Jiří Sichler

$\mathfrak{A}(1,1)$ can be strongly embedded into category of semigroups

*Commentationes Mathematicae Universitatis Carolinae*, Vol. 9 (1968), No. 2, 257--262

Persistent URL: [http://dml.cz/dmlcz/105178](http://dml.cz/dmlcz/105178)

**Terms of use:**

© Charles University in Prague, Faculty of Mathematics and Physics, 1968

Institute of Mathematics of the Academy of Sciences of the Czech Republic provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these *Terms of use.*

This paper has been digitized, optimized for electronic delivery and stamped with digital signature within the project *DML-CZ: The Czech Digital Mathematics Library* [http://project.dml.cz](http://project.dml.cz)
\( \mathcal{A}(1,1) \) can be strongly embedded into category of semigroups

Jiří SICHLER, Praha

Z. Hedrlín and J. Lambek have constructed in [1] a full embedding of category of graphs and their compatible mappings into the category of semigroups. Above result together with results contained in [2] imply, in particular, that every category of algebras can be fully embedded into the category of semigroups.

Strong embedding is defined in paper [3] of A. Pultr, where is also proved that every category of algebras can be strongly embedded into \( \mathcal{A}(1,1) \) - the category of all algebras with two unary operations, as well as into the category \( \mathcal{A}(2) \) of all groupoids.

The aim of the present note is to construct a strong embedding of \( \mathcal{A}(1,1) \) into the category of semigroups \( \mathcal{S}(2) \).

Note, that V. Trnková has constructed independently a strong embedding of category \( \mathcal{A}(2) \) into \( \mathcal{S}(2) \). Her result, as well as the present construction, together with [3], yield the following

**Corollary** Any category of algebras can be strongly embedded into \( \mathcal{S}(2) \).
The first step of our construction is given by

**Theorem 1.** Let $\mathcal{X} \subseteq \mathcal{A}(1, 1)$ be the primitive class of all the algebras $(X; \varphi, \psi)$ satisfying

$$(\varphi \cdot \varphi^2 \cdot \varphi^2)(x) = (\varphi^2 \cdot \psi \cdot \varphi)(y)$$

for every $x$ and $y$ in $X$. Then there exists a strong embedding $\Phi: \mathcal{A}(1, 1) \rightarrow \mathcal{X}$.

**Proof.** Let $A = (X; \alpha, \beta)$ be an object in $\mathcal{A}(1, 1)$. Put $Z = (X \times 3 \cup \{ \alpha_X, \beta_X \}$, where $(X \times 3 \cup \{ \alpha_X, \beta_X \} = \emptyset$. Put $\Phi(A) = (Z; \varphi, \psi)$, $\varphi$, $\psi$ being unary operations on $Z$ defined as follows:

- $\varphi(\alpha_X) = \varphi(\alpha_X) = \psi(\alpha_X) = \varphi(\alpha_X, 2) = \alpha_X$ for $x$ in $X$,
- $\varphi(\alpha_X) = \beta_X$,
- $\psi(\alpha_X, 0) = \langle x, 1 \rangle$, $\psi(\alpha_X, 1) = \langle x, 2 \rangle$,
- $\psi(\alpha_X, 2) = \langle x, 0 \rangle$, $\varphi(\alpha_X, 1) = \langle \beta(x), 2 \rangle$,
- $\varphi(\alpha_X, 0) = \langle \alpha(x), 2 \rangle$ for $x$ in $X$.

Let $A' = (X'; \alpha', \beta')$ be another object in $\mathcal{A}(1, 1)$. Thus, $\Phi(A') = ((X' \times 3 \cup \{ \alpha_X, \beta_X \}; \varphi', \psi')$.

Let $f: A \rightarrow A'$ be a morphism of $\mathcal{A}(1, 1)$. Define $\Phi(f): \Phi(A) \rightarrow \Phi(A')$ by $\Phi(f)(\alpha_X, i) = \langle f(\alpha_X), i \rangle$ for $i = 0, 1, 2$ and $x$ in $X$,

- $\Phi(f)(\alpha_X) = \alpha_X$,
- $\Phi(f)(\beta_X) = \beta_X$.

Clearly, $\Phi$ is one-to-one functor, $\Phi: \mathcal{A}(1, 1) \rightarrow \mathcal{X}$. It remains to prove that its image is a full subcategory of $\mathcal{X}$.

- 258 -
Take \( \varphi \) : \( \Phi(A) \rightarrow \Phi(A') \) \( \text{a morphism} \) in \( \mathcal{K} \). Write \( a' \) instead of \( a_{A} \), etc.

