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FIXED POINT THEOREMS BASED ON LERAY-SCHAUDER DEGREE

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1. Introduction. This note deals with the existence of the solution of the equation $Fx = x$. The technique used here is the so-called Leray-Schauder topological degree. This method has been largely used by Altman [1], J. Cronin [2], Leray and Schauder [6] and others.

In this paper there is given a generalization of Altman's result [1] (see Remark 2). Theorem 8 is the extending of Theorem 7 for a special case.

Other theorems are well known, but the proofs are based on topological degree theory (see Theorem 1 and Theorem 10). These theorems were proved in [7], [4] respectively, by an another way.

Using the theory of topological degree and the properties of Minkovski's function we prove Schauder's fixed-point theorem for a convex set (compare [2], p.139).

2. Terminology and notations used in this paper. Banach space is denoted by X, its norm by $\| \, \|$ and θ is the zero element in X. The symbol $f: M \rightarrow X$ denotes mapping defined on $M \subset X$ with range in X. Let be I identity mapping in X and F be completely continuous operator defined on $M \subset X$. The ball K_R in Banach space is set of all x,

$\| x \| < R$ and S_R is its boundary. Let M be subset of X and $f: M \to X$. The symbol $\overline{M}$ denotes closure of M (in norm of X), ∂M its boundary and $f(M)$ the image under the transformation f.

3. Brouwer's and Leray-Schauder's degree. Let G be a bounded and open subset of X, $f: \overline{G} \to X$ and $z_0 \in X - f(\partial G)$ (arbitrary but fixed). If X is finite dimensional Banach space let f be continuous mapping on $\overline{G}$ and in another case let be $f = I - F$, where F is a completely continuous operator.

Degree $d[f, G, z_0]$ has following properties:

I. $d[I, G, z_0] = 1$ for $z_0 \in G$.

II. $d[I, G, z_0] = 0$ for $z_0 \notin \overline{G}$.

III. Let G_1, G_2 be bounded and open subsets of X, $f: \overline{G_1 \cup G_2} \to X$ and $z_0 \in X - (f(\partial G_1) \cup f(\partial G_2))$, $G_1 \cap G_2 = \emptyset$. Then is $d[f, G_1 \cup G_2, z_0] = d[f, G_1, z_0] + d[f, G_2, z_0]$.

IV. Let $d[f, G, z_0] \neq 0$. Then exists $x_0 \in G$ such that $f(x_0) = z_0$.

V. Finite dimensional case. Let $f(x,t): \overline{G} \times \langle 0, 1 \rangle \to X$ be continuous mapping such that for all $x \in \partial G$ and for all $t \in \langle 0, 1 \rangle$ is $f(x,t) \neq z_0 \in X$. Then $d[f(x,t), G, z_0]$ is constant on $\langle 0, 1 \rangle$.

Infinite dimensional case. Let $F: \overline{G} \to X$ be completely continuous operator such that for all $x \in \partial G$ and for all $t \in \langle 0, 1 \rangle$ is $(I - tF)x \neq z_0 \in X$. Then $d[I, G, z_0] = d[I - F, G, z_0]$.

4. Fixed point theorems in finite dimensional space.

Theorem 1 ([7]): Let f be a continuous mapping of $\overline{K}_R$ into finite dimensional space X and let m be any constant. Suppose that f satisfies one of the following two conditions:

(1) For $x \in S_R$ such that $f(x) = ax$ is $a \leq m$,

(2) For $x \in S_R$ such that $f(x) = ax$ is $a \geq m$.

Then there exists at least one element $x_0 \in \overline{K}_R$ such that $f(x_0) = mx_0$.

Proof: Let the condition (1) be held. We assume that for all $x \in S_R$ is $f(x) \neq mx$ and we define

$$H(x,t) = tx - (1-t)(f(x) - mx)$$

for all $x \in \overline{K}_R$ and for $t \in \langle 0,1 \rangle$. The function $H(x,t)$ satisfies assumptions in V and we have (by V and I):

$$d[-f(x) + mx, K_R, \theta] = d[I, K_R, \theta] = 1 .$$

The theorem is proved by the property IV. Analogously for condition 2.

Theorem 2 ([7]): Let f be continuous mapping of $\overline{K}_R$ into X satisfying one of the following two conditions:

(3) For each $x \in S_R$ there is $\|x - f(x)\|^2 \geq$
$\geq \|f(x)\|^2 + \|x\|^2$.

