Lev Bukovský
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∇ - MODEL AND DISTRIBUTIVITY IN BOOLEAN ALGEBRAS
Lev BUKOVSKÝ, Košice

§ 0. Preliminaries. The main purpose of this paper is to study the addition of "new" sets in a ∇ -model constructed over a complete Boolean algebra and to investigate connections between it and the distributive laws in the Boolean algebra. A main part of this paper was presented on the 3rd Congress for Logic in Amsterdam 1967 (see [2]). Independently, Prof. D. Scott has presented similar results at the Summer Institute on Axiomatic Set Theory in Los Angeles 1967.

The reader is supposed to be familiar with the paper [12]. We remind some definitions, facts and introduce some notations.

A couple ⟨c,t⟩ is called a topological space iff
a) t ∈ ℙ(c) (ℙ(x) is the set of all subsets of x),
b) ∅, c ∈ t, c) t is closed under finite intersections and arbitrary unions. t is called a topology on c. Other topological terminology is used in an obvious way (see e.g. [5]).

If B is a Boolean algebra (see [10]), then 0,1 denote the zero and unit element of B respectively. The symbols −, ∨, , , ∧ are used for the Boolean complement, join and meet. ℘(B) denotes the set of all
ultrafilters on $B$ and $s$ is the isomorphism of $B$ on­
to $\mathcal{F}(B)$, a field of subsets of $\mathcal{F}(B)$. $t(B)$ is the 
weakest topology on $\mathcal{F}(B)$ containing $\mathcal{F}(B)$. The to­
pological space $\langle \mathcal{F}(B), t(B) \rangle$ is called the Stone 
space of the Boolean algebra $B$. A Boolean algebra $B$ is 
called complete iff there exists the join of any subsets of $B$.

For a topological space $\langle c, t \rangle$, let $R(c, t)$ de­
note the set of all regular open subsets of $c$, i.e. the 
set of all $u \in t$ for which $u = \text{Int } \overline{u}$. It is well known 
that the set $R(c, t)$ ordered by inclusion is a complete 
Boolean algebra. Moreover, for a system $u_\xi \in R(c, t)$, 
$\xi \in T$ we have:

$$\wedge_{\xi \in T} u_\xi = \text{Int } \bigcap_{\xi \in T} u_\xi, \bigvee_{\xi \in T} u_\xi = \text{Int } \bigcup_{\xi \in T} u_\xi, -u = \text{Int } (c - u),$$

$$0 = \emptyset, 1 = c.$$ (see [10], p.66). For other consultations the theory of Boo­
olean algebras, see [10].

All our considerations concern the Gödel-Bernays set 
theory $\Sigma^*$ with the axioms of groups A - E (see [3]).
From the text, it will be clear when our considerations a­
re mathematical (i.e. we construct a proof in $\Sigma^*$) and 
when they are metamathematical (i.e. we investigate proper­
ties of the theory $\Sigma^*$).

The set theoretical notations are used in an obvious 
way. An ordinal is the set of all less ordinals, a cardi­
nal is an initial ordinal, $\text{card}(x) = x$ is a cardinal -
the power of the set \( x \), \( x^y \) is the set of all functions defined on \( x \) with values in \( y \). Small greek letter always denotes an ordinal, the letters \( m,n,k \) (with indices eventually) denote cardinals. \( m^n = \text{card}(\bigcup_{k \leq n} k) = \sum_{k \leq n} m^k \), \( \text{cf}(m) \) is the least cardinal \( n \) such that \( m \) is cofinal with \( n \).

If \( \varphi \) is a normal formula (see [3]), then
\[
\mathcal{Z} \in \{ \mathcal{Z} : \varphi(\mathcal{Z}, \mathcal{X}, \ldots, \mathcal{X}_n) \} \equiv \varphi(\mathcal{Z}, \mathcal{X}_1, \ldots, \mathcal{X}_n).
\]
The existence of the class \( \{ \mathcal{Z} : \varphi(\mathcal{Z}, \mathcal{X}_1, \ldots, \mathcal{X}_n) \} \) is proved in [3].

