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CONTINUITY PROPERTIES OF NONLINEAR MAPPINGS

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1. Introduction. In the present paper we give some "general" counterexamples x) and assertions in the theory of continuity properties of nonlinear mappings between linear topological spaces. The starting point is Lemma 1. The proofs are constructive and are based on the existence of a nonconvergent but weakly convergent sequence in the range space.

I am indebted to J.Kolomý for the suggestion of these problems.

2. Notations and definitions. The convergence in the original and the weak topology of a locally convex linear topological space is denoted by " $\longrightarrow$ " and " $\rightharpoonup$ ", respectively. $\langle 0, \infty \rangle$ denotes the one-point compactification of the set of all nonnegative real numbers. The symbols ∂M and $int\ M$ denote the boundary and the interior of the subset M of a topological space. $\langle x, y \rangle$ is the closed line-segment between two points in a linear space. All topological spaces are supposed to be separated (with exception of the space X in Theorem 8).

Let X be a linear topological space and Y a local-

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x) For some special examples see Vajnberg [4] and Petryshyn [5].

ly convex linear topological space. Then a mapping f of a subset M of X into Y is called demicontinuous if f is continuous from M with the topology induced by the topology of X into (Y, w) (the space Y with the weak topology). If X is locally convex, then the mapping f is called weakly continuous if f is continuous from (M, w) (M with the topology induced by the weak topology of X) into (Y, w), and strongly continuous if f is continuous from the weak topology on M to the original topology of Y .

3. <u>Results</u>. We prove the following basic

<u>Lemma 1</u>. Let Y be a locally convex linear topological space. Let there exist a sequence $\{e_n\}_{n=1}^{\infty}$ in Y such that we have $e_n \rightharpoonup 0$ but not $e_n \to 0$. Then there exists a continuous mapping h from the compact space $\langle 0, \infty \rangle$ to the space Y with the weak topology which is continuous from $\langle 0, \infty)$ to Y (with the original topology) but it is not continuous at the point $x = \infty$.

<u>Proof</u>. Select a strictly increasing sequence $\{x_n\}_{n=1}^{\infty}$ of nonnegative real numbers such that $x_1 = 0$ and $x_n \to \infty$. Let $\{e_n\}_{n=1}^{\infty}$ be a sequence in Y with $e_n \rightharpoonup 0$ and $e_n \not\to 0$ We define a mapping $h : \langle 0, \infty \rangle \to Y$ in the following way:

$$h(x) = \begin{cases} 0 & \text{if } x = \infty , \\ \dfrac{(x_{n+1} - x) e_n + (x - x_n) e_{n+1}}{x_{n+1} - x_n} & \text{if } x \in \langle x_n, x_{n+1} \rangle , \end{cases}$$

$n = 1, 2, \dots$.

Suppose that $x_\alpha \longrightarrow x$ is a net in $\langle 0, \infty \rangle$ converging to an element x in $\langle 0, \infty)$. Then there are α_0 and n such that

$$\alpha \succ \alpha_0 \Longrightarrow x_\alpha \in (\varkappa_n, \varkappa_{n+2}) .$$

Since h is continuous from $\langle \varkappa_n , \varkappa_{n+2} \rangle$ to Y , we have $h(x_\alpha) \longrightarrow h(x)$ (and $h(x_\alpha) \rightharpoonup h(x)$, of course). Hence the mapping h is continuous from $\langle 0, \infty)$ to Y (and to (Y, w) too).

Let $x_\alpha \longrightarrow \infty$. It follows that, if V is a convex neighborhood of o in Y with the weak topology, there is an integer n_0 such that $e_n \in V$ for all $n \geqslant n_0$. Then there exists an index α_0 such that $x_\alpha > \varkappa_{n_0}$ for all $\alpha \succ \alpha_0$. The definition of h and the fact that $h(\langle \varkappa_n , \infty))$ is contained in the convex hull of the sequence $\{e_m\}_{m=n}^{\infty}$ imply that $h(x_\alpha) \in V$ for each $\alpha \succ \alpha_0$. Hence $h(x_\alpha) \rightharpoonup o$ and h is continuous from $\langle 0, \infty \rangle$ to the weak topology of Y .

Now, since $\varkappa_n \longrightarrow \infty$ and $h(\varkappa_n) = e_n \not\longrightarrow o$, we have that h is not continuous at the point $\varkappa = \infty$. Q.E.D.

Theorem 1. Let X and Y be locally convex linear topological spaces. Let there exist a sequence $\{e_n\}_{n=1}^{\infty}$ in Y such that $e_n \rightharpoonup o$ but not $e_n \longrightarrow o$. If $\dim X \geqslant 1$ there exists a weakly continuous mapping $f : X \rightarrow Y$ whose set of points of noncontinuity is a closed hyperplane in X .
(This closed hyperplane can be given before.)

