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Mean value theorems in the theory of lattice points with weight

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§ 1. Introduction. Let $\kappa$ be a natural number, $\kappa \geq 2$, and let

$$Q(x, \alpha_{\kappa}) = \sum_{j_1, \ldots, j_{\kappa}} a_{j_1} \ldots a_{j_{\kappa}} x_{j_1} \ldots x_{j_{\kappa}}$$

be a positive definite quadratic form with integer coefficients and the determinant $D$. Let further $M_1, M_2, \ldots, M_\kappa$ be natural numbers and $b_1, b_2, \ldots, b_\kappa$ integers. For arbitrary real numbers $\alpha_1, \alpha_2, \ldots, \alpha_\kappa$ and $\chi > 0$ let

$$A(x, \alpha_{\kappa}) = \sum e^{2\pi i \alpha_j x_j},$$

where the summation is over all systems $\mu_1, \mu_2, \ldots, \mu_\kappa$ of real numbers satisfying

$$\mu_j \equiv b_j \text{ (mod } M_j \text{)} \quad (j = 1, 2, \ldots, \kappa)$$

and

$$0 < \alpha_j \leq \chi.$$

Let us put as usually

$$V(x) = V(x, \alpha_{\kappa}) = \frac{M^{\frac{\kappa}{2}} \epsilon^{\frac{\kappa}{2} \sum \alpha_j b_j}}{\Gamma(\frac{\kappa}{2} + 1)} \sigma \quad (M = \frac{\pi^{\frac{\kappa}{2}}}{\sqrt{D^{\frac{\kappa}{2}} M_{\kappa}}}; \quad \sigma = 1 \text{ if all numbers } \alpha_1 M_1, \alpha_2 M_2, \ldots, \alpha_\kappa M_\kappa \text{ are integers, } \sigma = 0 \text{ otherwise})$$

and let us consider the "lattice rest"
(1) \( P(\alpha) = P(\alpha; \alpha_2) = A(\alpha) - V(\alpha). \)

As is known (see [5] pp. 11-84), we have

\[ P(\alpha) = o(\alpha^{\frac{k}{k+1}}) \]

and (if \( A(\alpha) \neq 0 \) - we shall exclude from our considerations the case where \( A(\alpha) = 0 \) identically)

\[ P(\alpha) = \Omega(\alpha^{\frac{k}{k+1}}). \]

In the papers [6] - [11] there were proved the following results:

I. Let \( k > 4 \).

a) There always holds

\[ P(\alpha) = o(\alpha^{\frac{k}{k-1}}). \]

b) If \( \alpha_1, \alpha_2, \ldots, \alpha_k \) are rational numbers we have either

\[ P(\alpha) = \Omega(\alpha^{\frac{k}{k-1}}) \]

or

\[ P(\alpha) = O(\alpha^{\frac{k}{k-1}}). \]

c) If at least one of the numbers \( \alpha_1, \alpha_2, \ldots, \alpha_k \) is irrational, then

\[ P(\alpha) = \sigma(\alpha^{\frac{k}{k-1}}). \]

d) If \( \varphi(\alpha) \) is a positive non-increasing function, \( \varphi(\alpha) = \sigma(1) \), there exists a system \( \alpha_1, \alpha_2, \ldots, \alpha_k \) such that

\[ P(\alpha) = \sigma(\alpha^{\frac{k}{k-1}}) \quad \text{and} \quad P(\alpha) = \Omega(\alpha^{\frac{k}{k-1}} \varphi(\alpha)) \]

hold.
e) For almost all systems $\alpha_1, \alpha_2, \ldots, \alpha_\kappa$ (in the sense of the Lebesgue measure in the $\kappa$-dimensional Euclidean space $E_\kappa$) there is

$$P(x) = 0 \left( x^{\frac{\kappa}{\kappa-\gamma}} \log^{\frac{\kappa}{\kappa-\gamma}} x \right)$$

(see [6], Theorems 3, 4, 5 and [10], p. 67).

II. Let $\kappa > 5$, $\alpha_1 = \alpha_2 = \ldots = \alpha_\kappa = \alpha$, and let $\gamma = \gamma(\alpha)$ be the supremum of all numbers $\beta > 0$ for which the inequality

$$\min_{\nu \in \text{integers}} |x \cdot \kappa - \nu| \leq \frac{c}{x^{\beta}}$$

is satisfied for infinitely many natural $\kappa$'s, $c$ being a positive constant depending at most on $\alpha$ and $\beta$. Let us put

$$f = \left( \frac{\kappa}{4} - \frac{1}{2} \right) \frac{2 \gamma + 1}{\gamma + 1}$$

(for $\gamma = +\infty$ put $f = \frac{\kappa}{2} - 1$). Then

$$P(x) = 0 \left( x^{f+\varepsilon} \right)$$

for every $\varepsilon > 0$. If $\ell_1 = \ell_2 = \ldots = \ell_\kappa = 0$, then we have, for every $\varepsilon > 0$,

$$P(x) = \Omega \left( x^{f-\varepsilon} \right)$$

(see [7], Theorem 4).

