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CERTAIN GENERALIZATIONS OF THE KATĚTOV THEOREM

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If X is any set, the vector space of all formal finite linear combinations $\sum \lambda_i x_i$, λ_i scalars, $x_i \in X$, will be denoted by $E(X)$. For any function f on X is defined in a unique manner the linear extension $\tilde{f}$ of f to $E(X)$. As in [9] we identify the function f with its linear extension $\tilde{f}$. By a Λ-structure on X we mean (cf.[5]) the space $E(X)$ endowed with a locally convex topology. This may be done by a vector space $\mathcal{F}(X)$ of functions on X and by a suitable collection $\mathcal{C}$ of subsets in $\mathcal{F}(X)$. The topology in $E(X)$ is defined as a locally convex topology of uniform convergence on the family $\{C, C \in \mathcal{C}\}$.

In some earlier papers (cf.[9],[10],[11]) we have developed a theory of Λ-structures corresponding to spaces of all uniformly continuous (continuous) functions on a uniform (completely regular) space X. Following a general idea of M. Katětov (cf.[6]), we are now concerned with the spaces of functions on X which arise in the theory of distributions.

In that what follows we mean by X a completely regular space, $\mathcal{F}(X)$ is a vector space of continuous functions on X separating points of X in the strong sense (i.e. for any finite family of points $\{x_i, 1 \leq i \leq n\}$ in X there exists f in $\mathcal{F}(X)$ such that $f(x_1) = 1$, $f(x_i) = 0$ for $2 \leq i \leq n$). In this case the system $\langle \mathcal{F}(X), E(X) \rangle$ is, of course, a dual pair. It will be assumed that $\mathcal{F}(X)$ is a topological (pseudotopological) space with a topology (pseudotopology) for which any $x \in X$ defines a continuous function $\hat{x} : f \rightarrow \langle f, x \rangle$ on $\mathcal{F}(X)$. In this case the space E(X) may be imbedded in the topological dual space $\mathcal{F}^*(X)$ of $\mathcal{F}(X)$.

A family $\mathcal{A}$ of continuous functions on X is said to be regular if for any $x \in X$ and for each neighborhood U(x) of x in X there exists a function $f \in \mathcal{A}$ with $f(x) = 1$ and $f(y) = 0$ for all y in $X \setminus U(x)$.

If $\mathcal{C}$ is a covering of $\mathcal{F}(X)$ with subsets bounded in the topology of pointwise convergence on X, then the topology in E(X) of uniform convergence on the system $\{C, C \in \mathcal{C}\}$ will be denoted by $t(\mathcal{C})$.

<u>Theorem 1</u>. Let $\mathcal{F}(X)$ be a locally convex $(\mathcal{L}\mathcal{F})$-space (cf.[2]), $\mathcal{C}$ a collection of subsets in $\mathcal{F}(X)$ satisfying the above mentioned conditions.

(a) If any subset $C \in \mathcal{C}$ is equicontinuous on X, then the canonical imbedding $\mathcal{W} : X \rightarrow (E(X), t(\mathcal{C}))$ is a continuous mapping. If $\mathcal{F}(X)$ is a regular system, then $\mathcal{W}$ is a homomorphic imbedding of X into $(E(X), t(\mathcal{C}))$.

(b) Let the closed and absolutely convex envelope in the topology of pointwise convergence on X be a compact subset in $\mathcal{F}(X)$ in the same topology. Then the topological dual space of $(E(X), t(\mathcal{C}))$ may be identified with $\mathcal{F}(X)$.

(c) If any sequence $\{f_n\}$ convergent to the origin in $\mathcal{F}(X)$ is a part of some $C \in \mathcal{C}$, then the completion $(\hat{E}(X), t(\mathcal{C}))$ is canonical isomorphic (in the algebraic sense) to a subspace of $\mathcal{F}^*(X)$.

(d) If the collection $\mathcal{C}$ satisfies the condition of (c) and any $C \in \mathcal{C}$ is weakly relatively compact in $\mathcal{F}(X)$, then the completion $(\hat{E}(X), t(\mathcal{C}))$ is canonical isomorphic (in the algebraic sense) to the dual space $\mathcal{F}^*(X)$.