(4) For each $x \in S_R$ there is $\|x - f(x)\|^2 \leq$
$\leq \|f(x)\|^2 + \|x\|^2$.

Then there exists a point $x_0 \in \overline{K}_R$ such that $f(x_0) =$
$= \theta$.

Proof: Let the condition (3) be held. We suppose there exists $x \in S_R$ such that $f(x) = ax$. The condition (3) implies that $a \leq 0$ and the condition (1) is satisfied. Analogously for the condition (4).

<u>Theorem 3</u> ([7]): Let f be a continuous mapping of $\overline{K}_R$ into X satisfying one of the following conditions:

(5) For each $x \in S_R$ there is $\| f(x) - x \|^2 \geq$
$\geq \| f(x) \|^2 - \| x \|^2$.

(6) For each $x \in S_R$ there is $\| f(x) - x \|^2 \leq$
$\leq \| f(x) \|^2 - \| x \|^2$.

Then there exists a point $x_0 \in \overline{K}_R$ such that $f(x_0) = x_0$.

<u>Proof</u>: Define for all $x \in \overline{K}_R$ $g(x) = x - f(x)$. The function g satisfies the conditions of Theorem 2.

<u>Theorem 4</u> (Rothe): If f is a continuous mapping of $\overline{K}_R$ into X such that $f(S_R) \subset \overline{K}_R$, then there exists $x_0 \in \overline{K}_R$ such that $f(x_0) = x_0$.

<u>Proof</u>: The assumptions in this theorem imply the condition (5) of Theorem 3.

<u>Theorem 5</u> (Brouwer): Let $M \subset X$ be homeomorphic with $\overline{K}_R$ and f be a continuous mapping of M into X such that $f(M) \subset M$. Then there exists $x_0 \in M$ such that $f(x_0) = x_0$.

<u>Proof</u>: Let be $M = \overline{K}_R$. The assertion is valid by Theorem 4. Let be $M = h(\overline{K}_R)$ where h is homeomorphism. Then $hfh^{-1}(\overline{K}_R) \subset \overline{K}_R$ and there exists $w_0 \in \overline{K}_R$ such that $hfh^{-1}(w_0) = w_0$. The point $x_0 = h^{-1}(w_0)$ satisfies the equation $f(x_0) = x_0$.

<u>Theorem 6</u>: If f is a continuous mapping of $\overline{K}_R$ into X , there exist a point $x_0 \in \overline{K}_R$ and a real number $\lambda_0 \geq 0$ such that $f(x_0) = \lambda_0 x_0$.

<u>Proof</u>: If there exists point $x_0 \in \overline{K}_R$ such that

$f(x_0) = \theta$ we have $\lambda_0 = 0$. Let for each $x \in \bar{K}_R$ be $f(x) \neq \theta$. Define function g for every $x \in \bar{K}_R$ as $g(x) = R \cdot \frac{f(x)}{\|f(x)\|}$. We obtain $g(\bar{K}_R) \subset \bar{K}_R$ and $x_0 \in \bar{K}_R$ such that $g(x_0) = x_0$ i.e. $f(x_0) = \lambda_0 x_0$, where $\lambda_0 = \frac{\|f(x_0)\|}{R}$.

5. Fixed point theorems in infinite dimensional space.

Lemma 1. Let p be a real number, $p > 0$, $p \neq 1$ and $a \geq b > 0$. Let the inequality

$$[\operatorname{sgn}(p-1)](a-b)^p \geq [\operatorname{sgn}(p-1)](a^p - b^p)$$

be satisfied. Then $a = b$.

Lemma 2. Let p be a real number, $p \neq 0, 1$ and $a > b > 0$. Then

$$[\operatorname{sgn}(p-1)](a-b)^p < [\operatorname{sgn}(p-1)](a^p - b^p).$$

Theorem 7: Let F be a completely continuous operator of $\bar{K}_R$ into X and p be a function of S_R into real numbers such that $p(x) > 0$ and $p(x) \neq 1$ for every $x \in S_R$. Let the following condition be fulfilled for each $x \in S_R$:

$$[\operatorname{sgn}(p(x)-1)]\,\|x - Fx\|^{p(x)} \geq [\operatorname{sgn}(p(x)-1)](\|Fx\|^{p(x)} - \|x\|^{p(x)}).$$

Then there exists $x_0 \in \bar{K}_R$ such that $Fx_0 = x_0$.