From the results presented above there follow corresponding $0$-estimates of the function

$$T(x) = \sqrt{M(x)} / x,$$

where

$$M(x) = \int_0^x |P(y)|^2 \, dy.$$
The direct investigation of the function \( M(x) \) provides often results which are even sharper:

III. It is always

\[
\lim \inf_{x \to +\infty} \frac{M(x)}{\sqrt[4]{x^2 + \frac{1}{2}}} > 0
\]

and thus

\[
M(x) = \Omega \left( x^{\frac{3}{2}+\frac{1}{2}} \right).
\]

Further,

\[
M(x) = O(x^{\frac{3}{2}-1})
\]

for \( \kappa \geq 4 \),

\[
M(x) = O(x^2 \log x)
\]

for \( \kappa = 3 \) and

\[
M(x) = O(x^{3/2})
\]

for \( \kappa = 2 \).

(See [9], Theorem 3.)

These results cannot be improved as it may be seen from the following assertions:

IV. a) Let the numbers \( \alpha_1, \alpha_2, \ldots, \alpha_\kappa \) be rational. Then

\[
M(x) = H_\kappa x^{\frac{3}{2}-1} + o(x^{\frac{3}{2}-1})
\]

for \( \kappa \geq 4 \),

\[
M(x) = H_3 x^2 \log x + O(x^2 \log x^{\frac{3}{2}})
\]

for \( \kappa = 3 \), where \( H_\kappa \) are nonnegative constants depending only on \( Q, M_j, L_j \) and \( \alpha_j \) (\( j = 1, 2, \ldots, \kappa \)).

1) We have \( H_\kappa > 0 \), if, e.g., \( L_1 = L_2 = \cdots = L_\kappa = 0 \) (see [7], Lemma 9 and [9], Theorem 1).

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If we have for some form $Q$, and suitable numbers $M_j$, $\ell_j$, and $\alpha_j$ ($j = 1, 2, \ldots, \kappa$), $H_\kappa = 0$, then even 2) 

$$M(x) = O(x^{\frac{3}{4} + \frac{1}{4}}) .$$

b) For almost all systems $\alpha_1, \alpha_2, \ldots, \alpha_\kappa$ (again in the sense of Lebesgue measure in $E_\kappa$) there is

$$M(x) = O(x^{\frac{3}{4} + \frac{1}{4}} \log^{3\kappa^2} x) .$$

(See [9], Theorems 1, 2 and [8], Theorem 1.)

The main aim of the presented paper is to complete the results on the $O$-estimations of function $M(x)$. Our examinations will be based on the following Theorem, which shall be proved using Jarník's method (see [1] - [3]):

Main Theorem. Let $\tilde{Q}$ be the form conjugated to $Q$, and, for a natural number $\kappa$, let

$$R_\kappa = \min \tilde{\alpha} \left( \frac{\tilde{m}_j}{m_j} - \alpha_j \tilde{m} \right) ,$$

the minimum being taken over all systems $m_1, m_2, \ldots, m_\kappa$ of integers. Then

$$M(x) = O(x^{\frac{3}{4} \sum_{i=1}^\kappa \min \frac{1}{\tilde{m}_j} \left( \frac{x}{\tilde{m}_j^{1/2}} , \frac{1}{R_\kappa} \frac{A}{x} \right) \frac{A}{x} } .$$

2) Let us remark that in this case Walfisz ([11]) has shown with help of the theory of modular forms that for $\kappa \geq 4$, even $M(x) = K_\kappa x^{\frac{3}{4} + \frac{1}{4}} + O(x^{\frac{1}{2}} \log^2 x)$, $K_\kappa$ being a positive constant depending only on $Q$, $M_j$, $\ell_j$, and $\alpha_j$ ($j = 1, 2, \ldots, \kappa$) (see [11] and [10], Lemma 11).
§ 2. Notations and auxiliary Theorems. In the whole paper we shall preserve the following notations and agreements:

The letter $C$ means (eventually, also various) positive constants, which depend on $Q$, $M_j$, $L_j$, and $\alpha_j$ ($j = 1, 2, \ldots, \kappa$). $C(\varepsilon)$, $C(\beta_j)$, respectively, etc. are positive constants (various) depending moreover on $\varepsilon$, $\beta_j$, $\beta_2$, $\ldots$, $\beta_\kappa$, respectively, etc. The symbols $O$, $\sigma$, and $\Omega$ have the usual meaning, i.e., they refer to the limiting process $x \to +\infty$ and the constants involved are of the "type" $C$. We express the validity of the relation $|A| \leq C B$ shortly by $A << B$.

