Commentationes Mathematicae Universitatis Carolinae

Jaroslav Ježek

Principal dual ideals in lattices of primitive classes

Commentationes Mathematicae Universitatis Carolinae, Vol. 9 (1968), No. 4, 533--545

Persistent URL: http://dml.cz/dmlcz/105197

Commentationes Mathematicae Universitatis Carolinae
9, 4 (1968)

PRINCIPAL DUAL IDEALS IN LATTICES OF PRIMITIVE CLASSES

Jaroslav JEŽEK, Praha

Consider a type Δ of universal algebras and the lattice $\mathcal{L}_\Delta$ of all primitive classes of algebras of type Δ. J. Rebane [2] has shown that if Δ contains at least one at least unary operation, then each proper principal dual ideal J of $\mathcal{L}_\Delta$ is infinite. It will be shown in the present paper that if Δ contains either at least two unary operations or at least one at least binary operation, then each J is uncountable. (It will follow that if, in addition, Δ is finite, then each J has exactly $2^{\aleph_0}$ elements; the continuum hypothesis is not used here.) Let us remark that if Δ consists of one unary and a finite number of nullary operations, then (as it is shown in [1]) $\mathcal{L}_\Delta$ and hence each J is countable; if Δ consists of one unary and an infinite number of nullary operations, then it is easy to prove that $\mathcal{L}_\Delta$ contains both countable and uncountable proper principal dual ideals.

Some terminology will be given in § 1. However, the reader is supposed to know the definitions and fundamental properties of absolutely free algebras and primitive classes. See Słomiński [3].

§ 1. Lattices of primitive classes

By a type we mean an arbitrary family $\Delta = (n_i)_{i \in I}$ of non-negative integers. Let us make a convention: if a type is denoted by Δ, then its definition set is denoted by I and the integer corresponding to $i \in I$ by n_i.

Algebra of type Δ is a set A together with a family $(f_i)_{i \in I}$ where f_i is an n_i-ary operation in A. We call f_i the i-th fundamental operation of this algebra. If $n_i = 0$, then f_i is simply an element of A.

Let us fix an infinitely countable set X; its elements are called variables. For each type Δ let us fix an absolutely free algebra W_Δ of type Δ freely generated by X. If $i \in I$, then the i-th fundamental operation of W_Δ is denoted by f_i.

Let us define a set $S(w)$ for each $w \in W_\Delta$: if $w \in X$, then $S(w) = \{w\}$; if $i \in I$, $w = f_i(w_1, \ldots, w_{n_i})$, then $S(w) = \{w\} \cup S(w_1) \cup \ldots \cup S(w_{n_i})$. The elements of $S(w)$ are called subwords of w. It is easy to prove that if w_1 is a subword of w_2, then $\varphi(w_1)$ is a subword of

$\varphi(w_2)$ for any endomorphism φ of W_Δ.

Let us define a non-negative integer $\kappa(w)$ for each $w \in W_\Delta$: if $w \in X$ or $w = f_i$ for some $i \in I$, $n_i = 0$, then $\kappa(w) = 0$; if $i \in$ $\in I$, $n_i \neq 0$, $w = f_i(w_1, \dots, w_{n_i})$, then $\kappa(w) = 1 + \kappa(w_1) + \dots$ $\dots + \kappa(w_{n_i})$. It is easy to prove $\kappa(w) \leq \kappa(\varphi(w))$ for any endomorphism φ of W_Δ.

By a Δ-equation we mean an ordered pair $\langle w_1, w_2 \rangle$ of elements of W_Δ. By a Δ-theory we mean any set of Δ-equations, i.e. any binary relation in W_Δ. A Δ-equation e is identified with the Δ-theory $\{e\}$. A Δ-equation $\langle w_1, w_2 \rangle$ is called trivial if $w_1 = w_2$.

By a fully invariant congruence relation (shortly: FI-congruence relation) of W_Δ we mean a congruence relation E such that $\langle w_1, w_2 \rangle \in E$ implies $\langle \varphi(w_1), \varphi(w_2) \rangle \in E$ for any endomorphism φ of W_Δ.

<u>Lemma 1</u>. Let m be a non-negative integer. The set of all Δ-equations $\langle w_1, w_2 \rangle$ such that either $w_1 = w_2$ or $\kappa(w_1) \geq m$ & $\kappa(w_2) \geq m$ is a FI-congruence relation of W_Δ.

The proof is evident.

For any Δ-theory E, the least FI-congruence relation of W_Δ containing E is denoted by $Cn(E)$.

We shall write $E_1 \vdash E_2$ instead of $E_2 \subseteq$ $\subseteq Cn(E_1)$.