The notion of a constructive set is defined in [3]. The axiom of constructivity \( V = L \) is sometimes used (\( V \) is the class of all sets, \( L \) is the class of all constructive sets). GCH stands for the formula
\[
(\forall \kappa) (2^{\kappa} = \kappa^{\kappa+1}), \text{ i.e. the Generalized Continuum Hypothesis.}
\]

Let \( f \in x^y, \varphi \in x^P(y) \). We define \( f \leq \varphi \) \( \equiv (\forall x) (x \in x \rightarrow f(x) \in \varphi(x)) \). Let \( \langle x_i, r_i \rangle, i = 1,2 \) be partial ordered sets, i.e. \( x_i \) is a set and \( r_i \) is a binary relation on \( x_i \); which is a partial ordering. We denote
\[
\text{Map}(x_1, \kappa_1, x_2, \kappa_2) = \{ f : f \in x_1^x_2 \land \langle \mu v \rangle \in \kappa_1 \rightarrow \langle f(\mu)f(\nu) \rangle \in \kappa_2 \},
\]
\( \text{i.e. Map}(x_1, \kappa_1, x_2, \kappa_2) \) (shortly \( \text{Map}(x_1, x_2) \)) is the set of all non decreasing functions from \( x_1 \) into \( x_2 \).

The notations and results of the paper [12] will be used without references. One notion will be denoted in
different way only, namely, we shall write $\hat{x}$ instead of $k^x$, i.e. $\hat{x}$ is defined as follows: $\hat{x} \in C(B)$ and $D(\hat{x}) = \{ \hat{y}: \forall y \in \hat{x}\}$ and $\hat{x}(\hat{y}) = 1$ for $y \in x$. Let us remember, that $\mathcal{F}^*$ denotes the translation of the formula $\mathcal{F}$ in the model $\mathcal{V}(B, z)$ (see [12], p.157).

1. **Distributive laws in a complete Boolean algebra**

In this paragraph, $B$ denotes always a complete Boolean algebra.

**Definition 1.1.** Let $\langle x^i, r^i \rangle$ be partially ordered set, $i = 1, 2$, $\mathcal{F} \subseteq x^1 \mathcal{P}(x^2)$, Let $z$ be a filter on $B$. The algebra $B$ is called $(z, \mathcal{F})$-distributive (or more precisely $(z, \mathcal{F}, x^1, r^1, x^2, r^2)$-distributive) iff for every system $a_{ij} \in B$, $i \in x^1$, $j \in x^2$ such that

\[(1.1) \sum_{i \in x^1} \bigvee_{j \in x^2} a_{ij} \in z, \]

\[(1.2) \text{if } \langle i_1, i_2 \rangle \in \kappa_1, a_{i_1 j_1} \land a_{i_2 j_2} \neq 0, \text{ then } \langle j_1, j_2 \rangle \in \kappa_2, \text{ the following condition holds true} \]

\[(1.3) \bigvee_{i \in \mathcal{F}} \bigwedge_{i \in x^1} \bigvee_{j \in \mathcal{(j)}} a_{ij} \in z. \]

The $\{1\}, \mathcal{F}$-distributivity is called simply $\mathcal{F}$-distributivity.

**Remarks:**

a) By (1.2) we have $a_{i_1 j_1} \land a_{i_2 j_2} = 0$ for $j_1 \neq j_2$.

b) Always $\bigwedge_{i \in x^1} \bigvee_{j \in x^2} a_{ij} \geq \bigvee_{i \in \mathcal{F}} \bigwedge_{i \in x^1} \bigvee_{j \in \mathcal{(j)}} a_{ij}$.

c) If $\mathcal{F} \subseteq \mathcal{F}'$, $B$ is $(z, \mathcal{F})$-distributive, then $B$ is also $(z, \mathcal{F}')$-distributive.
Let us remember that $B$ is called homogenous iff $B$ is isomorphic to the Boolean algebra $\{x : x \in B \land x \leq a \}$ for every $a \in B$, $a \neq 0$.

**Theorem 1.2.** Let $B$ be a homogenous complete Boolean algebra. The following conditions are equivalent:

(i) $B$ is $(\{f, F\})$-distributive,

(ii) $B$ is $(z, F)$-distributive for every filter $z$ on $B$,

(iii) there is no system $a_{i,j} \in B$ satisfying (1.2) such that

\begin{align*}
\text{a)} & \quad \bigwedge_{i \in x_1} \bigvee_{j \in x_2} a_{i,j} = 1, \\
\text{b)} & \quad \bigvee_{i \in \mathcal{F}} \bigwedge_{i \in x, j \in \mathcal{F}(i)} a_{i,j} = 0.
\end{align*}

