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# GENERATORS OF THE BOOLEAN ALGEBRA OF REGULAR OPEN SETS IN LINEAR METRIC SPACES

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In the present paper, systems of generators in special Bnolena algebras are investigated. In addition to generally known notions we use some definitions adopted from [2],[3]. In the first part we prove a general theorem on Boolean algebras (the method of its proof is due to Solovay [1]). This theorem is then applied to Boolean algebras of regular open sets. We prove that the complete Boolean algebra of all regular open sets in a metric linear space has a denumerable system of generators. In conclusion, the similar theorem for some other metric spaces is also proved.

<u>Definition 1</u>. (cf. [2],[3]) Let $B$ be a complete Boolean algebra, $\alpha$ and $\beta$ ordinal numbers. A set $A_{\alpha\beta} = \{w(\gamma,\delta);\, \gamma\in\omega_\alpha, \delta\in\omega_\beta\} \subseteq B$ is called an $(\alpha,\beta)$-system on $B$, if the following holds:

(1) $\gamma_1 \neq \gamma_2 \longrightarrow w(\gamma_1,\delta) \wedge w(\gamma_2,\delta) = 0$ for any $\delta \in \omega_\beta$.

(1') $\delta_1 \neq \delta_2 \longrightarrow w(\gamma,\delta_1) \wedge w(\gamma_1,\delta) = 0$ for any $\gamma \in \omega_\alpha$.

(2) For each $\gamma\in\omega_\alpha$ $\bigvee_{\delta\in\omega_\beta} w(\gamma,\delta) = 1$.

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$0, 1$ being the zero and unit element of $B$ respectively.

Theorem 1. Let $B$ be a complete Boolean algebra, $A_{\alpha\beta} = \{w(\gamma,\delta);\ \gamma\in\omega_\alpha,\ \delta\in\omega_\beta\}$ be an $(\alpha,\beta)$-system on $B$. Let us construct $v(\lambda_1,\lambda_2)$, $\lambda_1,\lambda_2\in\omega_\alpha$ as follows:

$$v(\lambda_1,\lambda_2)=\bigvee_{\delta_1\neq\delta_2}(w(\lambda_1,\delta_1)\wedge w(\lambda_2,\delta_2)) .$$

Then the collection $\{v(\lambda_1,\lambda_2);\ \lambda_1,\lambda_2\in\omega_\alpha\}$ generates the whole $(\alpha,\beta)$-system $A_{\alpha\beta}$.

Proof. Let $B'\subseteq B$ be the least complete Boolean algebra containing the family $\{v(\lambda_1,\lambda_2);\ \lambda_1,\lambda_2\in\omega_\alpha\}$. It has to be proved that all $w(\gamma,\delta)$'s are in $B'$. We prove it by induction on $\delta$. It suffices to show that, for any $\gamma\in\omega_\alpha$, $\xi\in\omega_\beta$, the joins $\bigvee_{\delta<\xi} w(\gamma,\delta)$, $\bigvee_{\delta\leq\xi} w(\gamma,\delta)$ are in $B'$.

Let us suppose that, for some $\xi\in\omega_\beta$ and for all $\lambda\in\omega_\alpha$, all $w(\lambda,\delta)$'s with $\delta<\xi$ are in $B'$. Since apparently $\bigvee_{\delta<\xi} w(\lambda,\delta)\in B'$, the only thing to prove is $\bigvee_{\delta\leq\xi} w(\lambda,\delta)\in B'$. Let us choose $\lambda\in\omega_\alpha$ arbitrarily. Defining

$$c_{\lambda_1}=\bigvee_{(\delta_1<\delta_2\rightarrow\delta_1<\xi)}(w(\lambda_1,\delta_1)\wedge w(\lambda,\delta_2))$$

for any $\lambda_1\in\omega_\alpha$, $\lambda_1\neq\lambda$ we obtain, from Definition 1 property (2), the following equality