$n$, $k$, $k'$, and $k''$ mean natural numbers, $m_1$, $m_2$, $\ldots$, $m_\kappa$, $h$, $h'$, $h''$, $\mu$ integers. If $h$ and $k$ ($h'$ and $k'$ etc.) are to appear simultaneously then always $(h, k) = 1$ ($h', k'$ etc.) etc.). For a real $t$ let $<t>$ be the distance of $t$ to the nearest integer, i.e.,

$$<t> = \min_{n} \{ t - \mu \} .$$

Further, let us put

$$P_k = \max_{j=1,2,\ldots,\kappa} \{ \alpha_j M_j \, k \} .$$

It is easy to show (see [6], Remark 2) that

$$P_k^2 << R_k < R_k^2 .$$

In the whole work it will be assumed that the number $\kappa$ is sufficiently large, i.e., $\kappa > C$. Let us put
and let
\[ M_1(y) = M_1(\gamma) , \]
and let
\[ M_2(x) = \int M_1(\gamma) d\gamma . \]
For a complex number \( S, \Re S > 0 \), let
\[ \Theta(s, \alpha) = \sum e^{-sQ(m_j M_3 + \alpha_j^2) + 2\pi i \alpha_j (m_j M_3 + \alpha_j^2)} , \]
where the summation is over all systems \( m_1, m_2, ..., m_k \).
As known, the function \( \Theta(s) \) is a holomorphic function in the half plane \( \Re S > 0 \). By an integral we always mean the (absolute convergent) Lebesgue integral; for real \( a \) we put
\[ \int f(s) ds = i \int f(a + it) dt , \]
and (for \( s = \frac{1}{x} + it, -\infty \leq a \leq x \leq +\infty \))
\[ \int f(s) ds = \int f(\frac{1}{x} + it) dt , \]
if the integrals on the right hand sides exist.

Let us remind some known properties of the Farey's fractions corresponding to \( \sqrt{x} \), i.e. the fractions of the form \( \frac{h}{k} \), where \( k \leq \sqrt{x} \) (see [5] pp.249-250): If \( \frac{h_1}{k_1} < \frac{h}{k} < \frac{h_2}{k_2} \) are three succeeding fractions of this form (i.e. between \( \frac{h}{k} \) and \( \frac{h_2}{k_2} \) lies just one Farey's fraction corresponding to \( \sqrt{x} - \) that is \( \frac{h}{k} \) ) then necessarily \( h_1 k' - k_1 h' = 1, h_2 k' - h_2 k'' = 1, k_1 + k' > \sqrt{x}, k_2 + k'' > \sqrt{x} \).

If we put thus, for \( k \leq \sqrt{x} \),
\[ L_{h, k} = \left\{ 2 \pi \frac{k + h'}{k + h}, 2 \pi \frac{k + h''}{k + h'} \right\}, \]

then, for \( t \in L_{h, k} \), the relation

\[ |t - \frac{2\pi h}{k} | \leq \frac{2\pi}{kV \lambda} \]

holds. The intervals \( L_{h, k} \) are, of course, disjunctive and they cover the entire real axis. If we put

\[ \omega = \frac{2\pi}{[V \lambda] + 1} \]

(for real \( t \), \( \lfloor t \rfloor \) is the integral part of the number \( t \)) then clearly

\[ \Theta_{0,1} = (-\omega, \omega). \]

At the end of this paragraph let us present several auxiliary assertions.

**Lemma 1.** For \( a > 0 \) and \( V \lambda > 0 \), we have

\[ M_2(x) = \int \int F(s) G(s') e^{x(s + s')} ds ds' + O(x), \]

where

\[ F(s) = \Theta(s) - M e^{2\pi k \lambda x} \frac{\sigma'}{s^{\sigma'}} \sigma', \quad G(s) = \overline{F(\bar{s}')}. \]

The proof can be carried out almost in the same way as in the papers [1] - [3].

**Lemma 2.** Let \( s = \frac{1}{x} + it \). If \( t << \omega \) then

\[ \frac{F(s)}{s} << x^{\frac{1}{2} + \frac{1}{2}}. \]

If \( |t - \frac{2\pi h}{k}| << \frac{1}{k \sqrt{V \lambda}} \) (this being accomplished, according to (2) for \( t \in L_{h, k} \)) and \( h = 0 \), then
Analogous assertions hold for the function \( G(\beta') \).