Proof. Let u be a nontrivial element in X^*. Denote $H = u^{-1}(0)$. We define a mapping $g : X \to \langle 0, \infty \rangle$ as follows:

$$g(x) = \begin{cases} \dfrac{1}{|u(x)|} & \text{if } x \in X \setminus H, \\ \infty & \text{if } x \in H. \end{cases}$$

Since u is a weakly continuous functional on X, the mapping $g : X \longrightarrow \langle 0, \infty \rangle$ is continuous from the weak topology in X to $\langle 0, \infty \rangle$. Let $h : \langle 0, \infty \rangle \to Y$ be the mapping from the proof of Lemma 1. Let $f = h \circ g$. Then f is the composition of two continuous mappings g and h (with the weak topologies in X and Y) and hence f is weakly continuous. Since $h | \langle 0, \infty)$ and g are continuous (with original topologies in X and Y) $f | X \setminus H$ is continuous.

Let $x \in u^{-1}(1)$ and $x_n = \frac{1}{\kappa_{n+1}} \cdot x$ $(n = 1, 2, \ldots)$.

Then $f(x_n) = h\left(\frac{1}{|u(x_n)|}\right) = h(\kappa_{n+1}) = e_{n+1} \not\to 0$.

Hence f is noncontinuous at each point of the hyperplane H. Q.E.D.

Theorem 2. Let X and Y be as in the preceding Theorem. Further, let D be a convex closed body in X (i.e. $\operatorname{int} D$ is non-empty). Then there exists a demicontinuous mapping $f : X \to Y$ whose set of points of noncontinuity is the boundary ∂D of the set D.

Proof. Clearly, we can suppose that $0 \in \operatorname{int} D$. Let d be the Minkowski functional of the convex body D. Since $d : X \longrightarrow \langle 0, \infty \rangle$ is continuous, the mapping $g : X \longrightarrow \langle 0, \infty \rangle$ defined by $g(x) = d(x) | 1 - d(x) |^{-1}$

for $x \in X \setminus \partial D$ and by $g(x) = \infty$ for $x \in \partial D$, is continuous. Let h be the mapping from Lemma 1. Then $f = h \circ g : X \to Y$ is demicontinuous as the composition of the continuous mapping g and the demicontinuous mapping h . Further, the restriction of f to the set $X \setminus \setminus \partial D$ is continuous, since this restriction is the composition of two continuous mappings: g and $h \mid \langle 0, \infty)$.

Let x be a point of ∂D. Since h is non-continuous at the point $r = \infty$, there exists a sequence $\{r_n\}_{n=1}^{\infty}$ in $\langle 0, \infty)$ such that $r_n \to \infty$ and $h(r_n) \not\to h(\infty)$. If we set $x_n = r_n (1 + r_n)^{-1} \cdot x$ for $n = 1, 2, \dots$, then $x_n \in \operatorname{int} D$ (since $d(x_n) = r_n (1 + r_n)^{-1} < 1$) and $f(x_n) = h(g(x_n)) = h(d(x_n) |1 - d(x_n)|^{-1}) = h(r_n) \not\to h(\infty)$. This completes the proof that f is noncontinuous at each point x of ∂D. Q.E.D.

<u>Theorem 3</u>. Let X and Y be as in Theorem 1. Let D be a convex closed w-body in X (i.e. the interior of D in the weak topology of X is non-empty). Then there exists a weakly continuous mapping $f : X \to Y$ whose set of points of non-continuity is the boundary of D in the weak topology of X .

<u>Proof</u> of this Theorem is very similar to that of the Theorem 2 (d is now the Minkowski functional of the convex w-body D) and it can be omitted. Q.E.D.

<u>Theorem 4</u>. Let X be a nontrivial metrizable linear topological space, Y a locally convex space and M a non-empty subset of X such that $M = \partial M$ (i.e. M is closed "boundary" set in X). Suppose that there exists

a sequence $\{e_n\}_{n=1}^{\infty}$ in Y such that the set $\bigcup_{n=1}^{\infty} \langle e_n, e_{n+1} \rangle$ lies in the exterior of a neighborhood of o in Y and $e_n \longrightarrow o$, $e_n \not\longrightarrow o$. Then there exists a demicontinuous mapping $f: X \longrightarrow Y$ whose set of points of non-continuity is the set M.