$$\begin{aligned}c_{\lambda_1}&=\bigvee_{(\delta_1\geq\delta_2)\vee(\delta_1<\xi)}(w(\lambda_1,\delta_1)\wedge w(\lambda,\delta_2))=v(\lambda,\lambda_1)\vee\\&\vee\bigvee_{\substack{\delta_1<\xi\\ \delta_2\in\omega_\beta}}(w(\lambda_1,\delta_1)\wedge w(\lambda,\delta_2))=v(\lambda,\lambda_1)\vee\\&\vee(\bigvee_{\delta_1<\xi}w(\lambda_1,\delta_1)\wedge\bigvee_{\delta_2\in\omega_\beta}w(\lambda,\delta_2))=v(\lambda,\lambda_1)\vee\bigvee_{\delta_1<\xi}w(\lambda_1,\delta_1)\end{aligned}$$

which shows that $c_{\lambda_1}\in B'$ for any $\lambda_1\in\omega_\alpha$, $\lambda_1\neq\lambda$. We show that the following equality holds: $\bigvee_{\delta\leq\xi} w(\lambda,\delta)=$

$= \bigwedge_{\lambda_1 \neq \lambda} c_{\lambda_1}$ . Let $\sigma \leq \xi$ . Using (2) from Definition 1 we obtain

$$c_{\lambda_1} \geq \bigvee_{\sigma_1 < \sigma \to \sigma_1 < \xi} (w(\lambda_1, \sigma_1) \wedge w(\lambda, \sigma)) = w(\lambda, \sigma) \wedge$$

$$\wedge \bigvee_{\sigma_1 \geq \sigma \vee \sigma_1 < \xi} w(\lambda_1, \sigma_1) = w(\lambda, \sigma) \wedge \bigvee_{\sigma_1 \in \omega_\beta} w(\lambda_1, \sigma_1) = w(\lambda, \sigma)$$

for any $\lambda_1 \neq \lambda$ .

This fact immediately implies $\bigvee_{\sigma \leq \xi} w(\lambda, \sigma) \leq \bigwedge_{\lambda_1 \neq \lambda} c_{\lambda_1}$ .

Suppose $u = \bigwedge_{\lambda_1 \neq \lambda} c_{\lambda_1} - \bigvee_{\sigma \leq \xi} w(\lambda, \sigma) \neq 0$ . Then there exists $\xi_0 > \xi$ such that $u \wedge w(\lambda, \xi_0) \neq 0$, since $\bigvee_{\sigma} w(\lambda, \sigma) = 1$. But $\bigvee_{\gamma} w(\gamma, \xi) = 1$ holds by (2') of Definition 1. Then there exists $\lambda_1$ such that

$$u \wedge w(\lambda, \xi_0) \wedge w(\lambda_1, \xi) \neq 0 \, .$$

But $\xi_0 \neq \xi$ implies $\lambda \neq \lambda_1$ by the assumption (1') in Definition 1. Then we obtain $w(\lambda, \xi_0) \wedge w(\lambda_1, \xi) \wedge$ $\wedge \, c_{\lambda_1} \neq 0$ . From this inequality we obtain the following assertion:

There exist $\xi_1$ , $\xi_2$ such that $(\xi_1 < \xi_2 \longrightarrow \xi_1 < \xi)$ and

$$w(\lambda, \xi_0) \wedge w(\lambda_1, \xi) \wedge w(\lambda_1, \xi_1) \wedge w(\lambda, \xi_2) \neq 0 \, .$$

But $\xi_0 = \xi_2$ and $\xi = \xi_1$ by Definition 1 (1') - a contradiction.

This proves that $\bigvee_{\sigma \leq \xi} w(\lambda, \sigma) = \bigwedge_{\lambda_1 \neq \lambda} c_{\lambda_1}$ , which completes the proof of Theorem 1.

Remark. In the preceding proof, we did not use (1) from the definition of an $(\alpha, \beta)$-system. It means that a slightly stronger theorem could be formulated. We do not do it because of the connection of Theorem 1 with the theory of $\nabla$-models (cf. [2]).

