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RELATIONS BETWEEN THE $\mathcal{V}$-COMPLETENESS AND THE PARACOMPACTNESS OF CLOSURE SPACES

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The main result in this paper is Theorem 1 (known for sequences and normal S-spaces ([2],theorem 9)).
On the other hand, the assertion of this theorem (or $\mathcal{M}$-completeness of $\langle P, u \rangle$) is sufficient for the paracompactness of $\langle P, u \rangle$ whenever u is a generalized order closure (Theorem 2).

Some definitions from [3] used in this paper. Let $\mathcal{V}$ be a (cofinal-closed) class of directed sets. A $\mathcal{V}$-net is a net whose domain belongs to $\mathcal{V}$. A $\mathcal{V}$-space is a closure space whose closure is determined (as in [1], 35 A.5) by some convergence relation $\mathcal{C}$ such that $\mathbf{D}\mathcal{C}$ consists of $\mathcal{V}$-nets. $\mathcal{P}$ is $\mathcal{V}$-complete iff $\mathcal{P}$ is a $\mathcal{V}$-regular (i.e. any $\mathcal{V}$-net N converges to Nx whenever $f \circ N \rightarrow fx$ for each continuous function f) $\mathcal{V}$-space and every $\mathcal{V}$-net remarkable in $\mathcal{P}$ converges in $\mathcal{P}$. $\mathcal{M}$ denotes the class of all monotone ordered sets. The $\mathcal{V}$-modification of a closure u is the coarsest $\mathcal{V}$-closure finer than u.

<u>Theorem 1</u>. Let $\mathcal{P} = \langle P, u \rangle$ be a paracompact space. Then every monotone net remarkable in $\mathcal{P}$ is convergent in $\mathcal{P}$; equivalently, any monotone net ranging

in P does not converge in $\beta\mathcal{P}$ to any point of $|\beta\mathcal{P}| - P$.

Proof. Let $\langle N_o, \preceq \rangle$ be a monotone net remarkable in $\mathcal{P}$ which does not converge in $\mathcal{P}$. Then there exists a bijective $\mathcal{N}^{\kappa}$-net N (i.e., $\mathbf{D}$ N is regularly ordered) remarkable in $\mathcal{P}$ which does not converge in $\mathcal{P}$ (we can choose a regularly ordered cofinal subset E of $\mathbf{D}\, N_o$ and a mapping n of $\alpha = \operatorname{card} E$ into E so that $m \succeq n\xi \Rightarrow N_o m \in P - N_o n[\xi]$, because N_o is not frequently constant and hence $\operatorname{card}(E \cap N_o^{-1} N_o n[\xi] < \alpha$; we can denote $\mathbf{D} N = n[\alpha]$).

Let us demote for each $n \in \omega_o$ and for each $f \in \mathcal{F}\mathcal{P}$ $k_f = \lim f \circ N$ and $U_{f,n} = P - f^{-1}[\![k_f - \frac{1}{n}, k_f + \frac{1}{n}]\!]$. ($\mathcal{F}\mathcal{P}$ is the collection of all continuous functions of $\mathcal{P}$ into $\mathbf{I}$, $\mathbf{I}$ is the unit interval $[\![0,1]\!]$ with the usual topology). Then $\mathcal{U} = \{ U_{f,n} \mid f \in \mathcal{F}\mathcal{P},\ n \in \omega_o \}$ is an open cover of $\mathcal{P}$ (whenever $x \in P$ then N does not converge to x in $\mathcal{P}$ and hence $fx \neq k_f$ for some $f \in \mathcal{F}\mathcal{P}$, $x \in U_{f,n}$ for this f and for any $n > \frac{1}{|fx - k_f|}$). Thus there exists a locally finite partition of the unity subordinated to $\mathcal{U}$ ([1],30 C.4), i.e. there exists $F \subset \mathcal{F}\mathcal{P}$ such that $\Sigma\{ fx \mid f \in F \} = 1$ for each $x \in P$ and the locally finite cover $\{ L_f = P - f^{-1}(0) \mid f \in F \}$ refines $\mathcal{U}$. If $f \in F$ then there exists $g \in \mathcal{F}\mathcal{P}$ and $n \in \omega_o$ so that $L_f \subset U_{g,n}$, the net N is

eventually in $P - U_{g,n}$, hence also in $P - L_f =$
$= f^{-1}(0)$, therefore $k_f = 0$ and we can choose $c_f \in$
$\in \mathbf{D} N$ such that $m \geqq c_f \Rightarrow f N m = 0$.

