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ON LATTICE POINTS IN HIGH-DIMENSIONAL ELLIPSOIDS

(Preliminary communication)

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Let n be an integer, $n \geqslant 5$ and

(1) $Q(u) = \sum_{j=1}^{n} \alpha_j u_j^2 , \quad \alpha_j > 0 \quad (j = 1, 2, \ldots, n)$.

For $x > 0$ let $A_Q(x)$ be a number of lattice points in a closed ellipsoid $Q(u) \leq x$, the volume of which is expressed by

$$V_Q(x) = \frac{\pi^{\frac{n}{2}} x^{\frac{n}{2}}}{\sqrt{\alpha_1 \alpha_2 \ldots \alpha_n}\, \Gamma(\frac{n}{2}+1)} \quad ;$$

We shall put

$$P_Q(x) = A_Q(x) - V_Q(x) .$$

The quadratic form $Q(u)$ is said to be "rational" if there is such a real number α that all α_j $(j = 1, 2, \ldots, n)$ are integer multiples of α ; otherwise we say that $Q(u)$ is "irrational".

The following results are well-known:

I. $P_Q(x) = O(x^{\frac{n}{2}-1})$ when Q is rational (see [6])

and this estimate is definitive; namely the following result is true:

II. $P_Q(x) = \Omega(x^{\frac{n}{2}-1})$ for Q rational (see [7]).

On the other hand:

III. $P_Q(x) = o(x^{\frac{r}{2}-1})$ for Q irrational (for $r \geq 6$ see [1], for $r = 5$ see [5]).

Furthermore we know that it is:

IV. $P_Q(x) = \Omega(x^{\frac{r-1}{4}})$ (see [8]).

V. For almost all systems $\alpha_1, \alpha_2, \dots, \alpha_r$ of positive real numbers (in the sense of the Lebesgue measure in the r-dimensional Euclidean space E_r) even the following is true: $P_Q(x) = O(x^{\frac{r}{4}+\varepsilon})$ for every $\varepsilon > 0$ (see [2]).

It is unknown if the estimates IV or V can be improved but we know that III cannot be, in general, improved, as it can be seen from the assertion:

VI. If $\varphi(x) > 0$ for $x > 0$ and $\varphi(x) \to 0$ for $x \to +\infty$, then for arbitrary $r \geq 5$ there exists an irrational form Q of the type (1) such that

$$P_Q(x) = \Omega(x^{\frac{r}{2}-1}\varphi(x)) \qquad \text{(see [9])}.$$

For a deeper and more detailed study of the function $P_Q(x)$ further specialization of the form Q turns out advantageous. Let σ and r_j be integers, $\sigma \geq 2$; $r_j \geq 4$ $(j = 1, 2, \dots, \sigma)$; $r = r_1 + r_2 + \dots + r_\sigma$, let $\alpha_j > 0$ $(j = 1, 2, \dots, \sigma)$. We shall consider the following forms:

$$(2)\quad Q(u) = \sum_{j=1}^{\sigma} \alpha_j (u_{1,j}^2 + u_{2,j}^2 + \dots + u_{r_j,j}^2),\ \alpha_j > 0\ (j = 1, 2, \dots, \sigma).$$

The assertions I - IV, of course, remain true also for the forms of the type (2); moreover, IV can be essentially strengthened due to the special choice (2) of the forms Q :

VII. Let σ and κ_j $(j = 1, 2, \dots, \sigma)$ be integers, $\kappa_j \geq 4$ $(j = 1, 2, \dots, \sigma)$, $\sigma \geq 2$, $\kappa = \kappa_1 + \kappa_2 + \dots + \kappa_\sigma$. The following estimate holds for the forms of the type (2):

$$P_Q(x) = \Omega(x^{\frac{\kappa}{2} - \sigma}) \qquad \text{(see [2]).}$$

We have now:

VIII. Let σ and κ_j be integers, $\sigma \geq 2$, $\kappa_j \geq 4$ $(j = 1, 2, \dots, \sigma)$, $\kappa = \kappa_1 + \kappa_2 + \dots + \kappa_\sigma$. Then for almost all systems of positive numbers $\alpha_1, \alpha_2, \dots, \alpha_\sigma$ (in the sense of the Lebesgue measure in E_σ) the estimate

$$P_Q(x) = O(x^{\frac{\kappa}{2} - \sigma + \varepsilon})$$

for every $\varepsilon > 0$ holds for the forms (2). (See [2].)

For any form Q of the type (1) let $f = f(Q)$ be the infimum of those real numbers ω for which

$$P_Q(x) = O(x^{\omega}),$$

i.e., for every $\varepsilon > 0$

$$P_Q(x) = O(x^{f+\varepsilon}), \quad P_Q(x) = \Omega(x^{f-\varepsilon}).$$

Then, according to I, III, VII there is

$$\frac{\kappa}{2} - \sigma \leq f(Q) \leq \frac{\kappa}{2} - 1$$

for the forms (2); σ, κ_j being integers, $\sigma \geq 2$, $\kappa_j \geq 4$ $(j = 1, 2, \dots, \sigma)$, $\kappa = \kappa_1 + \kappa_2 + \dots + \kappa_\sigma$.

It is obvious that for Q rational there is $f(Q) = \frac{\kappa}{2} - 1$ owing to II and by VIII it follows that for "almost all" Q there is $f(Q) = \frac{\kappa}{2} - \sigma$.