**Proof.** Evidently (ii) $\Rightarrow$ (i) $\Rightarrow$ (iii). Let us suppose that (ii) does not hold, i.e. there is a filter $z$ and a system $a_{i,j} \in B$ satisfying (1.2) such that

\begin{align*}
\mu &= \bigwedge_{i \in x_1} \bigvee_{j \in x_2} a_{i,j} \in z, \\
\nu &= \bigvee_{i \in \mathcal{F}} \bigwedge_{i \in x, j \in \mathcal{F}(i)} a_{i,j} \notin z.
\end{align*}

Let $\mathcal{L} = \mu - \nu (\neq 0)$. We define $\mathcal{L} a_{i,j} = \mathcal{L} \land a_{i,j}$. This system possesses properties (1.2) and (iii)a), but

\begin{align*}
\bigvee_{i \in \mathcal{F}} \bigwedge_{i \in x, j \in \mathcal{F}(i)} \mathcal{L} a_{i,j} = 0.
\end{align*}

However, $B \mid \mathcal{L} = \{x : x \in B \land x \leq \mathcal{L} \}$ is isomorphic to $B$, thus $B$ does not fulfil the condition (iii).

Q.E.D.

Now, we shall consider some special cases of distributive laws.
(m,n,k)-distributivity. Let \( r_h \) be the trivial ordering of the cardinal \( h \), i.e. \( \langle \xi, \eta \rangle \in r_h \iff \xi, \eta \in h \) \& \( \xi = \eta \).
Let \( F = \{ f : f \in mP(n) \land (\forall \xi) (\xi \in m \rightarrow f(\xi) \subseteq h) \} \).
In this case, the \( F \)-distributivity is called \((m,n,k)\)-distributivity (\( m,n \) are ordered by \( r_m, r_n \) respectively).

\((m,n,2)\)-distributivity is the obvious \((m,n)\)-distributivity, \((m,n,\omega)\)-distributivity is the weak \((m,n)\)-distributivity in the sense of [10].

\((m \neq n)\)-distributivity is the \( F \)-distributivity, where \( m \), \( n \) are ordered by \( r_m, r_n \) respectively and
\( F = \{ f \in mP(n) : (\exists \eta) (\forall \xi) (\xi \in m \rightarrow f(\xi) \subseteq n - \{ \eta \}) \} \).

Lebesgue distributivity. Let \( X \) be the set of all closed segments \([a,b]\), where \( 0 \leq a < b \leq 1 \) are rational numbers. Let \( X_0 = \{ f : f \in \bigcup_n \omega_n X \} \) and Lebesgue measure of \( \cup W(f) \) be less than one half. \( X_0 \) is ordered by inclusion, \( \omega_0 \) is ordered by " \( \in \) " . Let \( F = \{ \phi_X : \phi_X(n) = n \& f \in X_0 \land \phi_X(n) \subseteq \cup W(f) \} \) \( (F \subseteq \omega_0 P(X_0)) \).

In this case, \( F \)-distributivity is called Lebesgue distributivity.

For the next, the following notation will be useful.
\( B \) is called \((< m)\)-distributive iff \( B \) is \((n,k)\)-distributive for every \( n < m \) and every \( k \).

The definition of \( z \)-Lebesgue distributivity, \((z, m,n,k)\)-distributivity etc. is clear.
§ 2. Some criteria for distributivity. In this paragraph we prove some theorems which allow us to show some distributive laws in a Boolean algebra.

In the paper [11], P. Vopěnka has proved that

\[ \mu(B) = \min \{ \aleph_\alpha : B \text{ does not have a subset of pairwise disjoint elements of power } \aleph_\alpha \} \]

is a regular cardinal. Vopěnka's proof was given for topological spaces, but the fact mentioned above follows from it directly. A direct proof may be given using some results of R.S. Pierce (see [7]). \( \mu \) is a cardinal property, thus \( B \) may be decomposed in \( \mu \)-homogenous factors. For \( \mu \)-homogenous complete Boolean algebra, this proof is trivial.

The characteristic \( \mu \) may be used for proving some distributive laws (almost the same results are proved in [12], p. 161).

**Theorem 2.1.** If \( n \geq \mu(B) \), then \( B \) is \((m \neq n)\)-distributive.

**Proof.** Let \( n \geq \mu(B) \). By remark a), \( \xi_1 \neq \xi_2 \in n \) implies \( a_{\xi_1} \land a_{\xi_2} = 0 \). Therefore, \( A_\xi = \{ \xi : a_{\xi} \neq 0 \} \) has cardinality less than \( \mu(B) \). We define \( f(\eta) = A_\eta \).