Let us construct (by induction) a family $M =$
$= \{ M_\xi \mid \xi \in \alpha = \operatorname{card} \mathbf{D} N \}$ of points of the set $\mathbf{E} N \subset P$ and disjoint neighborhoods V_ξ of M_ξ by the following way.

Let $\eta \in \alpha$ and let the sets $V_\xi \subset P$ and
$F_\xi = \{ f \in F \mid V_\xi \cap L_f \neq \emptyset \}$ be chosen for all $\xi \in \eta$.
Because $\bigcup \{ F_\xi \mid \xi \in \eta \}$ is finite if $\alpha = \aleph_0$ and $\operatorname{card} \bigcup \{ F \mid \xi \in \eta \} \leqq$
$\leqq \eta \cdot \aleph_0 < \alpha$ if $\alpha > \aleph_0$ the set $\mathbf{E} \{ c_f \mid f \in \bigcup \{ F_\xi \mid \xi \in \eta \} \}$ is bounded in $\mathbf{D} N$; let us denote d_η some its upper bound and $M_\eta = N d_\eta$. Let us choose $g_\eta \in F$ so that $g_\eta M_\eta \neq 0$. Then $g_\eta \notin F_\xi$ and hence $L_{g_\eta} \cap V_\xi =$
$= \emptyset$ for each $\xi \in \eta$. L_{g_η} is a neighborhood of M_η and therefore we can choose a neighborhood $V_\eta \subset L_{g_\eta}$ of the point M_η so that the set $F_\eta = \{ f \in F \mid V_\eta \cap L_f \neq \emptyset \}$ is finite.

We shall prove that $\mathbf{E} M$ is discrete in $\mathcal{P}$.
Let $A \subset \mathbf{E} M$. If $y \in \mathbf{E} M$ then the point $y = M_\xi$ has the neighborhood V_ξ and $V_\xi \cap A \subset (M_\xi)$. Let us consider that $y \in u A - \mathbf{E} M$. Let us denote $\beta =$
$= \min \{ \gamma \in \alpha \cup (\alpha) \mid y \in u(A \cap M[\gamma]) \}$; obviously β is a limit ordinal number. Let us choose a net $\varkappa = \langle \{ \varkappa j \mid j \in$
$\in K \}, \leqq \rangle$ ranging in $B = \beta \cap M^{-1}[A]$ such that $M \circ \varkappa$ converges to y in $\mathcal{P}$. Let $f \in F$. If $\mathbf{E}(f \circ M \circ \varkappa) = (0)$ then $fy = 0$. Let $f M \varkappa j \neq 0$.
Then $f \in F_{\varkappa j}$. Because $\varkappa$ is not frequently in γ

whenever $\gamma \in \beta$ (by definition of β) there exists $\ell \in K$ such that $i \geqq \ell \Rightarrow \varkappa i \geqq \varkappa j + 1$. Therefore $i \geqq \ell \Rightarrow d_{\varkappa i} \geqq d_{\varkappa j+1} \geqq c_f \Rightarrow f M \varkappa i =$ $= f N d_{\varkappa i} = 0$ for each $i \in K$; thus $fy = 0$. But this is the contradiction with the assumption that F is a partition of the unity.

Because $\mathcal{P}$ is paracompact, there exists ([1], 30 C.10) a discrete family $\{ W_\xi \mid \xi \in \alpha \}$ of open sets so that $M_\xi \in W_\xi$ for each $\xi \in \alpha$.

