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$\mathfrak{A}(1,1)$ can be strongly embedded into category of semigroups

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Z. Hedrlín and J. Lambek have constructed in [1] a full embedding of category of graphs and their compatible mappings into the category of semigroups. Above result together with results contained in [2] imply, in particular, that every category of algebras can be fully embedded into the category of semigroups.

Strong embedding is defined in paper [3] of A. Pultr, where is also proved that every category of algebras can be strongly embedded into $\mathfrak{A}(1,1)$ - the category of all algebras with two unary operations, as well as into the category $\mathfrak{A}(2)$ of all groupoids.

The aim of the present note is to construct a strong embedding of $\mathfrak{A}(1,1)$ into the category of semigroups $\mathcal{S}(2)$.

Note, that V. Trnková has constructed independently a strong embedding of category $\mathfrak{A}(2)$ into $\mathcal{S}(2)$. Her result, as well as the present construction, together with [3], yield the following

Corollary Any category of algebras can be strongly embedded into $\mathcal{S}(2)$.

The first step of our construction is given by

Theorem 1. Let $\mathcal{K} \subseteq \mathcal{O}\!\iota\,(1,1)$ be the primitive class of all the algebras $(X;\varphi,\psi)$ satisfying

$$(\varphi\circ\psi^2\circ\varphi^2)(x) = (\varphi^2\circ\psi\circ\varphi)(y)$$

for every x and y in X. Then there exists a strong embedding $\Phi : \mathcal{O}\!\iota\,(1,1) \to \mathcal{K}$.

Proof Let $A = (X; \alpha, \beta)$ be an object in $\mathcal{O}\!\iota\,(1,1)$. Put $Z = (X \times 3) \cup \{a_X, b_X\}$, where $(X \times 3) \cap \{a_X, b_X\} = \emptyset$. Put $\Phi(A) = (Z;\varphi,\psi)$, φ, ψ being unary operations on Z defined as follows:

$$\varphi(b_X) = \varphi(a_X) = \psi(b_X) = \varphi(\langle x, 2\rangle) = a_X \quad \text{for } x \text{ in } X,$$

$$\psi(a_X) = b_X,$$

$$\psi(\langle x, 0\rangle) = \langle x, 1\rangle,\ \psi(\langle x, 1\rangle) = \langle x, 2\rangle,$$

$$\psi(\langle x, 2\rangle) = \langle x, 0\rangle,\ \varphi(\langle x, 1\rangle) = \langle \beta(x), 2\rangle,$$

$$\varphi(\langle x, 0\rangle) = \langle \alpha(x), 2\rangle \quad \text{for } x \text{ in } X.$$

Let $A' = (X'; \alpha', \beta')$ be another object in $\mathcal{O}\!\iota\,(1,1)$. Thus, $\Phi(A') = ((X' \times 3) \cup \{a_{X'}, b_{X'}\}; \varphi', \psi')$. Let $f : A \to A'$ be a morphism of $\mathcal{O}\!\iota\,(1,1)$. Define $\Phi(f) : \Phi(A) \to \Phi(A')$ by $\Phi(f)(\langle x, i\rangle) = \langle f(x), i\rangle$ for i = 0,1,2 and x in X,

$$\Phi(f)(a_X) = a_{X'},$$

$$\Phi(f)(b_X) = b_{X'}.$$

Clearly, Φ is one-to-one functor, $\Phi : \mathcal{O}\!\iota\,(1,1) \to \mathcal{K}$. It remains to prove that its image is a full subcategory of $\mathcal{K}$.

Take $g : \Phi(A) \to \Phi(A')$ - a morphism in $\mathcal{K}$. Write a' instead of $a_{X'}$, etc.

As $a \in \Phi(A)$, $a' \in \Phi(A')$ are the only fixed points of φ, φ' respectively, we have $g(a) = a'$. Thus, $g(b) = g(\psi(a)) = \psi'(a') = b'$.

Assuming $g(\langle x, 0\rangle) = a'$, we have $b' = \psi'^3(a') = g(\psi^3(\langle x, 0\rangle)) = g(\langle x, 0\rangle)$ - a contradiction; similarly, $g(\langle x, 0\rangle) \neq b'$.

If $g(\langle x, 0\rangle) = \langle y, 1\rangle$, then $g(\langle x, 2\rangle) = g(\psi^2(\langle x, 0\rangle)) = \langle y, 0\rangle$, consequently, $g(a) = g(\varphi(\langle x, 2\rangle)) = \langle \alpha'(y), 2\rangle$. But $g(a) = a'$. By a similar argument we get $g(\langle x, 0\rangle) = \langle y, 0\rangle$. Finally, $g(\langle x, 1\rangle) = g(\psi(\langle x, 0\rangle)) = \psi'(\langle y, 0\rangle) = \langle y, 1\rangle$, $g(\langle x, 2\rangle) = \langle y, 2\rangle$. We can define $f : X \to X'$ by $\langle f(x), i\rangle = g(\langle x, i\rangle)$, $i = 0,1,2$.

We have $\langle f(\alpha(x)), 2\rangle = g(\langle \alpha(x), 2\rangle) = g(\varphi(\langle x, 0\rangle)) = \varphi'(\langle f(x), 0\rangle) = \langle \alpha'(f(x)), 2\rangle$. Consequently, $f(\alpha(x)) = \alpha'(f(x))$. An analogous computation applied on $\langle f(\beta(x)), 2\rangle$ gives $f(\beta(x)) = \beta'(f(x))$. We conclude $g = \Phi(f)$, $f \in \mathcal{O}\!\mathcal{L}(1, 1)$.