The set of all FI-congruence relations of W_Δ is a complete lattice with respect to the set-theoretic inclusion. The dual of this lattice is denoted by $\mathcal{L}_\Delta$. This is the set of all FI-congruence relations of W_Δ with the relation $\leq_\Delta$ defined by $E_1 \leq_\Delta E_2$ if and only if $E_2 \subseteq E_1$.

A Δ-equation $\langle w_1, w_2 \rangle$ is called valid in an algebra A of type Δ if $\varphi(w_1) = \varphi(w_2)$ for all homomorphisms φ of W_Δ into A. If E is a Δ-theory, then $\mathcal{M}od(E)$ denotes the primitive class of all algebras of type Δ in which all equations from E are valid. If $\mathcal{O}\!\mathcal{L}$ is a class of algebras of type Δ, then $\mathcal{E}q(\mathcal{O}\!\mathcal{L})$ denotes the set of all Δ-equations that are valid in each $A \in \mathcal{O}\!\mathcal{L}$.

The following three properties are well-known:

i) If E_1 and E_2 are two different elements of $\mathcal{L}_\Delta$, then the primitive classes $\mathcal{M}od(E_1)$ and $\mathcal{M}od(E_2)$ are different, too.

ii) Any primitive class of algebras of type Δ can be expressed as $\mathcal{M}od(E)$ for some $E \in \mathcal{L}_\Delta$.

iii) If $E_1, E_2 \in \mathcal{L}_\Delta$, then $E_1 \leq_\Delta E_2$ if and only if $\mathcal{M}od(E_1) \subseteq \mathcal{M}od(E_2)$.

This shows that the name "lattice of primitive classes" for $\mathcal{L}_\Delta$ is available.

Let us denote by ι_Δ the greatest element of $\mathcal{L}_\Delta$.

If $E \in \mathcal{L}_\Delta$, then the set of all $H \in \mathcal{L}_\Delta$ such that $E \leq_\Delta H$ is called the principal dual ideal (of $\mathcal{L}_\Delta$) generated by E. It is called proper if $E \neq \iota_\Delta$.

§ 2. The uncountability of proper principal dual ideals of $\mathcal{L}_\Delta$ for large types Δ

Let us call a type Δ large if either

(1) $n_i \leq 1$ for all $i \in I$; there exist two different elements i_1, i_2 of I such that $n_{i_1} = n_{i_2} = 1$ or

(2) there exists an $i_1 \in I$ such that $n_{i_1} \geq 2$.

In my paper [1] it is shown that for each finite type Δ, the lattice $\mathcal{L}_\Delta$ is uncountable if and only if Δ is large. Here we shall prove this

<u>Theorem</u>. Let Δ be a large type. Then each proper principal dual ideal of $\mathcal{L}_\Delta$ is uncountable. Moreover, it contains a subset which (considered as partially ordered by $\leq_\Delta$) is isomorphic to the lattice of all subsets of an infinite set.

First a definition. A Δ-theory E is called separated if it is infinite and $Cn(E_1) = Cn(E_2)$ implies $E_1 = E_2$ for all $E_1, E_2 \subseteq E$.

It is easy to prove that if E is separated, then the mapping φ defined by $\varphi(E_1) = Cn(E - E_1)$ is an order-isomorphism of the lattice of all subsets of E onto a subset of the principal dual ideal of $\mathcal{L}_\Delta$ generated by $Cn(E)$.

We have further evidently: if $H \in \mathcal{L}_\Delta$, $H \neq \iota_\Delta$, then there exists in H at least one non-trivial equation e, and it is $H \vdash e$.

Hence, to prove the Theorem, it is enough to prove that for each non-trivial Δ-equation e there exists a separated Δ-theory E such that $e \vdash E$. This will be proved in the following two lemmas.

Lemma 2. If a type Δ satisfies (1), then for each non-trivial Δ-equation e there exists a separated Δ-theory E such that $e \vdash E$.

Proof. The elements $i \in I$ such that $n_i = 1$ are called unary symbols. If $s = s_1 s_2 \dots s_n$ is a finite (not necessarily non-empty) sequence of unary symbols and $w \in W_\Delta$, then w^s is defined in this way: if s is empty, then $w^s = w$; further, $w^{s_1 \dots s_n s_{n+1}} = f_{s_{n+1}}(w^{s_1 \dots s_n})$. The special unary symbols i_1 and i_2 (see (1)) are denoted by |

and $+$, respectively. We shall denote by $\overset{n}{+}$ the sequence consisting of n symbols $+$.

Let N be the set of all positive integers. Put $e = \langle u, v \rangle$, so that $u \neq v$. For each $n \in N$ put $e_n = \langle u^{1\overset{n}{+}1}, v^{1\overset{n}{+}1} \rangle$. For each $M \subseteq N$ let E_M be the set of all e_m with $m \in M$. Put $E = E_N$. We have evidently $e \vdash E$.