Proof. Let d be an invariant metric for the metrizable linear topological space X. Define

$$g(x) = \begin{cases} \dfrac{1}{d(x, M)} & \text{if } x \in X \setminus M, \\ \infty & \text{if } x \in M. \end{cases}$$

Since $d: X \longrightarrow \langle 0, \infty \rangle$ is continuous the mapping $g: X \longrightarrow \langle 0, \infty \rangle$ is continuous.

Come back to the proof of Lemma 1. For the sequence $\{e_n\}_{n=1}^{\infty}$ in the proof of Lemma 1 we take a sequence $\{e_n\}_{n=1}^{\infty}$ in Y satisfying the condition $\bigcup_{n=1}^{\infty} \langle e_n, e_{n+1} \rangle \subset Y \setminus V$ where V is a neighborhood of o in Y. Now we construct the mapping h as in the proof of Lemma 1. It is easily to prove that the mapping h has the following property:

(P) $\{a_n\}_{n=1}^{\infty} \subset \langle 0, \infty), \ a_n \longrightarrow \infty \implies h(a_n) \not\rightarrow h(\infty)$.

In the sequel we deal with <u>this</u> mapping h.

Let $f = h \circ g$. Then $f: X \longrightarrow Y$ is demicontinuous and the restriction of f to the exterior of M in X is continuous.

Let x be a point in M and $\{x_n\}_{n=1}^{\infty}$ a sequence in $X \setminus M$ with $x_n \longrightarrow x$. Then $g(x_n) \longrightarrow \infty$ and, by the property (P), we have:

$$f(x_n) = h(g(x_n)) \not\to h(\infty) = f(x).$$

Hence the mapping f is non-continuous at each point x in M . Q.E.D.

Remark. It is clear that the preceding Theorem can be formulated with X from a class of metric spaces.

Theorem 5. Let X be a nontrivial metrizable linear topological space and Y a locally convex space. Suppose that there exists a sequence $\{e_n\}_{n=1}^{\infty}$ in Y with $e_n \rightharpoonup 0$ and $e_n \longrightarrow 0$. Let M be a nonempty subset of X such that $M = \partial M$ and each point x in M is "attainable" from the exterior of the set M , i.e. for each point x in M there exists a continuous injective mapping $\varphi_x : \langle 0, 1 \rangle \to X$ with $\varphi_x(\langle 0, 1)) \subset X \setminus M$ and $\varphi_x(1) = x$. Then there exists a demicontinuous mapping $f : X \to Y$ with M as the set of points of non-continuity.

Proof. Let d be an invariant metric for X and define

$$g(x) = \begin{cases} \dfrac{1}{d(x, M)} & \text{for } x \in X \setminus M, \\ \infty & \text{for } x \in M. \end{cases}$$

Let $f = h \circ g$ (h is as in the proof of Lemma 1). Then f is demicontinuous on X and continuous on $X \setminus M$.

Let $x \in M$ and let φ_x be a continuous injective mapping from $\langle 0, 1 \rangle$ to X with $\varphi_x(\langle 0, 1)) \subset X \setminus M$ and $\varphi_x(1) = x$. Since φ_x is a homeomorphism from $\langle 0, 1 \rangle$ onto $\varphi_x(\langle 0, 1 \rangle)$, there are an integer n_0

and a sequence $\{x_n\}_{n=n_0}^{\infty}$ in $\varphi_x(\langle 0,1))$ such that $d(x_n, M) = \kappa_n^{-1}$, and $\kappa_n^{-1} \leq d(\varphi_x(0), M)$ for all $n \geq n_0$. Then $x_n \to x$ and $f(x_n) = h(g(x_n)) = h(\kappa_n) \not\to h(\infty) = h(g(x)) = f(x)$. Hence the mapping f is non-continuous at each point x in M. Q.E.D.

Remarks. The hypothesis on the locally convex space Y in Lemma 1 and Theorems 1,2,3 and 5 satisfies, for example, each normed linear space Y in which there exists (strongly) non-convergent, weakly convergent sequence. In fact, let $\{y_n\}_{n=1}^{\infty}$ be a sequence in Y with $y_n \rightharpoonup 0$ and $y_n \not\to 0$. Clearly, we can assume that $y_n \neq 0$ for each n. Since $\{y_n\}_{n=1}^{\infty}$ is bounded, it suffices to take $e_n = \| y_n \|^{-1} y_n$.

But there are normed linear spaces in which the convergence of sequences coincide with the weak convergence of sequences. The case of finite dimensional normed linear spaces is not interesting since in these spaces the strong and weak topologies are the same. The simplest nontrivial example of a space of this type is the space ℓ_1 of sequences $x = (x_1, x_2, \dots)$ with $\| x \| = \sum_{i=1}^{\infty} |x_i|$ finite; more generally, to this class of spaces belong the spaces $L_1(S, \Sigma, \mu)$, where (S, Σ, μ) is a measure space with positive measure μ such that every point of S has a positive measure. (See Banach [1] and Dunford-Schwartz [2].)

