\bullet}(x, u_1, v) \neq \tau^{\bullet}(x, u_2, v)$,

($6^{\tau^{\bullet}}$) the equation $\tau^{\bullet}(x, u, v) = y$ has a unique solution $x \in S$ ($v \in S$) for any given $u, v, y \in S$ with $u \neq 0$ ($x, y, u \in S$) ,

($7^{\tau^{\bullet}}$) there is an element $0 \in S$ such that $\tau^{\bullet}(a, 0, b) = \tau^{\bullet}(0, a, b)$ for all $a, b \in S$ and

($8^{\tau^{\bullet}}$) there is an element $1 \in S$ such that

$\tau^{\cdot}(a, 1, 0) = a$ **for all** $a \in S$.

<u>Proposition 3.2</u>. If $T = (S, \tau)$ is a natural ternary groupoid then $\mathbb{P}(T)$ is a natural parallel system. If $P = (\mathcal{P}, \mathcal{L}, /\!/)$ is a natural parallel system then there is a $T = \mathbb{T}(P, \alpha, (\beta_\iota)_{\iota \in S})$ which is natural.

<u>Proposition 3.3</u>. Let $T = (S, \tau)$ be a natural ternary groupoid. Define the <u>derived binary operations</u> $\overset{\tau^{\cdot}}{+}$, $\overset{\tau^{\cdot}}{\cdot}$ by $a \overset{\tau^{\cdot}}{+} b = \tau^{\cdot}(a, 1, b)$, $a \cdot b = \tau^{\cdot}(a, b, 0)$. Then $(S, \overset{\tau^{\cdot}}{+})$ is a loop and $(S \setminus \{0\}, \overset{\tau^{\cdot}}{\cdot})$ is a groupoid having the right unity and admitting the division from left; further it holds $a \overset{\tau^{\cdot}}{\cdot} 0 = 0 \overset{\tau^{\cdot}}{\cdot} a = 0$ for all $a \in S$.

<u>Proposition 3.4</u>. Let $T = (S, \tau)$ be a ternary groupoid satisfying $(7^{\tau^{\cdot}})$ and $(8^{\tau^{\cdot}})$. Let the <u>linearity property</u>

$(9^{\tau^{\cdot}})$ $\tau^{\cdot}(a, b, c) = a \overset{\tau^{\cdot}}{\cdot} b \overset{\tau^{\cdot}}{+} c$ for all $a, b, c \in S$

be valid. Then T is natural iff $(S, \overset{\tau^{\cdot}}{+})$ is a loop, $(S \setminus \{0\}, \overset{\tau^{\cdot}}{\cdot})$ is a groupoid with the right unity and with the division from left and, for $u_1 \neq u_2$, the right multiplications $R_{u_1} : x \to x \overset{\tau^{\cdot}}{\cdot} u_1$, $R_{u_2} : x \to x \overset{\tau^{\cdot}}{\cdot} u_2$ are distinct.

<u>Proposition 3.5</u>. Let $(S, +)$ be a loop with $\operatorname{card} S \geq 2$. Then each natural ternary groupoid $T = (S, \tau)$ with $\overset{\tau^{\cdot}}{+} = +$ and with $(9^{\tau^{\cdot}})$ may be constructed as follows: Choose an injection $f : S \to S^S$ such that $S^{f(0)} = \{0\}$, $f(a) : S \to S$ is a bijection for each $a \in S \setminus \{0\}$ and $f(1) : S \to S$

is the identity mapping. Define the binary operation $\cdot$ by $x \cdot y = x^{f(y)}$ for all $x, y \in S$. Then T is determined by $\overset{\tau^{\cdot}}{\cdot} = \cdot$.

<u>Proposition 3.6</u>. Let $T = (S, \tau)$ be a natural ternary groupoid. T satisfies $(9^{\tau^{\cdot}})$ and $(S, \overset{\tau^{\cdot}}{+})$ is a group iff there is a group of translations [3] of $P = \overline{\mathbb{P}}(T)$ acting transitively on $y(0)$.

§ 4. <u>Definition 4.1</u>. Let $T^{\cdot} = (S, \tau^{\cdot})$ be a ternary groupoid satisfying $(6^{\tau^{\cdot}})$, $(7^{\tau^{\cdot}})$ and $(8^{\tau^{\cdot}})$. $T^{\cdot}$ is said to be <u>ordered</u> if there is an ordering [4] $<$ on S such that

$(10^{\tau^{\cdot}})$ $v_1 < v_2 \Rightarrow \tau^{\cdot}(x, u, v_1) < \tau^{\cdot}(x, u, v_2)$ and

$(11^{\tau^{\cdot}})$ if $x_0, u_1, v_1, u_2, v_2 \in S$ satisfy $u_1 < u_2$ and $\tau^{\cdot}(x_0, u_1, v_1) = \tau^{\cdot}(x_0, u_2, v_2)$ then $x \gtrless x_0 \Rightarrow \tau^{\cdot}(x, u_1, v_1) \lessgtr \tau^{\cdot}(x, u_2, v_2)$.