<u>Proof</u>. The statement (a) is trivial. Any function $f \in \mathcal{F}(X)$ is obviously continuous on $E(X)$ in the topology $t(\mathcal{C})$. From the assumption of the statement (b) it follows that $t(\mathcal{C})$ is compatible with the duality of the pair $\langle \mathcal{F}(X), E(X) \rangle$. This implies (b). To prove (c) it suffices to note that $\mathcal{F}^*(X)$ is a complete uniform space in the extended topology $t(\mathcal{C})$. Without going into details (cf.[10]) we recall that a linear function on $\mathcal{F}(X)$ is continuous if and only if it is continuous on each subspace defining the inductive limit topology of $\mathcal{F}(X)$. If any subset $C \in \mathcal{C}$ is relatively weakly compact, then $t(\mathcal{C})$ is compatible with the duality of the pair $\langle \mathcal{F}(X), \mathcal{F}^*(X) \rangle$. Hence, $E(X)$ is a dense subset in the topology $t(\mathcal{C})$ in $\mathcal{F}^*(X)$.

<u>Remark 1</u>. If the condition of the statement (d) in theorem 1 is not satisfied, then, of course, the equality in (d) need not be true. An example of this sort may be found in [10].

<u>Remark 2</u>. Especially, if X is a compact subset of

the Euclidean finite dimensional space, $\mathcal{F}(X)$ the vector space of all indefinitely differentiable functions on X, then the Mackey topology $\tau = \langle E(X), \mathcal{F}(X) \rangle$ is identical (cf.[6]) on E(X) with that one of the pair $\langle \mathcal{F}^*(X), \mathcal{F}(X) \rangle$. Hence, the theorem of M. Katětov (cf.[6]) is a special case of theorem 1. Some corresponding results of [10] may be also considered as a special case of theorem 1.

As an illustration of theorem 1 we state explicitely some elementary examples. The theorems 2 - 4 follow by specialization of what has just been proved.

I. Let R^n be the Euclidean n-dimensional space, $\mathcal{D}$ the vector space of all indefinitely differentiable functions of compact support on R^n with the usual topology (cf.[8]). It is well known that $\mathcal{D}$ is a regular system (cf.[7],[8]). Let $\mathcal{C}_1$ denote the collection of all sequences convergent to the origin in $\mathcal{D}$. It holds

<u>Theorem 2</u>. (a) The canonical mapping w is a homomorphic imbedding of X into $(E(X), t(\mathcal{C}_1))$.
(b) The topological dual space $(E(X), t(\mathcal{C}_1))^*$ is (algebraically) identical with $\mathcal{D}$.
(c) The completion $(\hat{E}(X), t(\mathcal{C}_1))$ is canonical isomorphic (in the algebraic sense) with the space $\mathcal{D}^*$ of all distributions.

It should be noticed that a subset A of E(X) is bounded in $(E(X), t(\mathcal{C}_1))$ if and only if there exists an integer n such that $A \subseteq n \Gamma X$. The strong dual space of $(E(X), t(\mathcal{C}_1))$ is in such a way isomorphic to $\mathcal{D}$ with the uniform topology. For the proof of these statements we refer to [9].

II. Let $\mathcal{E}$ be the vector space of all indefinitely differentiable functions on R^n with the usual topology (cf.[8]). We denote by $\mathcal{C}_2$ the family of all sequences convergent to the origin in $\mathcal{E}$. Similarly as in the case I we have

Theorem 3. (a) The canonical mapping w is a homomorphism of X into $(E(X, t(\mathcal{C}_2))$.
(b) It holds $\mathcal{E} = (E(X), t(\mathcal{C}_2))$.
(c) The completion $(\hat{E}(X), t(\mathcal{C}_2))$ is identical with the space of all distributions of compact support on R^n.

Proof. The statement (c) follows from the fact that $\mathcal{E}^*$ may be identified with the space of all distributions of compact support (cf.[8]).

Remark 3. The topology $t(\mathcal{C}_2)$ may be defined as the topology of uniform convergence on the family of all precompact subsets in $\mathcal{E}$. This follows from the fact that in a metrizable locally convex space E any precompact subset is contained in the closed absolutely convex envelope of a sequence convergent to the origin and, conversely, any such sequence form a precompact subset in E.

III. Let Ω be an open region in the open complex plane, $\mathcal{A}(\Omega)$ the space of all holomorphic functions on Ω. With the topology of compact convergence on Ω the space $\mathcal{A}(\Omega)$ is $(\mathcal{F})$-space. The family $\mathcal{C}_3$ is defined similarly as in the case II.