Let us denote by $\beta = \beta(\alpha)$ the supremum of those real ω for which the inequality

$$\left| \frac{p}{q} - \alpha \right| < \frac{1}{q^{\omega}}$$

is satisfied for infinitely many pairs of integers $\{p_n;\ q_n\}_{n=1}^{\infty}$, $q_n \geq 1$, $q_n \to +\infty$. Notice that $\beta(\alpha) = \beta(\frac{1}{\alpha})$ and that always $2 \leq \beta(\alpha) \leq +\infty$; moreover, for almost all α (in the sense of the Lebesgue measure in E_1) we have $\beta(\alpha) = 2$.

Now, the following statement is true:

IX. Let r_j be integers, $r_j \geq 4$ $(j = 1, 2)$, $r = r_1 + r_2$, $\alpha_j > 0$ $(j = 1, 2)$; let $\beta = \beta(\frac{\alpha_1}{\alpha_2})$,

$$Q(u) = \alpha_1 (u_{1,1}^2 + \cdots + u_{r_1,1}^2) + \alpha_2 (u_{1,2}^2 + \cdots + u_{r_2,2}^2).$$

Then

$$f(Q) = \frac{r}{2} - 1 - \frac{1}{\beta - 1} \qquad \text{(see [3])}$$

where for $\beta = +\infty$ we put $\frac{1}{\beta - 1} = 0$. Notice that $\beta(\frac{\alpha_1}{\alpha_2}) = \beta(\frac{\alpha_2}{\alpha_1})$. Thus, the assertion IX solves the question of finding $f(Q)$ for the forms (2) in the case $\sigma = 2$. For $\sigma > 2$ the following result is known:

X. Let σ, r_j be integers, $\sigma > 2$, $r_j \geq 4$ $(j = 1, 2, \ldots, \sigma)$, let $r = r_1 + r_2 + \cdots + r_\sigma$, $\frac{r}{2} - \sigma \leq f \leq \frac{r}{2} - 1$. Then there exists a form (2) such that $f(Q) = f$ (see [4]).

Now, let us denote by $\beta = \beta(\alpha_1, \ldots, \alpha_k)$ the supremum of those real numbers ω for which the system of inequalities

$$\left| \frac{p_j}{q} - \alpha_j \right| < \frac{1}{q^{\omega}} \qquad (j = 1, 2, \dots, k)$$

is satisfied for infinitely many (k+1)-tuples $\{p_{1n}, \dots, p_{kn}; q_n\}_{n=1}^{\infty}$ of integers, $q_n \geq 1$, $q_n \to +\infty$. Notice that $\frac{k+1}{k} \leq \beta(\alpha_1, \dots, \alpha_k) \leq +\infty$ and that for almost all systems $\alpha_1, \alpha_2, \dots, \alpha_k$ (in the sense of the Lebesgue measure in E_k) it is $\beta(\alpha_1, \dots, \alpha_k) = \frac{k+1}{k}$.

Our contribution is the following

Theorem 1. Let σ be an integer, $\sigma \geq 2$, let $\alpha_j > 0$ $(j = 1, 2, \dots, \sigma)$, let $\beta = \beta\left(\frac{\alpha_2}{\alpha_1}, \frac{\alpha_3}{\alpha_1}, \dots, \frac{\alpha_\sigma}{\alpha_1}\right)$; let $r_j \geq \frac{2\beta}{\beta - 1}$, r_j integers $(j = 1, 2, \dots, \sigma)$; $r = r_1 + r_2 + \dots + r_\sigma$.

Then for the forms (2) we have

$$f(Q) = \frac{r}{2} - 1 - \frac{1}{\beta - 1}.$$

We put $\frac{1}{\beta - 1} = 0$, $\frac{2\beta}{\beta - 1} = 2$ for $\beta = +\infty$.

Assymmetry of the assumptions of Theorem 1 is only seeming because $\beta\left(\frac{\alpha_2}{\alpha_1}, \frac{\alpha_3}{\alpha_1}, \dots, \frac{\alpha_\sigma}{\alpha_1}\right) = \beta\left(\frac{\alpha_1}{\alpha_2}, \frac{\alpha_3}{\alpha_2}, \dots, \frac{\alpha_\sigma}{\alpha_2}\right) = \dots = \beta\left(\frac{\alpha_1}{\alpha_\sigma}, \frac{\alpha_2}{\alpha_\sigma}, \dots, \frac{\alpha_{\sigma-1}}{\alpha_\sigma}\right)$.

It is $\beta \geq \frac{\sigma}{\sigma - 1}$ so that $\frac{2\beta}{\beta - 1} \leq 2\sigma$. According to this fact we see that the assumption $r_j \geq \frac{2\beta}{\beta - 1}$ is automatically satisfied as soon as $r_j \geq 2\sigma$. If β

passes through the interval $\langle \frac{\sigma}{\sigma-1}, +\infty \rangle$, $f(Q)$ passes through the interval $\langle \frac{r}{2}-\sigma, \frac{r}{2}-1 \rangle$.

Thus, the Theorem 1 generalizes the assertion IX to the general case $\sigma \geqslant 2$ for which only the existence statement X was known up to now. The Theorem 1 solves the question of finding of $f(Q)$ for sufficiently large r_j for every form (2). Jarník's method was used for the proof (see V. Jarník [3]).

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