If there is an $(\alpha, \beta)$-system in any Boolean algebra, it has, as we have proved, $\omega_\alpha$ generators. In some special Boolean algebras, a suitably chosen $(\alpha, \beta)$-system generates the whole algebra. It means that such an algebra has $\omega_\alpha$ generators. In what follows, we deal with Boolean algebras in linear topological spaces. Let us recall some definitions: A set $\sigma \subset P$ in a topological space $(P, \tau)$ is called a regular open set if $\mathrm{Int}(cl\, \sigma) = \sigma$. The collection $B$ of all regular open sets of a space $(P, \tau)$ with operations $\bigvee \mathcal{U} = \mathrm{Int}(cl \bigcup_{u \in \mathcal{U}} u)$, $\bigwedge \mathcal{U} = \mathrm{Int}(\bigcap_{u \in \mathcal{U}} u)$, $-u = \mathrm{int}(P - u)$, where $\mathcal{U} \subseteq B$, is a complete Boolean algebra. If the sets from a family $a$ are mutually disjoint, we use the symbol $Ex(a)$. Souslin's number $\mu(P, \tau)$ of a topological space $(P, \tau)$ is introduced as follows:

$$\mu(P, \tau) = \min(\alpha;\ \neg \exists a (Ex(a) \,\&\, a \subset \tau \,\&\, \mathrm{card}\, a = \alpha));$$

it means that $\mu(P, \tau)$ is the least cardinal number with the property that in $(P, \tau)$ there is no disjoint family of open sets having this cardinality. A space $(P, \tau)$ is said to be saturated if $\mu(\sigma, \tau) = \mu(P, \tau)$ holds for any non-void open subset $\sigma$ of $P$. One can easily verify that each topological linear space is saturated. In the following, we use the following theorems proved in [3]:

1° $\mu(P, \tau)$ is a regular cardinal,

2° if $(P, \tau)$ is a metric space, then $\mu(P, \tau) = \aleph_{\alpha+1}$ for some $\alpha$; it means that $\mu(P, \tau)$ is an isolated cardinal.

**Theorem 2.** Let $(P, \tau)$ be a saturated metric space, $B$ the complete Boolean algebra of all open regular sets in $(P, \tau)$. Then $B$ has a denumerable set of generators.

**Proof.** The metric space $(P, \tau)$ is saturated and $\mu(P, \tau) = \aleph_{\alpha+1}$ holds for some $\alpha$. By induction on $n < \omega_0$ we construct a sequence of regular open sets $A_{\lambda_1 \dots \lambda_n}$ $(n < \omega_0;\ \lambda_1, \dots, \lambda_n < \omega_\alpha)$.

Let $B^* = \{B_{\lambda_1}\}_{\lambda_1 \in \omega_\alpha}$ be a family of non-empty open sets such that

(a) $d(B_{\lambda_1}) < 1$ for any $\lambda_1 \in \omega_\alpha$,

(b) $Ex(B^*)$,

(c) $cl(\bigcup_{\lambda_1} B_{\lambda_1}) = P$.

There is such a family because $\mu(P, \tau) = \aleph_{\alpha+1}$. Put $A_{\lambda_1} = Int(cl\, B_{\lambda_1})$ for any $\lambda_1 \in \omega_\alpha$ The family $\{A_{\lambda_1};\ \lambda_1 \in \omega_\alpha\}$ satisfies (a) - (c) as well. Having constructed the sets $A_{\lambda_1 \dots \lambda_n}$ for any $n \le n_0$, we construct the sets $A_{\lambda_1 \dots \lambda_{n_0+1}}$ as follows: Let $A_{\lambda_1 \dots \lambda_{n_0}}$ be an arbitrary already constructed set, and let $\{A_{\lambda_1, \dots \lambda_{n_0}, \lambda_{n_0+1}};\ \lambda_{n_0+1} \in \omega_\alpha \ \&\ \lambda_{n_0+1} \neq \lambda_n \ (n = 1, 2, \dots n_0)\}$ be a disjoint family of non-empty regular open sets such that