**Proof.** See [8], Lemma 3 and [9], Lemma 7. Let us remark that, if \( \sigma = 1 \), then necessarily \( R_{\kappa} = 0 \) for all \( \kappa \)'s.

**Lemma 3.** Using the notation of IV,a),§ 1, we have, for \( \kappa \equiv 4 \),

\[
H_\kappa = \frac{M^2 \chi^{\kappa-1}}{4 \pi^2 (\kappa - 1) \Gamma^2 \left( \frac{\kappa}{2} \right)} \sum_{\kappa \equiv 0 \ (\text{mod } H)} \sum_{\kappa \equiv 0} \frac{1}{\kappa^{2\kappa-2} \kappa^2},
\]

where

\[
S_{\kappa, \nu} = \sum_{a_1, a_2, \ldots, a_\nu} \frac{-2\pi i A(a_j M_j + B_j)}{\kappa^2} + 2\pi i \frac{\chi}{\kappa} \alpha_j(a_j M_j + B_j)
\]

and \( H \) is the least common denominator of the numbers \( \alpha_1, M_1, \alpha_2, M_2, \ldots, \alpha_\nu, M_\nu \). If \( (H, 2D, M_j, M_k) = 1 \), there is \( |S_{\kappa, \nu}| = \kappa^\frac{\nu}{2} \) for \( \kappa \equiv 0 \ (\text{mod } H) \).

**Proof.** See [9], Theorem 1 and [6], Lemma 2 and Definition 2.

**Lemma 4.** Let \( \kappa \equiv 6 \), \( \alpha_1 = \alpha_2 = \ldots = \alpha_\nu = \alpha \) and let the inequality

\[
< \alpha \kappa > = \kappa^{\frac{1}{\alpha^2}}
\]

be satisfied for all \( \kappa \)'s (and thus \( \beta \equiv 1 \)). Then
where \( q_\beta(x) = \log x \) for \( \beta = 6 \) and \( \beta = 1 \), \( q_\beta(x) = 1 \) in all other cases.

**Proof.** See [7], the proof of Theorem 1 (relations (36), (37), (41), (44), (49)-(51) and b) of this proof).

§ 3. Proof of the Main Theorem. We shall follow the considerations of the paper [8]. Let us always write

\[ s = \frac{4}{x} + i t, s' = \frac{4}{x} + it', \quad t \text{ and } t' \text{ being real numbers.} \]

From Lemma 1 (for \( \alpha = \beta = \frac{4}{x} \)) we have, taking regard to the obvious relation

\[ e^{x(s+s')} < < 1, \]

(5)

\[ M_2(x) < < \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{F(s) F(s')}{|ss'(s+s')|^2} \, dt dt' + O(x). \]

Because of the symmetry of the integrand we can write

(6)

\[ M_2(x) < < T_1 + T_2 + T_3 + O(x), \]

where

\[ T_1 = \int_{-2\pi}^{2\pi} \int_{-2\pi}^{2\pi} \cdots dt dt', \]

\[ T_2 = \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \cdots dt dt' + \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \cdots dt dt', \]

\[ T_3 = \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \cdots dt dt' \]
\[ T_3 = \int_0^\infty \int_0^\infty dt 
abla + \int_{-\infty}^0 \int_{-\infty}^0 dt \nabla \]

(the integrands which are not presented are the same as those in (5)). According to (3), there is

\[ (7) \quad T_i < \alpha \frac{x}{x} \int_0^\infty \left( \int_0^1 \frac{dt}{(1+x(t-x)^2)} \right) dt < \alpha \frac{x}{x} \]

To estimate \( T_2 \) and \( T_3 \), let us first of all consider the following assertions: Let \( \beta = c \), \( \beta = \frac{1}{2} \). Then

\[ (8) \quad \int_0^\infty \frac{e^{\frac{c}{R_x x}} - e^{\frac{R_x x}{(1+x^2 u^2)^{\frac{1}{2}}}}}{(1+x^2 u^2)^\beta} du \ll \int_0^\infty \frac{du}{(1+x^2 u^2)^\beta} \ll \begin{cases} \frac{1}{x} & \text{for } \beta > \frac{1}{2} \\ \frac{\ln x}{x} & \text{for } \beta = \frac{1}{2} \end{cases} \]

If however \( R_x \neq 0 \), we can write

\[ \int_0^\infty \frac{c^{\frac{1}{R_x x}} - e^{\frac{c}{R_x x}}}{e^{\frac{R_x x}{(1+x^2 u^2)^{\frac{1}{2}}}}} du \ll \left( \frac{\beta}{R_x x} \right)^\beta \int_0^\infty \frac{du}{(1+x^2 u^2)^\beta} e^{\frac{c}{R_x x}} du \cdot \]