The hypotheses on M in Theorem 5 satisfies, for example, each non-empty closed subset of X with $M^{(\alpha)}$ discrete for some denumerable ordinal α where $M^{(\alpha)}$ and $M^{(1)}$ is the derivation of $M^{(\alpha-1)}$ and M, respectively, if $\alpha - 1$ exists, and $M^{(\alpha)} = \bigcap_{\beta<\alpha} M^{(\beta)}$ otherwise, or each closed subset of the boundary of a convex subset of X. Specially, each convex compact subset of a metrizable linear topological space of infinite dimension is the set of points of non-continuity of some demicontinuous mapping (for example, with $Y = \ell_2$) since the closed linear hull of this set is not all of the space.

During the writing of the present paper the following two theorems have been obtained. They can be used to the study of continuity properties of mappings.

Theorem 6. Let $X = (X, \|\cdot\|)$ be a Banach space such that the dual space X^* is separable. Then there exists a norm $|||\cdot|||$ on X such that for each bounded subset M of X we have:

$$(M, w) = (M, |||\cdot|||) ,$$

i.e. the weak topology of X and the $|||\cdot|||$ -topology of X coincide on bounded subsets of X.

Theorem 7. Let X be a separable Banach space. Then there exists a norm $|||\cdot|||$ on X such that for each bounded subset M of X the identity mapping $id : (M, w) \to (M, |||\cdot|||)$ is continuous, i.e. the $|||\cdot|||$ -topology of X is on bounded subsets of X coarser than the weak topology of X.

The (simple) proofs are omitted. (Hint: It suffices to take a dense, respectively weakly dense, sequence $\{u_n\}_{n=1}^{\infty}$ of the unit ball of the dual space X^* and set $|||x||| = \sum_{n=1}^{\infty} 2^{-n} \cdot |u_n(x)| \cdot$)

<u>Example</u>. Let X and Y be normed linear spaces such that there are sequences $\{c_n\}_{n=1}^{\infty}$ and $\{e_n\}_{n=1}^{\infty}$ in X and Y, respectively, such that $c_n \rightharpoonup o$, $c_n \not\to o$ and $e_n \rightharpoonup o$, $e_n \not\to o$. We can assume that $\|c_n\| = 1 = \|e_n\|$ for each integer n and that e_1 and e_2 are linearly independent. Let h be the mapping from the proof of Lemma 1 constructed for our sequence $\{e_n\}_{n=1}^{\infty}$. Define

$$g(x) = \begin{cases} \frac{1}{\|x\|} & \text{if } x \in X,\ x \neq o, \\ \infty & \text{if } x = o. \end{cases}$$

Let $f = h \circ g$. Then $f: X \to Y$ is <u>demicontinuous</u> on X, continuous on $X \setminus \{0\}$, <u>non-continuous</u> and <u>weakly non-continuous</u> at the point $x = 0$. The proof of the first three assertions is similar to the proofs of preceding Lemma and Theorems. It remains to prove the weak non-continuity of f at the point $x = 0$. We have $f(c_n) = h(g(c_n)) = h(1) = \frac{(k_2 - 1)e_1 + e_2}{k_2} = e \neq o$

(since the elements e_1 and e_2 are linearly independent), and hence $f(c_n) = e \not\rightharpoonup o = f(o)$, but $c_n \rightharpoonup o$, which proves the weak non-continuity of f at $x = 0$.

<u>Remark</u>. For linear mappings between normed linear spaces the continuity, demicontinuity and weak continuity coincide (the proof is easy). (For more general re-

sults see Bourbaki, Espaces vectoriels, Actualités Sci. Ind.,Hermann,Paris,No.1229,1955.)

Now, we give some "positive" results for continuity properties of mappings which are trivial consequences of Namioka's result ([3],Cor.1.3).

<u>Theorem 8</u>. Let X be a pseudometrizable separable locally convex linear topological space, K a weakly compact subset X and Y a locally convex space. Then there exists a weak dense and weak G_δ subset K_0 of K such that for each continuous mapping $f : K \rightarrow Y$ and for each demicontinuous mapping $g : K \rightarrow Y$ the set of points of the strong continuity of the mapping f and the set of points of the weak continuity of the mapping g contain the set K_0 .

<u>Corollary</u>. The same conclusion as in Theorem 8 but with X a separable reflexive Banach space and K a bounded weakly closed subset of X is valid.

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