$d(A_{\lambda_1 \dots \lambda_{n_0+1}}) < \dfrac{1}{n+1}$ for any $\lambda_{n_0+1} \in \omega_\alpha$ and that $cl(\bigcup_{\lambda_{n_0+1}} A_{\lambda_1 \dots \lambda_{n_0} \lambda_{n_0+1}}) = A_{\lambda_1 \dots \lambda_{n_0}}$ holds. For $n \in \omega_0$, $\sigma \in \omega_\alpha$ we define: $w(n, \sigma) = \bigvee_{\lambda_n = \sigma} A_{\lambda_1 \dots \lambda_n}$. In [3] it is proved that $\mathcal{A}_{0\alpha} = \{w(n, \sigma);\ n \in \omega_0, \sigma \in \omega_\alpha\}$ is a $(0, \alpha)$-system on $B$. Following Theorem 1, the elements of $\mathcal{A}_{0\alpha}$ are generated by a denumerable family $\{v(n, m);\ n, m \in \omega_0\}$. We show that the sets $A_{\lambda_1 \dots \lambda_n}$

can be constructed from elements of $\mathcal{A}_{0\alpha}$ by induction on $n$. For any $\lambda_1 \in \omega_\alpha$, $A_{\lambda_1}$ is $w(1, \lambda_1)$. Having constructed all the sets $A_{\lambda_1, \dots, \lambda_n}$ for some $n$, we have

$$A_{\lambda_1 \dots \lambda_n, \lambda_{n+1}} = w(n+1, \lambda_{n+1}) \wedge A_{\lambda_1 \dots \lambda_n} .$$

This proves the theorem, for each regular open set in $(P, \tau)$ being the join of a family of sets $A_{\lambda_1 \dots \lambda_n}$. This follows from the construction of these sets.

Remark. Theorem 2 can be proved without verifying that $\mathcal{A}_{0\alpha}$ is a $(\alpha, \beta)$-system. Instead of this we have to prove that the sets

$$v(n, m) = \bigvee A_{\lambda_1 \dots \lambda_k} \quad (k \geq m, n \;\&\; \lambda_n \leq \lambda_m)$$

generate the sets $A_{\lambda_1 \dots \lambda_n}$. This can be done directly, by analogy to the proof of Theorem 1.

Corollary 1. Let $(P, \tau)$ be a linear metric space, $B$ the complete Boolean algebra of all regular open sets in $(P, \tau)$. Then $B$ has a denumerable set of generators.

Remark. It is possible to give some other conditions for a topological space to be saturated.

We say that a space $(P, \tau)$ is homogeneous if, for every two points $x, y \in P$, there exists a homeomorphism $\varphi$ of $P$ onto $P$ such that $\varphi(x) = y$. The following assertion gives a sufficient condition for a space $(P, \tau)$ to be saturated.

Lemma 1. Let $(P, \tau)$ be a homogeneous topological space. Let there exist a saturated set $\sigma_1 \subset P$ such that $\mu(\sigma_1, \tau) = \mu(P, \tau)$. Then $(P, \tau)$ is saturated.

Proof. Let there exist a non-empty set $\sigma_0 \subset P$ such that $\mu(\sigma_0, \tau) < \mu(P, \tau)$. We can easily verify that $\sigma_1 \neq 0$. Let us fix $x \in \sigma_1$, $y \in \sigma_0$. Then there is no homeomorphism $\varphi$ such that $\varphi(x) = y$, and this gives a contradiction. From this follows that $(P, \tau)$ is saturated.

Corollary 2. Let $(P, \tau)$ be a metric space which satisfies the assumptions of Lemma 1. Then the complete Boolean algebra of all regular open sets in $(P, \tau)$ has a denumerable set of generators.

References

[1] R.M. SOLOVAY: New proof of a Theorem of Gaifman and Hales. Bull. Amer. Math. Soc., Vol. 72, No 2 (1967).

[2] P. VOPĚNKA: General theory of $\nabla$-models. Comment. Math. Univ. Carolinae 8,1(1967),145-170.

[3] P. ŠTĚPÁNEK - P. VOPĚNKA: Decomposition of metric spaces into nowhere dense sets. Comment. Math. Univ. Carolinae 8,3(1967),387-404.

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