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A REMARK ON SELECTIVE FUNCTORS x)

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In the present remark, we show that the definition of selectivity from [1] may be simplified. Namely, we prove the sufficiency of $\emptyset$-selectivity for selectivity (i.e. Δ-selectivity for any type Δ).

We use the terminology and notation of [1] (set functor, I, Q_M, P^- etc.) with the following alterations and additions: We describe types by $(\alpha_\beta)_{\beta<\gamma}$ instead of the unprecise $\{\alpha_\beta \mid \beta < \gamma\}$. Similarly, relational systems are denoted by $(\kappa_\beta)_{\beta<\gamma}$ etc. We write $\mathcal{R}(\Delta)$ instead of $\mathcal{T}(I, \Delta)$ (as was originally done in [2]). For the void type, $\mathcal{R}(\emptyset)$ is identical with the category $\mathcal{T}$ of all sets and all their mappings. Without fear of confusion we write again $\square$ for the natural forgetful functors of various $\mathcal{R}(\Delta)$. (Thus, in the case of $\Delta = \emptyset$, $\square = I$.) If $\tau : F \to G$ is a transformation of functors, we say that it is a monotransformation (an epitransformation) if every τ^X is a monomorphism (an epimorphism).

In [1], a one-to-one set functor F is said to be Δ-

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x) Supported by the Alexander von Humboldt-Stiftung

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selective if there is a type Δ and a one-to-one functor Φ mapping $\mathcal{R}(\Delta)$ onto a full subcategory of $\mathcal{R}(\Delta')$ such that the diagram

$$\begin{tikzcd}
\mathcal{R}(\Delta) \arrow[r, "\Phi"] \arrow[d, "\square"] & \mathcal{R}(\Delta') \arrow[d, "\square"] \\
\mathcal{S} \arrow[r, "F"] & \mathcal{S}
\end{tikzcd}$$

commutes.

The aim of this note is to prove

<u>Theorem</u>: A set functor is selective if and only if it is $\emptyset$ -selective. I.e., a one-to-one set functor F is selective if and only if there is a type Δ and a one-to-one functor, mapping $\mathcal{S}$ onto a full subcategory of $\mathcal{R}(\Delta)$ such that the diagram

$$\begin{tikzcd}
 & \mathcal{R}(\Delta) \arrow[d, "\square"] \\
\mathcal{S} \arrow[ur, "\Phi"] \arrow[r, "F"'] & \mathcal{S}
\end{tikzcd}$$

commutes.

First, we formulate and prove some lemmas.

<u>Lemma 1</u>: Let F be a covariant one-to-one set functor. Then there exists a monotransformation $\mu : I \to F$.

Let F be a contravariant one-to-one set functor. Then there exist transformations $\mu : P^- \to F$ and $\varepsilon : F \to P^-$ such that $\varepsilon \cdot \mu$ is the identity transformation of P^- (consequently, μ is a mono- and ε an epitransformation).

<u>Proof</u>: We use the notation $1 = \{0\}$, $2 = \{0,1\}$ etc.

For every set X and every $x \in X$ define $\xi_x^X : 1 \to X$ by $\xi_x^X(0) = x$. We have obviously, for every $f : X \to Y$,

$$f \cdot \xi_x^X = \xi_{f(x)}^Y .$$

Let F be a one-to-one covariant set functor. Thus, there is an $a \in F(1)$ such that

$$F(\xi_0^2)(a) \neq F(\xi_1^2)(a) .$$

Put $\mu^X(x) = F(\xi_x^X)(a)$. For $f : X \to Y$ we obtain $F(f)\mu^X(x) = F(f \cdot \xi_x^X)(a) = F(\xi_{f(x)}^Y)(a) = \mu^Y(f(x))$ and, hence, μ is a transformation. If $x, y \in X$, $x \neq y$, there exists an $f : X \to 2$ such that $f(x) = 0$, $f(y) = 1$. Thus, $F(f)\mu^X(x) = \mu^Y(0) = F(\xi_0^2)(a) \neq F(\xi_1^2)(a) = F(f)\mu^X(y)$ and hence $\mu^X(x) \neq \mu^X(y)$.

Now, let F be a contravariant one- to-one set functor. Instead of for P^-, we shall prove the assertion for P_2 defined by:

$$P_2(X) = \{\alpha \mid \alpha \text{ a mapping of } X \text{ into } 2\} ,$$
$$P_2(f)(\alpha) = \alpha \cdot f ,$$

which is evidently naturally equivalent with P^-.