Put $H = Cn(e)$. Let us define a relation R_M in W_Δ for each $M \subseteq N$ in this way: $\langle w_1, w_2 \rangle \in R_M$ if and only if either $w_1 = w_2$ or there exists an equation $\langle u_1, u_2 \rangle \in H$, a number $m \in M$ and a finite (not necessarily non-empty) sequence s of unary symbols such that $w_1 = u_1^{1\overset{m}{+}1s}$ and $w_2 = u_2^{1\overset{m}{+}1s}$. Let us prove that R_M is a FI-congruence relation of W_Δ. It is evidently enough to prove transitivity. Let $\langle w_1, w_2 \rangle \in R_M$ and $\langle w_2, w_3 \rangle \in R_M$. If $w_1 = w_2$ or $w_2 = w_3$, then $\langle w_1, w_3 \rangle \in R_M$ evidently. In the opposite case there exist equations $\langle u_1, u_2 \rangle \in H$, $\langle v_2, v_3 \rangle \in H$, numbers $m, n \in M$ and sequences $s, \bar{s}$ such that $w_1 = u_1^{1\overset{m}{+}1s}$, $w_2 = u_2^{1\overset{m}{+}1s} = v_2^{1\overset{n}{+}1\bar{s}}$, $w_3 = v_3^{1\overset{n}{+}1\bar{s}}$.

It follows from the expression of w_2 that either $|\overset{m}{+}|s$ is an end of $|\overset{n}{+}|\bar{s}$ or $|\overset{n}{+}|\bar{s}$ is an end of $|\overset{m}{+}|s$. We shall consider the first case; the second could be handled similarly. There exists a sequence t such that $|\overset{n}{+}|\bar{s}$ is equal to $t\,|\overset{m}{+}|s$. Clearly $\langle v_2^t , v_3^t \rangle \in H$ and $v_2^t = u_2$; we get $\langle u_1, v_3^t \rangle \in H$. As $w_1 = u_1^{|\overset{m}{+}|s}$ and $w_3 = v_3^{t|\overset{m}{+}|s}$, we get $\langle w_1 , w_3 \rangle \in R_M$. The assertion on R_M is thus proved. We have evidently $R_M \supseteq E_M$ and hence $R_M \supseteq Cn\,(E_M)$.

To prove the Lemma, it is evidently enough to prove that if $n \in N - M$, then $e_n \notin Cn\,(E_M)$. Suppose on the contrary that $e_n \in C_n\,(E_M)$; we get $e_m \in R_M$. There exists an equation $\langle u_1 , v_1 \rangle \in$ $\in H$, a number $m \in M$ and a sequence s such that $u^{|\overset{n}{+}|} = u_1^{|\overset{m}{+}|s}$ and $v^{|\overset{n}{+}|} = v_1^{|\overset{m}{+}|s}$. We shall go on under the assumption $\kappa(u_1) \leq \kappa(v_1)$; in the contrary case the proof would be analogous. Evidently $\kappa(u) \leq \kappa(v)$, too. As $\langle u_1 , v_1 \rangle \in$ $\in Cn \langle u, v \rangle$ and $u_1 \neq v_1$, we get $\kappa(u_1) \geq$

$\geq r(u)$ evidently applying Lemma 1. From this and from $u^{|\overset{n}{+}|} = u_1^{|\overset{m}{+}|s}$ it follows easily that s is empty, $n = m$ and $u = u_1$.
But $n = m$ is in a contradiction with the assumption $n \notin M$.

Lemma 3. If a type Δ satisfies (2), then for each non-trivial Δ-equation e there exists a separated Δ-theory E such that $e \vdash E$.

Proof. Let us fix an $i_1 \in I$ with $n_{i_1} \geq 2$ and put $n_{i_1} = k$. If $w_1, w_2 \in W_\Delta$, then put $w_1 \cdot w_2 = f_{i_1}(w_1, w_2, \dots, w_2)$. If $w_1, \dots, w_n \in W_\Delta$, then the product $w_1 \dots w_n$ is defined in this way: if $n = 1$, it is equal to w_1; if $n > 1$, then $w_1 \dots w_n = (w_1 \dots w_{n-1}) \cdot w_n$. If $w_1 = \dots = w_n = w$, we write w^n instead of $w_1 \dots w_n$.

Put $e = \langle u, v \rangle$, so that $u \neq v$. Let N be the set of all integers $n \geq 2$. Let us fix a variable x. For each $n \in N$ put $e_n = \langle u \cdot x^n, v \cdot x^n \rangle$. For each $M \subseteq N$ let E_M be the set of all e_m with $m \in M$. Put $E = E_N$.

We have evidently $e \vdash E$.