As \( a \in \Phi(A) \), \( a' \in \Phi(A') \) \( \text{are the only fixed points of} \ \varphi \), \( \varphi' \) \( \text{respectively}, \) we have
\[
g_{\varphi}(a) = a'.
\]
Thus, \( g_{\varphi}(a') = \varphi(\varphi(a)) = \varphi'(a') = b'.\)

Assuming \( \varphi(\langle \alpha, 0 \rangle) = a' \), \( \text{we have} \)
\[
\varphi' = \varphi^{3}(a') = g_{\varphi^{3}(\langle \alpha, 0 \rangle)} = g_{\langle \alpha, 0 \rangle} - \text{a contradiction}; \text{similarly,} \ \varphi(\langle \alpha, 0 \rangle) = b'.
\]
If \( \varphi(\langle \alpha, 0 \rangle) = \langle y, 1 \rangle \), then \( \varphi(\langle \alpha, 2 \rangle) = \).
\[
\varphi_{\varphi^{2}(\langle \alpha, 0 \rangle)} = \langle y, 0 \rangle , \ \text{consequently,} \ \varphi_{\varphi}(a) = \)
\[
\varphi(\varphi(\langle \alpha, 2 \rangle)) = \langle \alpha'(y), 2 \rangle. \ \text{But} \ \varphi_{\varphi}(a) = a'. \ \text{By}
\]
a similar argument we get \( \varphi(\langle \alpha, 0 \rangle) = \langle y, 0 \rangle \). Finally, \( \varphi(\langle \alpha, 1 \rangle) = \varphi(\varphi(\langle \alpha, 0 \rangle)) = \varphi(\langle y, 0 \rangle) = \langle y, 1 \rangle, g_{\langle \alpha, 2 \rangle} = \langle y, 2 \rangle . \)

We can define \( f : X \rightarrow X' \) by \( \langle f(x), i) = \varphi(\langle x, i \rangle) \), \( i = 0, 1, 2 \).

We have \( \langle f(\alpha(x)), 2 \rangle = g_{\varphi}(\langle \alpha(x), 2 \rangle) = g_{\varphi}(\varphi(\langle x, 0 \rangle)) = \)
\[
= g_{\langle f(x), 0 \rangle} = \alpha'(f(x)). \ \text{Consequently,} \ f(\alpha(x)) = \alpha'(f(x)). \ \text{An analogous computation applied on} \)
\[
\langle f(\beta(x)), 2 \rangle \ \text{gives} \ f(\beta(x)) = \beta'(f(x)). \ \text{We conclude}
\]
\[
\varphi = \Phi(f), \ f \in \mathcal{G}(1, 1). \]

Let \( D \) be a semigroup with two generators \( a, b \) and with the defining relation \( a b^{2} = b a b a. \) There is proved in [4] that \( D \) is rigid. Another rigid semigroup was found by Z. Hedrlín; it is the semigroup \( H \) generated by \( c, d \) and satisfying the relation \( c^{2} dc = cd^{2} c^{2} . \) Both these semigroups will be used in the proof of

**Theorem 2** The primitive class \( \mathcal{K} \) can be strongly embedded into \( \mathcal{G}(2) . \)
Proof. \( A = (X; \varphi, \psi) \) being an object in \( \mathcal{K} \), put \( \Phi(A) = ((X \times D) \cup H_\times, \cdot) \), where \( H_\times \) is isomorphic with \( H \). \( H_\times \cap (X \times D) = \emptyset \). If \( \alpha_X(c_X, d_X) \in H_\times \), \( \alpha = c^k d^l \cdot \cdot \cdot c_m d_n \) (here and in the sequel we omit the indices), let us define a mapping \( \alpha_X = \alpha_X(c^k, d^l) : X \rightarrow X \) by
\[
\alpha_X(x) = (c^k \varphi^l \cdot \cdot \cdot c_m \varphi^n)(x).
\]
Note that \( \alpha = \beta \) in \( H \) implies \( \alpha_X = \beta_X \), because of \( (X; \varphi, \psi) \in \mathcal{K} \). In particular, \( \alpha_X \circ \beta_X = \alpha \beta_X \), where \( \alpha \beta \) is the product of \( \alpha, \beta \) in \( H \).