Denotation: $(S, \tau^{\cdot}, <)$; conditions $(6^{\tau^{\cdot}})$ to $(8^{\tau^{\cdot}})$ are here required automatically.

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[3] i.e., of coordinate automorphisms of P which preserve each $y(a)$, $a \in S$.

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[4] An <u>ordering</u> on a set S is meant here as a binary relation $<$ on S such that $a < b \Rightarrow a \neq b$; $a < b$ and $b < c \Rightarrow a < c$; $a \neq b \Rightarrow a < b$ or $b < a$.

__Proposition 4.1.__ Let $T^{\cdot} = (S, \tau^{\cdot}, <)$ be an ordered ternary groupoid. Then $(5^{\tau^{\cdot}})$ is valid, and the elements $0, 1$ from $(7^{\tau^{\cdot}})$ and $(8^{\tau^{\cdot}})$ respectively are determined uniquely.

__Proposition 4.2.__ Let $T = (S, \tau^{\cdot})$ be a ternary groupoid with $(6^{\tau^{\cdot}})$ to $(9^{\tau^{\cdot}})$ and such that $(S \setminus \{0\}, \overset{\tau^{\cdot}}{+})$ is a group. If $<$ is an ordering on S then $(10^{\tau^{\cdot}})$ is equivalent to

$(12_{1,2}^{\tau^{\cdot}})$ $a < b \Rightarrow a \overset{\tau^{\cdot}}{+} c < b \overset{\tau^{\cdot}}{+} c,\ c \overset{\tau^{\cdot}}{+} a < c \overset{\tau^{\cdot}}{+} b$

and $(11^{\tau^{\cdot}})$ is equivalent to

$(13^{\tau^{\cdot}})$ for $u_2 < u_1$, the mapping $x \to \overset{\tau^{\cdot}}{-} x \overset{\tau^{\cdot}}{\cdot} u_2 \overset{\tau^{\cdot}}{+} x \overset{\tau^{\cdot}}{\cdot} u_1$ is monotonically increasing.

__Proposition 4.3.__ There exists a ternary groupoid $(S, \tau^{\cdot})$ with $(6^{\tau^{\cdot}})$ to $(9^{\tau^{\cdot}})$ and with an ordering $<$ on S such that $(S \setminus \{0\}, \overset{\tau^{\cdot}}{\cdot})$ is not a loop and that one of the following three alternatives takes place:

(i) $(10^{\tau^{\cdot}})$, $(12_1^{\tau^{\cdot}})$ are valid; $(12_2^{\tau^{\cdot}})$ is not valid,

(ii) $(10^{\tau^{\cdot}})$, $(12_{1,2}^{\tau^{\cdot}})$ are valid; $(11^{\tau^{\cdot}})$ is not valid,

(iii) $(10^{\tau^{\cdot}})$, $(11^{\tau^{\cdot}})$ are valid.

Let $P = (\mathcal{P}, \mathcal{L}, /\!/)$ be a natural parallel system. By $Q_{(c,d)}$, we denote the set $\{\ell \in \mathcal{L} \setminus Y \mid (c,d) \in \ell\}$ for $(c,d) \in S \times S$. Each ordering $<$ on S determines naturally the __induced ordering__ on every $y(a)$, $a \in S$ on every $Q \in /\!/ \setminus \{Y\}$ and on every $Q_{(c,d)}$, $(c,d) \in S \times S$.

__Definition 4.2.__ Let $P = (\mathcal{P}, \mathcal{L}, /\!/)$ be a natural parallel system. P is said to be __ordered__ if there is an ordering $<$ on S such that (i) each mapping $Q \to y(a)$ defined by $\ell \to \ell \cap y(a)$ for

$Q \in /\!/ \setminus \{Y\}$, $a \in S$ preserves the induced ordering, (ii) each mapping $Q_{(c,d)} \rightarrow y(a)$ defined by $\ell \rightarrow \ell \cap y(a)$ for $(c, d) \in S \times S$, $a \in S$, $a < d$ preserves the induced ordering and (iii) each mapping $Q_{(c,d)} \rightarrow y(a)$ defined by $\ell \rightarrow \ell \cap y(a)$ for $(c, d) \in S \times S$, $a \in S$, $a > d$ reverses the induced ordering.

<u>Proposition 4.4.</u> If $T^{\cdot} = (S, \tau^{\cdot}, <)$ is an ordered ternary groupoid then $\overline{\mathbb{P}}(T)$ is ordered by $<$. If $P = (S \times S, \mathcal{L}, /\!/, <)$ is an ordered natural parallel system then $(\mathbb{T}(P))^{\cdot}$ is ordered by $<$.

R e f e r e n c e s

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