Let us choose a set $S \subset \alpha$ so that S and $\alpha - S$ are $\leqq$-cofinal in α, let us choose a function $f_\xi \in \mathcal{F}\mathcal{P}$ for each $\xi \in S$ so that $f_\xi M_\xi =$ $= 1$ and $f_\xi [P - W_\xi] = (0)$, let us denote by f the sum of functions f_ξ over S. If $y \in P$ then there exists a neighborhood U of y such that $U \cap W_\eta \neq \emptyset$ for at most one $\eta \in S$ ([1],26 A.8). For this η $f =$ $= f_\eta$ on U and f is continuous in y. Thus f is continuous in $\mathcal{P}$.

The sets $d[S]$ and $d[\alpha - S]$ are cofinal in $\mathbf{D}N$, because card $d[S]$ = card $S = \alpha$ = card $d[S -$ $- \alpha]$; along with it $fNz = 0$ for each $z \in d[\alpha - S]$ and $fNz = 1$ for each $z \in d[S]$. Therefore $f \circ N$ does not converge, N is not remarkable and it is the contradiction.

The second assertion of Theorem 1 is equivalent to the first one, because the net is remarkable in $\mathcal{P}$ if and only if the one is convergent in $\beta\mathcal{P}$.

Corollary 1. Every metrizable space (and every pseudo-metrizable space) is $\mathcal{N}$-complete.

Corollary 2. The $\mathcal{N}$-modification of any paracompact space is $\mathcal{N}$-complete.

Proof. Let $\langle P, u \rangle$ be a paracompact space, let v be the $\mathcal{N}$-modification of u, let N be a $\mathcal{N}$-net remarkable in $\langle P, v \rangle$. Because $\mathcal{F}\langle P, u \rangle \subset \mathcal{F}\langle P, v \rangle$, N is remarkable in $\langle P, u \rangle$, converges in $\langle P, u \rangle$ and hence converges in $\langle P, v \rangle$.

Theorem 2. Let u be a generalized order closure. Then $\langle P, u \rangle$ is $\mathcal{N}$-complete if and only if $\langle P, u \rangle$ is paracompact.

Proof. "If" is an immediate corollary of Theorem 1, because every generalized order closure space is a $\mathcal{N}$-space and obviously $\mathcal{N}$-regular ([4],3.11).

Let $\mathcal{P} = \langle P, u \rangle$ be not paracompact. Then there exists a well-ordered cover which is not uniformizable by [3], hence there exists a regularly ordered cover $\mathcal{U} = \{ U_\xi \mid \xi \in \gamma \}$ which is not uniformizable (a cofinal subcover of the preceding cover).

For each $x \in P$ let us denote $Q_x = \mathbf{E}\{ y \in P \mid \llbracket x, y \rrbracket \cup \llbracket y, x \rrbracket \subset U_\xi$ for some $\xi \in \gamma \}$. For any $x \in P$, $y \in P$ either $Q_x \cap Q_y = \emptyset$ (iff $\llbracket x, y \rrbracket \cup \llbracket y, x \rrbracket$ is not contained in U_ξ for any $\xi \in \gamma$) or $Q_x = Q_y$. For any $x \in P$ Q_x is interval-like (obviously) and open-closed in $\mathcal{P}$. (Let $x \in P$. Then U_ξ is a neighborhood of x for some $\xi \in \gamma$ and thus there exists an interval-like neighborhood $W_x \subset U_\xi$

of the point x. If a point y belongs to W_x, then $[x, y]\cup[y, x]\subset W_x\subset U_\xi$ and therefore the point y belongs to Q_x. Consequently, W_x is contained in Q_x, which proves that the set Q_x is open. Further, the set Q_x is closed as the intersection of the collection $\{P - Q_z \mid z\in P - Q_x\}$ of the closed sets.