The operation \( \cdot \) is defined as follows:
\[
\langle x, w \rangle \cdot \langle y, v \rangle = \langle x, w \cdot y \cdot v \rangle \quad \text{for } x \text{ and } y \text{ in } X, w \text{ and } v \text{ in } D,
\]
\[
\alpha, \cdot \langle x, w \rangle = \langle \alpha(x), w \rangle \quad \text{for } x \text{ in } X,
\]
in \( H \) and \( w \) in \( D \),
\[
\alpha \cdot \beta = \alpha \beta \quad \text{for } \alpha \text{ and } \beta \text{ in } H.
\]

A bit of computation, using the remark above, yields that \( \Phi(A) \) is in \( \mathcal{Y}(2) \).

For \( A' = (X'; \varphi', \psi') \) denote \( \Phi(A') = ((X' \times D) \cup H_\times', \cdot) \). Let \( \mathcal{F} : A \rightarrow A' \) be a morphism in \( \mathcal{K} \). Put
\[
\Phi(\mathcal{F})(\langle x, w \rangle) = \langle \mathcal{F}(x), w \rangle \quad \text{for } x \text{ in } X \text{ and } w \text{ in } D,
\]
\[
\Phi(\mathcal{F})(\alpha) = \alpha \quad \text{for } \alpha \text{ in } H.
\]
One can easily see that \( \Phi \) is a one-to-one functor, \( \Phi : \mathcal{K} \rightarrow \mathcal{Y}(2) \).

Now, let \( \mathcal{F} : \Phi(A) \rightarrow \Phi(A') \) be a morphism in
At first, suppose that \( F(\langle x, w \rangle) = \alpha \in H_x \), and take a \( \beta \) in \( H_x \). We have \( \alpha = F(\langle x, w \rangle) = F(\langle x, w \rangle) \cdot F(\beta) = \alpha \cdot F(\beta) \). Consequently, \( F(\beta) \in H_x \). Since \( H \) is rigid, we have \( F(\beta) = \beta \) for every \( \beta \) in \( H_x \). In particular, \( \alpha = F(\langle x, w \rangle) = F(\langle x, w \rangle \cdot \alpha) = \alpha \cdot \alpha \), but a rigid semigroup cannot have an idempotent element. We conclude that \( F(\langle x, D \rangle) \leq \langle x' \rangle \times D \).

Denote \( F(\langle x, a \rangle) = \langle x, t \rangle, F(\langle x, b \rangle) = \langle y, w \rangle \). It is \( F(\langle x, a \cdot b \rangle) = F(\langle x, a \cdot (x, b) \rangle) = \langle x, t \rangle \).

\( \langle y, w \rangle = \langle x, tw \rangle \), analogously \( F(\langle x, baba \rangle) = \langle y, wt \rangle \), \( F(\langle x, a, a^2 \rangle) = \langle x, tw^2 \rangle \).

As \( baba = a^2 \) in \( D \), then \( y = z \). If \( y_2 = a \cdot v_j \in D \), then \( F(\langle x, v_j \rangle) = \langle x, a,v_j \rangle = \langle y, w \rangle \). Similarly for \( v_2 = b \cdot v_j \), but this means that \( F(\{ x \times D \}) \leq \{ y \} \times D \). Both \( \{ x \} \times D \) and \( \{ y \} \times D \) are isomorphic with rigid semigroup \( D \), thus \( F(\langle x, w \rangle) = \langle y, w \rangle \) for any \( w \) in \( D \). Now, we may define a mapping \( f : X \to X' \) by

\[ f(x) = \langle y, w \rangle = F(\langle x, w \rangle). \]

Suppose \( F(x) = \langle y, w \rangle \) for some \( \alpha \in H_x, y_j \in X'. \)

Taking \( \langle x, w \rangle \in X \times D \), we get \( f(x), w \cdot w = F(\langle x, w \rangle) \cdot F(\alpha) = F(\langle x, w \rangle) = \langle f(x), w \rangle \).
while, D has no idempotent. Thus, \( F(H_X) \subseteq H_X \) and, by the rigidity of \( H \), \( F(\alpha) = \alpha \). For \( x \) in \( X \) we have \( \langle \varphi'(f(x)), \omega_r \rangle = C_x \cdot \langle f(x), \omega_r \rangle + F(C_x) \cdot F(\langle f(x), \omega_r \rangle) = F(C_x) \cdot F(\langle f(x), \omega_r \rangle) = \langle f(\varphi(x)), \omega_r \rangle \). Similarly for \( \psi \) using \( \lambda \). We conclude that \( f \in \mathcal{K} \), \( F = \Phi(f) \).

I am indebted to A. Pultr for suggestion of the problem and for his helpful comments.

References


(Received April 5, 1968)