The last integral, for \( \beta > \frac{1}{2} \), can be estimated by means of the expression

\[ \int_0^\infty c^{\frac{1}{R_x x}} du + \left( \frac{R_x x}{c^{\frac{1}{R_x x}}} \right)^\beta \int_0^\infty \frac{du}{(1+x^2 u^2)^\beta} \ll \sqrt{\frac{R_x}{x^4}} \]

(for \( c \geq 0 \) there is \( c \cdot e^{-c \beta} \ll 1 \)). For \( \beta > \frac{1}{2} \), \( \beta = c \), we thus obtain, according to (8),
First of all, let us estimate $T_2$. Let us remark that, according to (2) for $t \in L_{\rho \lambda}$, $\rho > 0$, $t > 2w$, $t' \leq w$, we have

$$|s + s'| > > \frac{\rho}{\lambda} \quad \text{and} \quad |s| > > \frac{\rho}{\lambda}.$$  

If we now use (3) and (4) (we decompose the integration path into intervals $L_{\rho \lambda}$ and in each of them we use the corresponding estimate (4)) we obtain, according to (9) (for $\beta = \frac{\rho}{\lambda}$, $\beta > \frac{1}{2}$, i.e., for $\rho > 2$) or according to (8) (for $\beta = \frac{\rho}{\lambda} = \frac{1}{2}$, i.e., for $\rho = 2$), successively (making use of (2))

$$T_2 < < \sum_{\rho \geq 1} \frac{1}{\lambda} \sum_{\rho \geq 1} \min \left( \frac{1}{\lambda}, \frac{\rho}{\lambda} \right) \frac{\rho}{\lambda}.$$

An easy rearranging provides

$$T_1 < < \sum_{\rho \geq 1} \min \left( \frac{\rho}{\lambda}, \frac{1}{\lambda} \right) \frac{\rho}{\lambda}.$$

Let us pass over to the estimation of $T_3$. Obviously,
\int \int \frac{F(s)F(\overline{s})}{s^*s'(s+s')^2} dt \, dt' < \int \int \frac{|F(s)|^2+|F(\overline{s})|^2}{|s^*s'(s+s')^2|} \, dt \, dt' ,

and an analogous inequality is obtained also for the second integral appearing in $T_3$. From the symmetry of the integrands there follows that

(11) $T_3 < \int \int \frac{|F(s)|^2+|F(\overline{s})|^2}{t\,t'(\frac{1}{x} + |t-t'|^2)} \, dt \, dt'$

Analogously as in [8] (relations (29)-(33) we find easily that, for $t \geq w$,

$$\int \frac{dt'}{t'(\frac{1}{x} + |t-t'|^2)} < \frac{\alpha}{t} ,$$

and thus substituting into (11) we obtain

$$T_3 < \frac{\alpha}{x} \int \frac{|F(s)|^2+|F(\overline{s})|^2}{t^2} \, dt .$$

We again decompose the integration path into intervals

$$\mathcal{L}_{h, h} \quad (h > 0, \ h \leq \sqrt{x}) \quad \text{and in each of them we use the corresponding estimate (4). According to (9) (for } \beta = \frac{h}{R} > \frac{1}{2} \), we successively obtain (for } t \in \mathcal{L}_{h, h} \ (h > 0 \ ), \text{ we have, according to (2), } t > \frac{h}{R} \ )$$

$$T_3 < \frac{\alpha}{h} \int \frac{1}{h^2} \min \left( \frac{1}{x}, \frac{R^2}{\sqrt{R^2 + x^2 \omega^2}} \right) \, du$$

(12) $$< \frac{\alpha}{h} \int \frac{1}{h^2} \min \left( \frac{1}{x}, \frac{R^2}{\sqrt{R^2 + x^2 \omega^2}} \right)$$
Let us now denote

\[ F(x) = x^\frac{3}{2} \sum_{\lambda \in \mathbb{Z}^n} \min \left( \frac{x}{4\lambda}, \frac{1}{R_{\lambda}} \right) \frac{R_{\lambda}}{\sqrt{x}}. \]

Obviously \( R_{\lambda} \ll 1 \)

\[ (13) \quad F(x) \gg x^{\frac{3}{2} + 1} \sum_{\lambda \in \mathbb{Z}^n} \frac{R_{\lambda}}{\sqrt{x}} \gg x^{\frac{5}{2} + 1}. \]

According to (6), (7), (10) and (12) we can write \( F(x) \gg x \) by (13))

\[ (14) \quad M_{2}(x) \ll F(x). \]