Let a set $M \subseteq N$ be given. A finite sequence $e^{(1)}, \dots, e^{(h)}$ of Δ -equations is called proof (with respect to M) if for each $j = 1, \dots, h$ one of the following cases takes place:

i) $e^{(j)}$ is trivial;

ii) there exists an $m \in M$ and an endomorphism φ of W_Δ such that either $e^{(j)} = \langle \varphi(u \cdot x^m), \varphi(v \cdot x^m) \rangle$ or $e^{(j)} = \langle \varphi(v \cdot x^m), \varphi(u \cdot x^m) \rangle$;

iii) there exists an $i \in I$ and a sequence $\langle w_1, \overline{w_1} \rangle, \dots \langle w_{n_i}, \overline{w_{n_i}} \rangle$ of Δ -equations such that all these equations occur among $e^{(1)}, \dots, e^{(j-1)}$ and $e^{(j)} = \langle f_i(w_1, \dots, w_{n_i}), f_i(\overline{w_1}, \dots, \overline{w_{n_i}}) \rangle$;

iv) there exist two equations $\langle w_1, w_2 \rangle$ and $\langle w_2, w_3 \rangle$ among $e^{(1)}, \dots, e^{(j-1)}$ such that $e^{(j)} = \langle w_1, w_3 \rangle$.

Let R_M be the set of all those Δ -equations that occur as the last member of a proof (with respect to M). It is easy to see that R_M is a FI-congruence relation of W_Δ , so that evidently

$R_M = Cn(E_M)$.

To prove the Lemma, it is evidently enough to prove that if $n \in N - M$, then $e_n \notin Cn(E_M)$. Suppose on the contrary that $e_n \in R_M$, so that e_n is the last member of a proof (with respect to M) $e^{(1)}, \ldots, e^{(h)}$. We may suppose $\kappa(u) \leq \kappa(v)$; in the contrary case the proof would be analogous. We can not receive $e^{(h)}$ applying only the rules i) and iv); hence, there exists a $j \leq h$ and a $w \in W_\Delta$ such that $w \neq u \cdot x^n$ and $e^{(j)} = \langle u \cdot x^n, w \rangle$ and such that $e^{(j)}$ can be got applying ii) or iii). (In the case $\kappa(u) \geq \kappa(v)$ we would seek $e^{(j)}$ in the form $\langle w, v \cdot x^n \rangle$.) Suppose that $e^{(j)}$ can be got by iii). There exist elements $w_1, w_2, \ldots, w_k \in W_\Delta$ such that $w = f_{i_1}(w_1, w_2, \ldots, w_k)$ and
$\langle u, w_1 \rangle \in R_M$, $\langle x^n, w_2 \rangle \in R_M$, ...
..., $\langle x^n, w_k \rangle \in R_M$.

Let us call an element $t \in W_\Delta$ special if it has a subword $f_{i_1}(t_1, \ldots, t_k)$ where t_k is not a variable. Evidently, we get a FI-congruence relation if we take the set of all those Δ-equations $\langle t, \bar{t} \rangle$ such that either $t = \bar{t}$ or t and $\bar{t}$ are both special. As E_M and hence $R_M = Cn(E_M)$ is contained in this FI-congruence relation and as $x^n = f_{i_1}(x^{n-1}, x, \dots, x)$, $x^{n-1} =$

$= f_{i_1}(x^{n-2}, x, \dots, x), \dots,$

$x^2 = f_{i_1}(x, x, \dots, x)$,

we get $x^n = w_2 = \dots = w_k$. As $\kappa(u) \leq \kappa(v)$,

$\kappa(u \cdot x^m) > \kappa(u)$ for all $m \in M$ and $\langle u, w_1 \rangle \in R_M$, we get $u = w_1$ easily by Lemma 1. We get $w = u \cdot x^n$, a contradiction. Hence, $e^{(j)}$ is as in ii). We have either $u \cdot x^n = \varphi(u \cdot x^m) =$

$= \varphi(u) \cdot (\varphi(x))^m$ or

$u \cdot x^n = \varphi(v \cdot x^m) = \varphi(v) \cdot (\varphi(x))^m$;

in both these cases $x^n = (\varphi(x))^m$. As n, $m \geq 2$, we have $f_{i_1}(x^{n-1}, x, \dots, x) =$

$= f_{i_1}((\varphi(x))^{m-1}, \varphi(x), \dots, \varphi(x))$

and hence $x = \varphi(x)$. From $x^n = x^m$ we get evidently $n = m \in M$, a contradiction.

The Theorem is thus proved.

References

[1] J. JEŽEK: Primitive classes of algebras with unary and nullary operations.(To appear in Colloquium Math.)

[2] J. REBANE: On primitive classes of algebras of one type.(Russian.)Eesti NSV Teaduste Akademia Toimetised.Füüs.-Mat.XVI(1967),Nr.2,143-145.

[3] J. SŁOMIŃSKI: The theory of abstract algebras with infinitary operations.Rozprawy Matematyczne 18(1959).

(Received March 12,1969)