Therefore there exists $x\in P$ such that the open cover $\mathcal{U}_x=\{U_\xi\cap Q_x\mid \xi\in\gamma\}$ of the subspace $\mathcal{Q}_x$ of $\mathcal{P}$ is not uniformizable, let us choose such x (if G_{Q_x} belongs to a continuous uniformity $\mathcal{G}_{Q_x}$ for $\mathcal{Q}_x$ and $\{G_{Q_x}[(y)]\mid y\in Q_x\}$ refines U_x, then $G = \bigcup\{G_{Q_x}\mid x\in P\}$ belongs to a continuous uniformity $\{\bigcup\{K_{Q_x}\mid x\in P\}\mid K_{Q_x}\in\mathcal{G}_{Q_x}\}$ for $\mathcal{P}$ and $\{G[(y)]\mid y\in P\}$ refines $\bigcup\{\mathcal{U}_x\mid x\in P\}$ which refines $\mathcal{U}$).

Let $z\dashv z_0$ be two different points of Q_x. Let us consider that the cover $\mathcal{V}=\{V_\xi\cap Q\mid\xi\in\gamma\}$ of the subspace $\mathcal{Q}$ with $|\mathcal{Q}|=Q=Q_x\cap\,]z,\rightarrow[$ of $\mathcal{Q}_x$ is not uniformizable; otherwise the cover $\{U_\xi\cap R\mid\xi\in\gamma\}$ of $R=Q_x\cap[\leftarrow, z_0[$ is not uniformizable (easy) and the other proof is analogical.

Let us define $\nu y=\min\{\xi\in\gamma\mid y\in V_\xi\}$ for each $y\in Q$. The set $\nu[Q]$ is cofinal in γ, because $\mathcal{V}$ is not uniformizable; and $\nu[\,]z, y]\,]$ is not cofinal in γ for any $y\in Q$. Therefore we can construct (by induction) the family $N=\{N_\xi\mid\xi\in\gamma\}$ of elements of Q and the family $\varkappa = \{\varkappa\xi \mid \xi \in \gamma\}$ of elements of γ so that $N\xi \prec N\eta$ and $\nu t \leqq \varkappa\xi <$ $< \nu N\eta$ whenever $\eta \in \gamma$, $\xi < \eta$, $x \prec t \trianglelefteq N\xi$. Indeed, let $\eta \in \gamma$ and let $N\xi$ and $\varkappa\xi$ be chosen for each $\xi \in \eta$; then $\varkappa[\eta]$ is not cofinal in γ and $N\eta$ can be chosen so that $\xi < \eta \Longrightarrow \varkappa\xi <$ $< \nu N\eta$, thus $N\eta \notin \rrbracket x, N\xi \rrbracket$, hence $N\xi \prec N\eta$ for each $\xi \in \eta$; seeing that $\nu[\rrbracket x, N\eta \rrbracket]$ is not cofinal in γ, we can choose $\varkappa\eta$ so that $t \trianglelefteq N\eta \Longrightarrow \nu t \leqq \varkappa\eta$. For each $t \in Q$ $t \prec N\xi \Longrightarrow \nu t \leqq \varkappa\xi \Longrightarrow t \in U_{\varkappa\xi} \Longrightarrow t \in V_{\varkappa\xi}$, hence the open cover $\mathcal{W} = \{W_\xi = \rrbracket x, N\xi \llbracket \mid \xi \in \gamma\}$ of the space $\mathcal{Q}$ refines $\mathcal{V}$ and therefore $\mathcal{W}$ is not uniformizable.

Obviously, the net $\langle N, \leqq \rangle$ does not converge in $\mathcal{Q}$ and, consequently, in $\mathcal{P}$, we shall prove that $\langle N, \leqq \rangle$ is remarkable in $\mathcal{P}$. Let f be a function on P ranging in $\llbracket 0,1 \rrbracket$ such that the net $\langle f \circ N, \leqq \rangle$ does not converge in $\mathbf{I}$. Then there exist sets B_0 and C_0 separated in $\mathbf{I}$ so that $f \circ N$ is frequently in both B_0 and C_0. Let us denote $B = Q \cap f^{-1}[B_0]$, $C =$ $= Q \cap f^{-1}[C_0]$. We can choose an increasing mapping h on γ into γ (by induction) so that $Nh\eta \in B$ if $\eta = 0$ or η is a limit ordinal or $Nh\eta - 1 \in C$ and $Nh\eta \in C$ if $Nh\eta - 1 \in B$, because $h[\eta]$ is not cofinal in γ for any $\eta \in \gamma$ and N is frequently in both B and C.

