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REMARK ON A PAPER OF LOVÁSZ

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L. Lovász [1] stated and proved the following theorem concerning finite undirected graphs without loops (multiple edges are allowed):

Let $\mathcal{G} = \langle g, G \rangle$ be a graph of valency k with n vertices. Let n_1, n_2 be non-negative integers such that $n_1 + n_2 = n$. Then there exist subsets g_1, g_2 of g such that $g = g_1 \cup g_2$, $n(g_i) = n_i$ $(i = 1, 2)$ and the sum of the valencies of the subgraphs $\mathcal{G}g_1$, $\mathcal{G}g_2$ is at most k.

We shall show that this theorem is sharp in the following sense:

<u>Theorem</u>. Let n, k, n_1, n_2 be non-negative integers, $k < n$, $n = n_1 + n_2$; put $a = \min(n_1, n_2)$. Let at least one of the following conditions be fulfilled:

(i) $n \geq 2k \cdot [\frac{a+k}{k}]$

(ii) k divides a, $n \geq 2a + k + 1$

(iii) there are non-negative integers p, q such that $q \geq 2$, $n = p \cdot 2k + q(k+1)$ and q does not divide $a - pk$.

Then there is a graph $\mathcal{G} = \langle g, G \rangle$ with n vertices and valency k such that, given any partition $g = g_1 \cup g_2$, $n(g_i) = n_i$ $(i = 1, 2)$, the sum of the valencies of the subgraphs $\mathcal{G}g_1$, $\mathcal{G}g_2$ is at least k.

Remark. If $k = 2$, $n \geqq 15$ or $k = 4$, $n \geqq 60$ then it is easy to find that the conditions of our theorem are fulfilled; see the following tables:

$a \backslash n$	$4m-1$	$4m$	$4m+1$	$4m+2$
$\leqq 2m-2$	i, ii	i	i	i
$2m-1$	iii $q = 5$	i	i	i
$2m$		iii $q = 4$	iii $q = 3$	iii $q = 6$
$2m+1$				iii $q = 2$

$k = 2, m \geqq 4$

$a \backslash n$	$8m-4$	$8m-3$	$8m-2$	$8m-1$	$8m$	$8m+1$	$8m+2$	$8m+3$
$\leqq 4m-5$	i	i	i	i	i	i	i	i
$4m-4$	iii $q=12$	ii	ii	ii	i	i	i	i
$4m-3$	iii $q=4$	iii $q=9$	iii $q=6$	iii $q=3$	i	i	i	i
$4m-2$	iii $q=4$	iii $q=9$	iii $q=6$	iii $q=11$	i	i	i	i
$4m-1$			iii $q=6$	iii $q=3$	i	i	i	i
$4m$					iii $q=8$	iii $q=5$	iii $q=10$	iii $q=7$
$4m+1$							iii $q=2$	iii $q=7$

$k = 4, \ m \geqq 8$

Recall that by a graph $\mathcal{G}$ we mean an unordered couple $\langle g, G \rangle$, g being the set of the vertices of $\mathcal{G}$, G being the set of the edges of $\mathcal{G}$. A graph $\mathcal{G}$ is said to have valency k, if k is the greatest integer such that $\mathcal{G}$ has a vertex of valency k. The subgraph of $\mathcal{G} = \langle g, G \rangle$ spanned by s ($s \subset g$) is denoted by $\mathcal{G}_{/s}$, the number of elements of a finite set g by n(g). $[x]$ is the greatest integer which does not exceed x.

__Proof__ of Theorem. If $\mathcal{G} = \langle g, G \rangle$ is a graph of valency k, we shall call a partition $g = g_1 \cup g_2$ good, if the respective sum is less than k.

Let $\mathcal{G} = \langle g, G \rangle$ be a graph such that there is a partition $g = g' \cup g''$, $n(g') = n(g'') = k$ and $(x, y) \in G$ if and only if $x \in g'$, $y \in g''$. It is easy to see that there is only one good partition of $\mathcal{G}$; it is the above partition $g = g' \cup g''$. We shall call $\mathcal{G}$ an even k-graph and g', g'' independent sets of $\mathcal{G}$.

If $\mathcal{G} = \langle g, G \rangle$ is a graph of valency k and $\mathcal{G}$ contains an k-even graph $\mathcal{G}'$, then given any good partition $g = g_1 \cup g_2$ of $\mathcal{G}$ it is possible to denote independent sets of $\mathcal{G}'$ by h_1, h_2 in such a way that $h_1 \subset g_1$, $h_2 \subset g_2$. Especially, if $\mathcal{G}$ contains m even k-graphs, then $\min n(g_i) \geq mk$.

To prove our theorem, consider (under the respective conditions) the following graphs

$\mathcal{G}_1$ - graph of valency k which contains $[\frac{a+k}{k}]$ even k-graphs,

$\mathcal{G}_2$ - a graph of valency k which contains $\frac{a}{k}$ even k-graphs and a complete graph with k + 1 vertices,

$\mathcal{G}_3$ consists of p even k-graphs and q complete graphs, each of them with $k+1$ vertices.

Suppose $g = g_1 \cup g_2$ to be a good partition of $\mathcal{G}_1$, $\min n(g_i) = a$. Then using the above results we have $a \geq [\frac{a+k}{k}] \cdot k$ which is false.

Suppose $g = g_1 \cup g_2$ to be a good partition of $\mathcal{G}_2$, $\min n(g_i) = a$. Then there is an index i such that all the vertices of the complete graph are contained in g_i which is a contradiction.

Finally, suppose $g = g_1 \cup g_2$ to be a good partition of $\mathcal{G}_3$, $\min n(g_i) = a$. It follows from (iii) that there is a couple of complete graphs $\langle g', G' \rangle$, $\langle g'', G'' \rangle$, $n(g') = n(g'') = k+1$ such that $n(g_1 \cap g') <$ $< n(g_1 \cap g'')$ holds. Then $n(g_2 \cap g') + n(g_1 \cap g'') \geq k+2$ holds and - as follows from the completeness of $\langle g', G' \rangle$, $\langle g'', G'' \rangle$ - it is a contradiction.

References

[1] L. LOVÁSZ: On decomposition of graphs, Studia Scient. Math. Hung. I(1966), 237-238.

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