Proof: We suppose that for each $x \in S_R$ is $Fx \neq x$. Define the operator $H(x,t) = x - tFx$ for every $x \in \bar{K}_R$ and each $t \in \langle 0, 1 \rangle$.

Lemma 1 proves that the assumptions of property V are valid and by property I and IV we obtain the assertion of Theorem 7.

Theorem 8: Let F be a completely continuous operator of $\overline{K}_R$ into X such that for every $x \in S_R$ there is $Fx \neq \theta$ and $Fx \neq x$.

Let p be a function of S_R into real numbers such that for each $x \in S_R$ there is $p(x) \neq 0$, $p(x) \neq 1$. Assume that for all $x \in S_R$

$$[\operatorname{sgn}(p(x)-1)]\,\|x - Fx\|^{p(x)} \geq [\operatorname{sgn}(p(x)-1)]\,(\|Fx\|^{p(x)} - \|x\|^{p(x)}).$$

Then there exists the point $x_0 \in K_R$ such that $Fx_0 = x_0$.

Proof: As Theorem 7 but by using Lemma 2.

Remark 1: In Theorem 7 and 8 we can assume that F is defined on set $\overline{G}$, $\theta \in G$ and the conditions are satisfied on ∂G.

Remark 2: If $p(x) = 2$ in Theorem 7, we obtain Altman's fixed point theorem [1]. This Altman's condition is equivalent to $(Fx, x) \leq (x, x)$ in Hilbert space, where $(\cdot,\cdot)$ is inner product.

Theorem 9 (Rothe): Let F be a completely continuous operator of $\overline{K}_R$ into X such that $F(S_R) \subset \overline{K}_R$. Then there exists $x_0 \in \overline{K}_R$ such that $Fx_0 = x_0$.

Proof: As in finite dimensional case.

Theorem 10 ([4]): Let X be Hilbert space with inner product $(\cdot,\cdot)$, $y \in X$, f a mapping of $\overline{K}_R$ into X satisfying one of the following two conditions:

(7) For each $x \in S_R$ there is $(x, f(x)) \leq (x, y)$ and $I - f$ is a completely continuous operator,

(8) For each $x \in S_R$ there is $(x, f(x)) \geq (x, y)$ and $I + f$ is a completely continuous operator.

Then there exists $x_0 \in \overline{K}_R$ such that $f(x_0) = y$.

Proof: Let the condition (7) be satisfied and let be

$F = I + f - y$. Then F is a completely continuous operator and for each $x \in S_R$ is $(Fx, x) = (x, x) + (f(x), x) - (x, y) \leq (x, x)$ and the remark 2 proves this theorem. Analogously for the condition (8).

Remark 3: Let M be an open bounded subset of X , $\theta \in M$, $\overline{M}$ a convex set. Set $J(x) = \{a, a > 0, \frac{x}{a} \in \overline{M}\}$ for every $x \in X$ and $t(x) = \inf_{a \in J(x)} a$. The function t has the following properties [3]:

$0 \leq t(x) < \infty$ for $\lambda \geq 0$ and $x \in K$ is $t(\lambda x) = \lambda t(x)$, for $x, y \in X$ is $t(x + y) \leq t(x) + t(y)$ and

(9) $x \in M$ if and only if $t(x) < 1$,

(10) $x \in \partial M$ if and only if $t(x) = 1$,

(11) $x \notin \overline{M}$ if and only if $t(x) > 1$.

Theorem 11 (Schauder): Let M be an open bounded set of X , $\overline{M}$ a convex set and F a completely continuous operator of $\overline{M}$ into X such that $F(\overline{M}) \subset \overline{M}$. Then there exists $x_0 \in \overline{M}$ such that $Fx_0 = x_0$.

Proof: We assume that for all $x \in \partial M$ is $Fx \neq x$ and $\theta \in M$. Let exist $\tau_1 \in (0, 1)$ and $x_1 \in \partial M$ such that $x_1 = \tau_1 F x_1$. Then $t(Fx_1) = t(\frac{x_1}{\tau_1}) = \frac{1}{\tau_1} > 1$, and (11) implies $Fx_1 \notin \overline{M}$, and this contradicts our assumptions. The assertion is valid by property V , I and IV. If $\theta \notin M$ we must use a translation for the set M .

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