Let us denote $m_t = \min\{\xi \in \gamma \mid t \trianglelefteq N h \xi\}$ and $\varphi t = h(m_t + 1)$ for each $t \in Q$. Then $t \in Q \Longrightarrow$ $\Rightarrow t \trianglelefteq N h m_t \prec N \varphi t \Rightarrow W_{\varphi t}$ is a neighborhood of t. There exists a set $R \subset Q$ and a point y so that $y \in u R - \bigcup\{W_{\varphi t} \mid t \in R\}$ ([1],24 E.4 & 24 E.2). Let us denote $S = \{N h \xi \mid \xi \leqq m_t$ for some $t \in R\}$. Seeing that for each $t \in R$ $t \trianglelefteq N h m_t \prec y$ and $N h m_t \in S$, y belongs to $u S$.

For each $r \in S$ there exists $t \in R$ so that $h^{-1} N^{-1} r \leqq m_t$, for this t $r \trianglelefteq N h m_t$, $\varphi r \leqq \varphi N h m_t = \varphi t$ and $r = N h m_r \prec N h(m_r + 1) \trianglelefteq N \varphi t \trianglelefteq y$; therefore $y \in$ $\in u B$ and $y \in u C$, the function f is not continuous and the net N is remarkable in $\mathcal{P}$.

Theorem 3. Let $\mathcal{Y}$ be a cofinal-closed class of directed sets. Let $\mathcal{P}$ be the cartesian product of a family $\{\mathcal{P}_a \mid a \in A\}$ of closure spaces. Every $\mathcal{Y}$-net remarkable in $\mathcal{P}$ converges in $\mathcal{P}$ if and only if every $\mathcal{Y}$-net remarkable in $\mathcal{P}_a$ converges in $\mathcal{P}_a$ for each $a \in A$. Consequently, $\mathcal{P}$ is $\mathcal{Y}$-complete if and only if $\mathcal{P}$ is a $\mathcal{Y}$-space and $\mathcal{P}_a$ is $\mathcal{Y}$-complete for each $a \in A$.

Proof. Let N be a $\mathcal{Y}$-net ranging in $|\mathcal{P}|$ which does not converge in $\mathcal{P}$. Then $\pi_a \circ N$ does not converge in $\mathcal{P}_a$ for some $a \in A$. For such a the $\mathcal{Y}$-net $\pi_a \circ N$ is not remarkable in $\mathcal{P}_a$ by assumption, $f \circ \pi_a \circ$ $\circ N$ does not converge in $\mathbf{I}$ for some $f \in \mathcal{F}\mathcal{P}_a$, hence N is not remarkable in $\mathcal{P}$.

On the other hand, let $a \in A$ and let N be remarkable in $\mathcal{P}_a$. Let $x \in |\mathcal{P}|$, let a mapping ψ on $\mathcal{P}_a$

into $\mathcal{P}$ be defined so that $\pi_a \psi y = y$ and $\pi_b \psi y = \pi_b \psi x$ for each $b \in A - (a)$. If f is a continuous function on $\mathcal{P}$, $f \psi$ is continuous on $\mathcal{P}_a$ and $f \psi N$ converges; hence ψN is remarkable in $\mathcal{P}$ and converges in $\mathcal{P}$ by assumption. Let z be its limit, then $N = \pi_a \psi N$ converges to $\pi_a z$.