The function \( M(x) \) being non-negative and non-decreasing, we have

\[ M(x) \leq \frac{4}{x} \int M(y) \, dy = \frac{4}{x} \left( M_{2}(4x) - M_{2}(x) \right), \]

and, according to (14),

\[ (15) \quad M(x) \ll \frac{4}{x} \left( F(4x) + F(x) \right). \]

Now we have

\[ F(4x) \ll x^{\frac{3}{2} + 1} \sum_{\lambda \in \mathbb{Z}^n} \min \left( \frac{x}{4\lambda}, \frac{1}{R_{\lambda}} \right) \frac{R_{\lambda}}{\sqrt{x}} \ll \]

\[ \ll x^{\frac{5}{2} + 1} \sum_{\lambda \in \mathbb{Z}^n} \min \left( \frac{x}{4\lambda}, \frac{1}{R_{\lambda}} \right) \frac{R_{\lambda}}{\sqrt{x}} = \]

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Finally, we obtain from (15)

\[ M(x) < \sum_{k} \min \left( \frac{x^{k-1}}{R_{k}}, \frac{1}{R_{k}} \right) \frac{1}{\sqrt{k}} \]

and thus, according to (13),

\[ F(4x) << F(x) \]

Finally, we obtain from (15)

\[ (16) \quad M(x) << \frac{1}{x} F(x) = \sum_{k} \min \left( \frac{x^{k-1}}{R_{k}}, \frac{1}{R_{k}} \right) \frac{1}{\sqrt{k}} \]

this proving the Main Theorem.

§ 4. Consequences of the Main Theorem. First of all, let us present two "exceeding" consequences of the relation (16). It always holds

\[ M(x) << \sum_{k} \frac{x^{k-1}}{R_{k}} \]

and thus the relation (16) yields immediately the \( O \) -estimates presented in III, § 1. On the other hand,

\[ M(x) << \sum_{k} \frac{1}{R_{k}^{\frac{k}{2}}} \]

if at least one of the numbers \( \alpha_{1}, \alpha_{2}, \ldots, \alpha_{n} \) is irrational (and thus \( R_{k} \neq 0 \) for all \( k \)). This relation was the starting point for the \( O \) -estimates IV.b), § 1 in the paper [8].
Let us now rearrange the relation (16) in the following way: because of

\[
\min \frac{1}{\kappa} \left( \frac{x}{R_k}, \frac{1}{R_k} \right) \frac{1}{\sqrt{x}} \leq \min \frac{1}{\kappa} \left( \frac{x}{R_k}, \frac{1}{R_k} \right)
\]

(if \( R_k = 0 \) or \( R_k \neq 0 \) and \( x/R_k^2 \leq \frac{1}{R_k} \), the equality takes place; if \( R_k \neq 0 \), \( x/R_k^2 > \frac{1}{R_k} \), i.e.,

for \( \frac{1}{\sqrt{x}} < \sqrt{R_k} \), we have \( \frac{1}{\sqrt{x}} \frac{1}{R_k^{1/2} - 1} < \frac{1}{R_k^{1/2}} \) on the left hand side, and the inequality takes place), we can write

\[
(17) \quad M(x) < x^{1/\kappa} \sum_{k=1}^{\kappa} \min \frac{1}{\kappa} \left( \frac{x}{R_k}, \frac{1}{R_k} \right) .
\]

From the assertion I.o.), § 1 there follows, for \( \kappa > 4 \):

If at least one of the numbers \( \alpha_1, \alpha_2, \ldots, \alpha_\kappa \) is irrational then

\[
(18) \quad M(x) = o(x^{1/\kappa}) .
\]

For \( \kappa = 4 \), we cannot derive this result from the results of the paper [6]. Therefore we shall use the relation (17).

**Theorem 1.** Let \( \kappa \geq 4 \) and let at least one of the numbers \( \alpha_1, \alpha_2, \ldots, \alpha_\kappa \) be irrational. Then (18) holds.

**Proof** (analogously to [6], Theorem 3). According to the assumptions, there is \( R_k \neq 0 \) for all \( k \)'s and \( \kappa - 2 \geq 2 \). If we produce, for every \( x > c \), a natural number \( \psi(x) \) such that
\[ \sum_{\mathbf{a} \in \Psi(A)} \frac{1}{R_{\mathbf{a}}^{\kappa-1}} \leq \frac{1}{\psi(A)} \leq \sum_{\mathbf{a} \in \Psi(A)} \frac{1}{R_{\mathbf{a}}^{\kappa-1}}, \]

then \( \psi(A) \) is non-decreasing function, \( \lim_{A \to +\infty} \psi(A) = +\infty \). But according to (17) we have

\[
M(A) \leq A \left( \sum_{\mathbf{a} \in \Psi(A)} \frac{1}{R_{\mathbf{a}}^{\kappa-1}} + \frac{1}{\psi(A)} \sum_{\mathbf{a} \in \Psi(A)} \frac{1}{R_{\mathbf{a}}^{\kappa-1}} \right) < \infty
\]

and the theorem is thereby proved.