Let D be a semigroup with two generators a,b and with the defining relation $ab^2 = baba$. There is proved in [4] that D is rigid. Another rigid semigroup was found by Z. Hedrlín; it is the semigroup H genereated by c,d and satisfying the relation $c^2dc = cd^2c^2$. Both these semigroups will be used in the proof of

Theorem 2 The primitive class $\mathcal{K}$ can be strongly embedded into $\mathcal{S}(2)$.

Proof. $A = (X; \varphi, \psi)$ being an object in $\mathcal{K}$, put $\Phi(A) = ((X \times D) \cup H_X, \cdot)$, where H_X is isomorphic with H, $H_X \cap (X \times D) = \emptyset$. If $\alpha_X(c_X, d_X) \in H_X$, $\alpha = c^{k_1} d^{l_1} \dots c^{k_n} d^{l_n}$ (here and in the sequel we omit the indices), let us define a mapping $\tilde{\alpha}_X = \tilde{\alpha}_X(\varphi, \psi) : X \to X$ by $\tilde{\alpha}_X(x) = (\varphi^{k_1} \circ \psi^{l_1} \circ \dots \circ \varphi^{k_n} \circ \psi^{l_n})(x)$. Note that $\alpha = \beta$ in H implies $\tilde{\alpha}_X = \tilde{\beta}_X$, because of $(X; \varphi, \psi) \in \mathcal{K}$. In particular, $\tilde{\alpha}_X \circ \tilde{\beta}_X = \widetilde{\alpha\beta}_X$, where $\alpha\beta$ is the product of α, β in H.

The operation $\cdot$ is defined as follows:

$\langle x, w \rangle \cdot \langle y, v \rangle = \langle x, wv \rangle$ for x and y in X, w and v in D,

$$\left.\begin{aligned} \langle x, w \rangle \cdot \alpha &= \langle x, w \rangle \\ \alpha \cdot \langle x, w \rangle &= \langle \tilde{\alpha}(x), w \rangle \end{aligned}\right\} \text{ for } x \text{ in } X,$$

in H and w in D,

$\alpha \cdot \beta = \alpha\beta$ for α and β in H.

A bit of computation, using the remark above, yields that $\Phi(A)$ is in $\mathcal{S}(2)$.

For $A' = (X'; \varphi', \psi')$ denote $\Phi(A') = ((X' \times D) \cup H_{X'}, \cdot)$. Let $f : A \to A'$ be a morphism in $\mathcal{K}$. Put

$\Phi(f)(\langle x, w \rangle) = \langle f(x), w \rangle$ for x in X and w in D,

$\Phi(f)(\alpha) = \alpha$ for α in H.

One can easily see that Φ is a one-to-one functor, $\Phi : \mathcal{K} \to \mathcal{S}(2)$.

Now, let $F : \Phi(A) \to \Phi(A')$ be a morphism in

$\mathcal{G}(2)$.

At first, suppose that $F(\langle x, w\rangle) = \alpha \in H_{X'}$ and take a β in H_X. We have $\alpha = F(\langle x, w\rangle) = F(\langle x, w\rangle)\cdot F(\beta) = \alpha \cdot F(\beta)$. Consequently, $F(\beta) \in H_{X'}$. Since H is rigid, we have $F(\beta) = \beta$ for every β in H_X. In particular, $\alpha = F(\langle x, w\rangle) = F(\langle x, w\rangle \cdot \alpha) = \alpha \cdot \alpha$, but a rigid semigroup cannot have an idempotent element. We conclude that $F(X \times D) \subseteq X' \times D$.

Denote $F(\langle x, a\rangle) = \langle z, t\rangle$, $F(\langle x, b\rangle) = \langle y, w\rangle$. It is $F(\langle x, ab\rangle) = F(\langle x, a\rangle \cdot \langle x, b\rangle) = \langle z, t\rangle \cdot \langle y, w\rangle = \langle z, tw\rangle$, analogously $F(\langle x, baba\rangle) = \langle y, wtwt\rangle$, $F(\langle x, ab^2\rangle) = \langle z, tw^2\rangle$. As $baba = ab^2$ in D, then $y = z$. If $v_0 = av_1 \in D$, then $F(\langle x, v_0\rangle) = F(\langle x, a\rangle \cdot \langle x, v_1\rangle) = \langle y, t\rangle \cdot \langle y', u_1\rangle = \langle y, tu_1\rangle$, similarly for $v_2 = bv_1$; but this means that $F(\{x\} \times D) \subseteq \{y\} \times D$. Both $\{x\} \times D$ and $\{y\} \times D$ are isomorphic with rigid semigroup D, thus $F(\langle x, w\rangle) = \langle y, w\rangle$ for any w in D. Now, we may define a mapping $f: X \to X'$ by

$$\langle f(x), w\rangle = F(\langle x, w\rangle).$$

Suppose $F(\alpha) = \langle y, w\rangle$ for some $\alpha \in H_X$, $y \in X'$. Taking $\langle x, w\rangle \in X \times D$, we get $\langle f(x), ww\rangle = \langle f(x), w\rangle \cdot \langle y, w\rangle = F(\langle x, w\rangle)\cdot F(\alpha) =$
$$= F(\langle x, w\rangle) = \langle f(x), w\rangle,$$

while, D has no idempotent. Thus, $F(H_X) \subseteq H_X$, and, by the rigidity of H, $F(\alpha) = \alpha$. For x in X we have $\langle \varphi'(f(x)), w \rangle = c_{X'} \cdot \langle f(x), w \rangle = F(c_x) \cdot F(\langle x, w \rangle) = F(\langle \varphi(x), w \rangle) = \langle f(\varphi(x)), w \rangle$. Similarly for ψ using b. We conclude that $f \in \mathcal{K}$, $F = \Phi(f)$.

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