Example. If $\mathcal{P}$ is the (naturally) ordered set of real numbers endowed with the closure of the right approximation, then the uniformizable space $\mathcal{P} \times \mathcal{P}$ is not normal ([1],30 C.14) and $\mathcal{P} \times \mathcal{P}$ is $\mathcal{N}$-complete. Indeed, $\mathcal{P}$ is $\mathcal{N}$-complete by Corollary 2 (or by an easy direct proof) and $\mathcal{P} \times \mathcal{P}$ is a S-space as the product of two S-spaces.

Theorem 4. Let $\mathcal{V}$ be a (cofinal-closed) class of directed sets, let α be a cardinal number. Then the following conditions are equivalent:

(a) The sum of any family $\{\mathcal{P}_a \mid a \in A\}$ of $\mathcal{V}$-complete spaces (resp. such that card $A < \alpha$) is $\mathcal{V}$-complete.

(b) Every discrete closure space $\mathcal{Q}$ (resp. such that card $|\mathcal{Q}| < \alpha$) is $\mathcal{V}$-complete.

(c) There exists no proper ultrafilter on any set A (resp. such that card $A < \alpha$) which has a base order-isomorphic to some element of $\mathcal{V}$.

In particular, the sum of $\mathcal{V}$-complete spaces is $\mathcal{V}$-complete whenever $\mathcal{V} \subset \mathcal{N}$.

Proof. (b) $\Longrightarrow$ (a): Let N be remarkable $\mathcal{V}$-net in $\mathcal{P} = \sum \{\mathcal{P}_a \mid a \in A\}$. Let ψ be a mapping on $\mathcal{P}$ onto the discrete space $\mathcal{Q}$ with $|\mathcal{Q}| = A$ such that $\psi[|\mathcal{P}_z|] = (z)$ for each $z \in A$. If $f \in \mathcal{F}\mathcal{Q}$ then $f \circ \psi \in \mathcal{F}\mathcal{P}$ (ψ is continuous) and $f \circ \psi \circ N$ converges in I. Thus $\psi \circ N$ is remarkable in $\mathcal{Q}$, converges to some z in $\mathcal{Q}$ by (b), hence $\psi \circ N$ is eventually in (z) and N is eventually in $\mathcal{P}_z$. The restriction of N on $|\mathcal{P}_z|$ is remarkable in $\mathcal{P}_z$, hence converges in $\mathcal{P}_z$ and N converges to the same point in $\mathcal{P}$. (a) $\Rightarrow$ (b) is trivial.

(c) $\Longrightarrow$ (b): Let $\langle N, \prec \rangle$ be a $\mathcal{V}$-net remarkable in $\mathcal{Q}$, let us denote $\mathcal{U}$ its limit in the ultrafilter space $\beta|\mathcal{Q}| = \beta\mathcal{Q}$, let us denote $Bm = \{Nn \mid m \prec n\}$ for each $m \in \mathbf{D}N$. $\mathbf{E}B$ is a base of the ultrafilter $\mathcal{U}$ ($\langle N, \prec \rangle$ is eventually in each $U \in \mathcal{U}$), further $\langle \mathbf{E}B, \supset \rangle$ and $\langle \mathbf{D}N, \prec \rangle \in \mathcal{V}$ are order-isomorphic. Therefore $\mathcal{U}$ is fixed and $\langle N, \prec \rangle$ is convergent in $\mathcal{Q}$.

(b) $\Longrightarrow$ (c): Let B be a base of an ultrafilter $\mathcal{U}$ on A, let h be an order-isomorphism of $\langle E, \sigma \rangle \in \mathcal{V}$ onto $\langle B, \supset \rangle$. Let us choose $Nb \in b$ for each $b \in B$. Then the $\mathcal{V}$-net $\langle N \circ h, \sigma \rangle$ converges to $\mathcal{U}$ in the ultrafilter space βA, hence in $\beta\mathcal{Q}$ (where $|\mathcal{Q}| = A$ and $\mathcal{Q}$ is discrete), thus $\langle N \circ h, \sigma \rangle$ is remarkable in $\mathcal{Q}$ and convergent in $\mathcal{Q}$ by (b). Therefore $\mathcal{U}$ is fixed.

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