The estimate (18) cannot be improved generally. Using the known method of categories (3) an assertion analogous to I.d), § 1 can be stated:

**Theorem 2.** Let \( \kappa \geq 4 \) and let \( \varphi(A) \) be a non-increasing positive function, \( \varphi(1) = \sigma(1) \). Then there exists a system \( \alpha_1, \alpha_2, \ldots, \alpha_\kappa \) such that (18) takes place and

\[(19) \quad M(A) = \Omega \left( A^{\kappa-1} \varphi(A) \right) \]

holds.

**Proof.** Let \( \mathbb{M} \) be a set of all points \( (\mu_1, \mu_2, \ldots, \mu_\kappa) \in E_\kappa \) such that \( 0 \leq \mu_j \leq \frac{1}{M_\kappa} \) \((j = 1, 2, \ldots, \kappa)\), \( \mathbb{M} \) let be a set of all points from \( \mathbb{M} \) having rational

3) The first one who used this method for \( \Omega \)-estimates in the theory of lattice points was Jarník in the paper [4].
coordinates. For a natural \( n \), let \( \mathcal{M}_n \) be a set of all points \( (\beta_1, \beta_2, \ldots, \beta_k) \in \mathcal{M}^\circ \) such that, for a suitable \( \lambda = \lambda(n, \beta_j) > n \), there is

\[
\frac{M(x, \beta_j)}{x^{n-1}g(x)} > n.
\]

From the continuity of the function \( M(x, \beta_j) \) for a steady \( \lambda \) on the set \( \mathcal{M}^\circ \), let us remark that, for \((\beta_1, \beta_2, \ldots, \beta_k) \in \mathcal{M}^\circ \), there is \( A(x, \beta_j) = P(x, \beta_j) \), and the function \( A(x, \beta_j) \) is, for a steady \( \lambda \), continuous in the entire space \( E_k \). There follows that every set \( \mathcal{M}_n \) is open.

Let \( \mathcal{L} \) be the set of all points \((\beta_1, \beta_2, \ldots, \beta_k) \in \mathcal{M}^\circ \) such that, for the least common denominator \( H \) of the numbers \( \beta_1 M_1, \beta_2 M_2, \ldots, \beta_k M_k \), we have \( (H, 2 \prod_{j=1}^{k} M_j^{1/m_j}) = 1 \). According to the Lemma 3 and the assertion IV.1), § 1, we obtain that, for every \((\beta_1, \beta_2, \ldots, \beta_k) \in \mathcal{L} \), there is

\[
M(x, \beta_j) \leq c(\beta_j) x^{n-1}
\]

for \( x > c(\beta_j) \). Thus, choosing a natural \( n \), we have, for every \((\beta_1, \beta_2, \ldots, \beta_k) \in \mathcal{L} \),

\[
\frac{M(x, \beta_j)}{x^{n-1}g(x)} \leq \frac{c(\beta_j)}{g(x)} > n
\]

for all sufficiently large \( x > c(\beta_j) \), and it immediately follows that \( \mathcal{L} \subseteq \bigcap_{n=1}^{\infty} \mathcal{M}_n \). Since the set \( \mathcal{L} \) is obviously dense in \( \mathcal{M} \), all the sets \( \mathcal{M}_n \) are
dense in \( M \), too.

Conclusively, the sets \( M - M_n \) are nowhere dense in \( M \), and thus the set

\[
\mathcal{N} \cup (M - M^0) \cup \bigcap_{n=1}^\infty (M - M_n)
\]

is of the first category in \( M \), i.e., \( M^0 \) is a complete space; there exists a point

\[
(a_1, a_2, \ldots, a_n) \in (M^0 - M) \cap \bigcap_{n=1}^\infty M_n.
\]

The relation (18) thus holds. Since \( (a_1, a_2, \ldots, a_n) \in \bigcap_{n=1}^\infty M_n \), the inequality

\[
\frac{M(x)}{x^{\alpha(x)}} > n
\]

is satisfied for every \( n \) for a suitable \( x = x(a_n, n) \), \( n > n \), and (19) holds, too.