There is a $b \in F(2)$ such that

$$a_0 = F(\xi_0^2)(b) \neq F(\xi_1^2)(b) = a_1 .$$

Define $\mu : P_2 \to F$ and $\varepsilon : F \to P_2$ as follows:

for $\alpha : X \to 2$, $\mu^X(\alpha) = F(\alpha)(b)$,

for $c \in F(X)$, $\varepsilon^X(c)(x) = 1 \Longleftrightarrow F(\xi_x^X)(c) = a_1$.

If $f : X \to Y$, we have $F(f)\mu^Y(\alpha) = F(f)F(\alpha)(b) = F(\alpha \cdot f)(b) = \mu^X(\alpha \cdot f) = \mu^X P_2(f)(\alpha)$,

$(P_2(f)\,\varepsilon^Y(c))(x) = \varepsilon^Y(c)(f(x)) = 1 \Longleftrightarrow a_1 = F(\xi^Y_{f(x)})(c) =$

$= F(f\cdot\xi^X_x)(c) = F(\xi^X_x)(F(f)(c)) \Longleftrightarrow (\varepsilon^X F(f)(c))(x) = 1.$

For any $\alpha : X\to 2$, $\varepsilon^X(\mu^X(\alpha))(x) = 1 \Longleftrightarrow a_1 =$

$= F(\xi^X_x)(\mu^X(\alpha)) = F(\alpha\cdot\xi^X_x)(b) \Longleftrightarrow \alpha\cdot\xi^X_x = \xi^2_1 \Longleftrightarrow \alpha(x) = 1$ and hence $\varepsilon^X\mu^X(\alpha) = \alpha$.

<u>Lemma 2</u>(Hedrlín): For $A\subset Q_M(X)$ define $qA\subset Q_M P^-(X)$ by

$$\varphi\in qA \Longleftrightarrow \forall\alpha : M\to X\ ((\forall m\ \alpha(m)\in\varphi(m)) \Rightarrow \alpha\in A).$$

Let $A\subset Q_M(X)$, $B\subset Q_M(Y)$, $f : X\to Y$. Then

$$Q_M(f)(A)\subset B \Longleftrightarrow Q_M P^-(f)(qB)\subset qA.$$

<u>Proof</u>: Let $Q_M(f)(A)\subset B$, $\varphi\in qB$. Let $\alpha(m)\in Q_M P^-(f)(\varphi)(m) = (P^-(f)\cdot\varphi)(m) = f^{-1}(\varphi(m))$ and hence $f\cdot\alpha(m)\in\varphi(m)$ for any $m\in M$. Then $f\cdot\alpha\notin B$ and hence $\alpha\notin A$.

Let $Q_M P^-(f)(qB)\subset qA$. Let $\alpha : M\to X$. If $f\cdot\alpha\notin B$, then $\varphi : M\to P^-(Y)$, defined by $\varphi(m) = \{f\cdot\alpha(m)\}$, is an element of qB. Thus, $Q_M(P^-(f))(\varphi) = P^-(f)\cdot\varphi\in qA$. Since $\alpha(m)\in f^{-1}\{f(\alpha(m))\} = P^-(f)(\varphi(m))$, $\alpha\notin A$.

<u>Lemma 3</u>. If g is a one-to-one mapping, then

$$g(f(A))\subset g(B) \Longleftrightarrow f(A)\subset B.$$

<u>Proof</u> is trivial.

<u>Proof of Theorem</u>: Let F be $\emptyset$-selective. Thus, there is a $\Delta' = (\alpha'_\beta)_{\beta<\gamma'}$ and a commutative diagram

$$\begin{tikzcd} & \mathcal{R}(\Delta') \arrow[d, "\square"] \\ \gamma \arrow[ur, "\Phi"] \arrow[r, "F"'] & \gamma \end{tikzcd}$$

with one-to-one functor Φ, mapping γ onto a full subcategory of $\mathcal{R}(\Delta')$.

I. Let F be a covariant. Let $\Delta = (\alpha_\beta)_{\beta<\gamma}$. Put $\bar{\alpha}_\beta = \alpha'_\beta$ for $\beta < \gamma'$, $\bar{\alpha}_{\gamma'+\iota} = \alpha_\iota$, $\bar{\Delta} = (\alpha_\beta)_{\beta<\gamma'+\gamma}$. We have $\Phi(X) = (F(X), (\varkappa^X_\beta)_{\beta<\gamma'})$. Now, we see easily that the following prescription:

$\Psi(X, (\varkappa_\beta)_{\beta<\gamma}) = (F(X), (\bar{\varkappa}_\beta)_{\beta<\gamma'+\gamma})$, where $\bar{\varkappa}_\beta = \varkappa^X_\beta$ for $\beta < \gamma'$, $\bar{\varkappa}_{\gamma'+\beta} = (Q_{\alpha_\beta}\mu^X)(\varkappa_\beta)$ (for objects $(X, (\varkappa_\beta)_{\beta<\gamma})$ of $\mathcal{R}(\Delta)$; μ from the first part of lemma 1),

$\square \cdot \Psi(f) = F \cdot \square(f)$ (for morphisms),

defines a one-to-one functor $\Psi : \mathcal{R}(\Delta) \to \mathcal{R}(\bar{\Delta})$. Let

$$\varphi : (F(X), (\bar{\varkappa}_\beta)_{\beta<\gamma'+\gamma}) \to (F(Y), (\bar{\lambda}_\beta)_{\beta<\gamma'+\gamma})$$

be a morphism in $\mathcal{R}(\bar{\Delta})$. Hence, according to $\bar{\varkappa}_\beta$, $\bar{\lambda}_\beta$ for $\beta < \gamma'$, $\square\varphi = F(f)$ for some $f : X \to Y$. For every $\beta < \gamma$ we obtain

$$Q_{\alpha_\beta}\mu^Y(Q_{\alpha_\beta}(f)(\varkappa_\beta)) = Q_{\alpha_\beta}F(f)(Q_{\alpha_\beta}\mu^X)(\varkappa_\beta) =$$

$$= Q_{\bar{\alpha}_{\gamma'+\beta}}(\square\varphi)(\bar{\varkappa}_{\gamma'+\beta}) \subset \bar{\lambda}_{\gamma'+\beta} = Q_{\alpha_\beta}\mu^Y(\lambda_\beta)$$

and hence, by lemma 3, $Q_{\alpha_\beta}(f)(\varkappa_\beta) \subset \lambda_\beta$.

II. Let F be contravariant. Take Δ, $\bar{\Delta}$ as in I, denote again $\Phi(X) = (F(X), (\varkappa^X_\beta)_{\beta<\gamma'})$. For an object $(X, (\varkappa_\beta)_{\beta<\gamma})$ of $\mathcal{R}(\Delta)$, put $\Psi(X, (\varkappa_\beta)_{\beta<\gamma}) = (F(X), (\bar{\varkappa}_\beta)_{\beta<\gamma'+\gamma})$, where $\bar{\varkappa}_\beta = \varkappa^X_\beta$ for $\beta < \gamma'$, $\bar{\varkappa}_{\gamma'+\beta} = (Q_{\alpha_\beta}\mu^X)(q\,\varkappa_\beta)$ (μ from the second part of lemma 1, q from lemma 2), for morphisms define Ψ by $\square \cdot \Psi(\varphi) = F \cdot \square(\varphi)$. Again, it is easy to see that this defines a one-to-one functor $\Psi : \mathcal{R}(\Delta) \to \mathcal{R}(\bar{\Delta})$. Let $\varphi : (F(Y), (\bar{\lambda}_\beta)_{\beta<\gamma'+\gamma}) \to (F(X), (\bar{\varkappa}_\beta)_{\beta<\gamma'+\gamma})$ be a morphism in $\mathfrak{R}(\bar{\Delta})$. According to $\bar{s}_\beta$, $\bar{\kappa}_\beta$ with $\beta < \gamma'$ we obtain $\Box\varphi = F(f)$ for some $f: X \to Y$. We have, for $\beta < \gamma$,

$$Q_{\alpha_\beta}\mu^X(Q_{\alpha_\beta}P^-(f)(q s_\beta)) = Q_{\bar{\alpha}_{\gamma'+\beta}}F(f)(Q_{\alpha_\beta}\mu^X(q s_\beta)) =$$
$$= Q_{\bar{\alpha}_{\gamma'+\beta}}(\Box\varphi)(\bar{s}_{\gamma'+\beta}) \subset \bar{\kappa}_{\gamma'+\beta} = Q_{\alpha_\beta}\mu^X(q\kappa_\beta).$$

Thus, by lemma 3, $Q_{\alpha_\beta}P^-(f)(q s_\beta) \subset q\kappa_\beta$ and hence finally by lemma 2 $Q_{\alpha_\beta}(f)(\kappa_\beta) \subset s_\beta$.

R e f e r e n c e s

[1] Z. HEDRLÍN, A. PULTR: On categorial embeddings of topological structures into algebraic, Comment. Math. Univ. Carolinae 7,3(1966),377-400.

[2] Z. HEDRLÍN, A.PULTR: On full embeddings of categories of algebras, Illinois J. of Math.,10,3(1966),392-406.

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