Theorem 4. (a) The canonical imbedding $w : \Omega \rightarrow$ $\rightarrow (E(\Omega), t(\mathcal{C}_3))$ is continuous on Ω.

(b) It hold similar statements to (b) and tp (c) of the theorem 3.

Remark 4. The above stated procedure may be applied, of course, to the spaces $K(M_p)$ (cf.[3]) and to the corresponding spaces of functions on a σ-compact indefinitely differentiable variety.

Let X be a locally compact space, $\mathcal{K} = \mathcal{K}(X)$ the space of all continuous functions on X of compact support. For any compact subset $K \subseteq X$ we denote by $\mathcal{K}(K, X)$ the vector space of all continuous functions of the support contained in K. The norm topology in $\mathcal{K}$ induces on each $\mathcal{K}(K, X)$ a Banach topology τ_K. Let τ be the inductive limit topology in $\mathcal{K}$ defined by the family $\mathcal{K}(K, X)$, K compact in X. We recall that on each $\mathcal{K}(K, X)$ the topology τ induces the uniform topology τ_K. The dual space $\mathcal{K}^*$ to $(\mathcal{K}, \tau)$ is identical with the family of all Radon measures on X (cf.[1]). Although the space $\mathcal{K}$ need not be an $(\mathcal{L}\mathcal{F})$-space, we may apply the above stated procedure due to the pseudotopological structure of $\mathcal{K}$. Let $\mathcal{C}_4$ be the family of all sequences convergent to the origin in $\mathcal{K}$ (i.e. any such sequence is contained in a suitable $\mathcal{K}(K, X)$, K being compact subset of X).

Theorem 5. Let X, $\mathcal{K}$ and $\mathcal{C}_3$ have the same meaning as stated above. Then it holds:

(a) The canonical imbedding w of X into $(E(X), t(\mathcal{C}_4))$ is a homomorphism.

(b) The topological dual space $(E(X), t(\mathcal{C}_4))^*$ may be identified with $\mathcal{K}(X)$.

(c) The completion $(\hat{E}(X), t(\mathcal{C}_4))$ is identical with the family $\mathcal{M}(X) = \mathcal{K}^*(X)$ of all Radon measures on X.

<u>Proof</u>. The mapping w of X into $(E(X), t(\mathcal{C}_4))$ is, evidently, continuous. The continuity of w^{-1} follows from $\sigma(E(X), \mathcal{K}(X)) \leq t(\mathcal{C}_4)$ and from theorem 6, § 2, chap. II of [1]. This proves (a). Any closed and absolutely convex envelope of a subset in $\mathcal{C}_4$ is closed in the topology of pointwise convergence on X, hence, the topology $t(\mathcal{C}_4)$ is compatible with the duality of the pair $(\mathcal{K}(X), E(X))$. From the Mackey theorem it follows (b).

Let ξ be an element of $(\hat{E}(X), t(\mathcal{C}_4))$. From a theorem of A. Grothendieck (cf. [4]) it follows that ξ is a linear function on $\mathcal{K}(X)$ continuous in the topology of pointwise convergence on X on each closed and absolutely convex envelope of a subset of $\mathcal{C}_4$. Hence, for any sequence $\{f_n\}$ in $\mathcal{K}$, $f_n \to 0$, it holds $\xi(f_n) \to 0$. This implies $\xi \in \mathcal{M}(X)$. Now, let μ be a Radon measure on X. Let C be an arbitrary element of $\mathcal{C}_4$. There exists a compact $K \subseteq X$ such that $C \subseteq \mathcal{K}(K, X)$. Because of the equicontinuity of C it follows from the generalized theorem of Ascoli (cf. [9]) that the topology of pointwise convergence and the norm topology τ_K coincide on the closed absolute convex envelope of C . Thus, μ is by the above mentioned theorem of A. Grothendieck an element of $(\hat{E}(X), t(\mathcal{C}_4))$. This completes the proof.

We notify that the statement (c) may be directly proved as in theorem 1.

The space of all Radon measures of compact support was described in [10] as a completion of a certain Λ-structure $(\hat{E}(X), t_{oc})$ (for X locally and σ-compact). The main part of these results was communicated in [12]. We shall return to some questions of this paper in another communication, especially, in connection with adequate applications.

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