Let further \( \alpha_1 = \alpha_2 = \ldots = \alpha_n = \alpha \), be an irrational number. From II, § 1, it follows (in the notation introduced there), for \( \kappa \geq 6 \), the estimate

\[
\lim_{x \to \infty} \sup_{x \to 6} \frac{\log M(x)}{\log x} \leq 2 \kappa + 1
\]

For \( \kappa = 4, 5 \), we obtain, using the estimate from the paper [7], weaker results:

\[
\lim_{x \to \infty} \sup_{x < 6} \frac{\log M(x)}{\log x} \leq \max \left( \frac{\kappa}{2}, 24 \right) + 1
\]

for \( \kappa = 5 \) and

\[
\lim_{x \to \infty} \sup_{x < 6} \frac{\log M(x)}{\log x} \leq 3
\]
for \( \kappa = 4 \), considering (17) we can prove the following generalization:

Theorem 3. Let \( \kappa \geq 4 \), \( \alpha_1 = \alpha_2 = \ldots = \alpha_\kappa = \alpha \). Let

\[
(20) \quad < \alpha \kappa > > \frac{1}{\kappa^k}
\]

for all \( \kappa \)'s. Then

\[
M(x) \leq x \left( \frac{\kappa - 1}{\kappa + 1} \right)^{\frac{2\alpha + 1}{\kappa + 1}} + g(x),
\]

where \( g(x) = \log x \) for \( \kappa = 4 \) and \( \beta = 1 \), \( g(x) = 1 \) simultaneously in other cases.

Proof. According to (17) and \( \kappa > \kappa^2 \) we can write

\[
M(x) \leq x \left( \frac{\kappa - 1}{\kappa + 1} \right)^{\frac{2\alpha + 1}{\kappa + 1}} + 1,
\]

i.e.,

\[
M(x) \leq x \sum_{n=1}^{\kappa - 2} \min \left( \frac{x}{n}, \frac{1}{P_n^2} \right),
\]

and the assertion follows from Lemma 4.

Connecting the \( \Omega \)-estimate III, § 1 and Theorem 3, we obtain the following result:

Theorem 4. Let \( \kappa = 4 \), \( \alpha_1 = \alpha_2 = \ldots = \alpha_\kappa = \alpha \). Let (20) with the value \( \beta = 1 \) hold for all \( \kappa \)'s (i.e., if \( \{ a_0, a_1, a_2, \ldots \} \) is the continued fraction expressing the number \( \alpha \), then \( a_n < 1 \) ). Then

\[
0 < \lim_{x \to +\infty} \inf \frac{M(x)}{\frac{x}{\log x}} < \lim_{x \to +\infty} \sup \frac{M(x)}{\frac{x}{\log x}} < +\infty.
\]

Remark. a) If at least one of the numbers \( \alpha_1 \),
\(\alpha_2, \ldots, \alpha_n\) is irrational then it follows from (17) that
\[
M(x) < \sum_{k=1}^{\infty} \frac{1}{\alpha_k \sqrt[k]{k}}.
\]

Using this estimate in the paper [8] we could slightly improve the \(O\)-estimate IV b), § 1.

b) An assertion analogous to Theorem 2 can be stated for \(k = 3\): If \(\varphi(x)\) is a positive and non-increasing function, \(\varphi(x) = \sigma(1)\), there exists a triplet of numbers \(\alpha_1, \alpha_2, \alpha_3\) such that at least one of them is irrational and moreover
\[
M(x) = \Omega \left( x^2 \varphi(x) \log x \right)
\]
(and obviously \(M(x) = O \left( x^2 \log x \right)\)). The proof is to be carried out analogously, we have only to mention that the constant \(H_3\) in IV a), § 1 is non-zero if the least common denominator of the numbers \(\alpha_{1}M_1, \alpha_{2}M_2, \alpha_{3}M_3\) is relatively prime to \(2DM_1^2 M_2^2 M_3^2\). The validity of this assertion follows from Theorem 1 of the paper [9] and Lemma 2 of the paper [6].

c) If \(\alpha_1 = \alpha_2 = \ldots = \alpha_n = \alpha\) and if \(\varphi\) is defined in the same way as in II, § 1 we obtain from Theorem 3 (and III, § 1 in the case of a rational \(\alpha\)) the estimate
\[
M(x) = O \left( x^{2\varphi + 1 + \varepsilon} \right)
\]
(for an arbitrary \(\varepsilon > 0\), the constants in the \(O\)-estimate are of the type \(c(\varepsilon)\)).

d) The proof of Theorem 3 could be carried out directly, analogously to the proof of Theorem 1 in the paper [7].
It is anyhow interesting to compare (17) with this result (see [6], Theorem 2): Let $\kappa > 4$, then

$$P(x) = O(x^{\frac{3}{4} - \frac{1}{4}} \sum_{\mathfrak{A} \subset \mathfrak{h}} \min_{\mathfrak{A} \subset \mathfrak{h}} \left( \frac{x}{\mathfrak{a}}, \frac{1}{\mathfrak{A}} \right) \log^2 \mathfrak{A}.$$ 

References


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