Evidently \( f \in \mathcal{F} \) (since \( \mu(B) \) is regular) and

\[ \bigwedge_{\eta \in \mathcal{F}} \bigvee_{\xi \in \eta} a_{\xi} = \bigwedge_{\eta \in \mathcal{F}} \bigvee_{\xi \in f(\eta)} a_{\eta} \quad Q.E.D. \]

In a similar way, one can prove

**Theorem 2.2.** Every complete Boolean algebra \( B \) is \((m,n,\mu(B))\)-distributive for any \( m,n \).

**Definition 2.3.** Let \( < c,t > \) be a topological space. We define:
\[ \mathcal{ND}_\alpha(c,t) = \{ x : x \in c \text{ and } x \subseteq \bigcup_{f \in \mathcal{A}_\alpha} x_f, \ x_f \text{ closed and } \operatorname{Int} x_f = \emptyset \}, \]

\[ \mathcal{B}_\alpha(c,t) = \{ (x \cup y) - z : x \in t \& y, x \in \mathcal{ND}_\alpha(c,t) \}, \]

\[ \mathcal{B}(c,t) = \mathcal{A}_\alpha \bigcup_{U(c,t)} \mathcal{B}_\alpha(c,t). \]

A concept \( \square \) is defined for a Boolean algebra \( B \) as \( \Box(B) = \Box(\mathcal{U}(B), t(B)), \) (compare \cite{11}.

**Lemma 2.4.** Let \( \omega_\beta < \mathcal{U}(c,t) \). Then \( \mathcal{B}(c,t) \) is an \( \omega_\beta \)-field of sets, \( \mathcal{ND}(c,t) \) is an \( \omega_\beta \)-ideal. Moreover, \( \mathcal{R}(c,t) \) is isomorphic to the factor algebras \( \mathcal{B}_\beta(c,t)/\mathcal{ND}_\beta(c,t) \) and \( \mathcal{B}(c,t)/\mathcal{ND}(c,t) \).

**Proof.** The first part is trivial. To prove the second part, it suffices to prove that \( \mathcal{R}(c,t) \) is isomorphic to \( \mathcal{B}(c,t)/\mathcal{ND}(c,t) \) for any \( \mathcal{A}_\alpha \leq \mathcal{U}(c,t) \).

Let \( \pi \) be the natural homomorphism of \( \mathcal{B}_\alpha(c,t) \) onto \( \mathcal{B}_\alpha(c,t)/\mathcal{ND}_\alpha(c,t) \). For every \( x \in \mathcal{R}(c,t) \) we define \( h(x) = \pi(x) \). By simple computation we have:

\[ h(\emptyset) = 0, \ h(c) = 1, \ h(\operatorname{Int}(c-u)) = 1 - h(u), \ h(\operatorname{Int} \overline{U x_f}) = \bigvee h(x_f). \]

If \( h(x) = 0 \), then \( x = \emptyset \) since \( \mathcal{ND}_\alpha(c,t) \cap t = \{ \emptyset \} \). For every \( \mu \in \mathcal{R}_\alpha(c,t) \) there is a \( v \in \mathcal{R}(c,t) \) such that \( v = \operatorname{Int} x \), where \( u = (x - y) \cup z \), \( -602- \)
Theorem 2.5. If \( (\omega_\alpha, \omega_\beta) \subset \gamma (c, t) \), then \( \mathfrak{R} (c, t) \) is \((\omega_\alpha, \omega_\beta)\)-distributive (compare [6], [8]).

Proof. Let \( \omega_\gamma = \omega_\beta \). By the lemma 2.4, \( \mathfrak{B}_\gamma (c, t) \) is \((\omega_\alpha, \omega_\beta)\)-distributive (as \( \omega_\gamma \)-field) and \( \mathfrak{N} \mathfrak{D}_\gamma (c, t) \) is an \( \omega_\gamma \)-additive ideal. Therefore, the factor algebra \( \mathfrak{B}_\gamma (c, t) / \mathfrak{N} \mathfrak{D}_\gamma (c, t) \) is also \((\omega_\alpha, \omega_\beta)\)-distributive.

Q.E.D.

Theorem 2.6. If \( (c, t) \) is an \( \omega_\gamma \)-additive to logical space (i.e. the intersection of less than \( \omega_\gamma \) open set is an open set, see [9], p. 125), then \( \mathfrak{R} (c, t) \) is \((< m)\)-distributive, \( m = \min \{ \omega_\gamma, \nu (c, t) \} \).

Proof. Let \( a_{\tilde{z}} \in \mathfrak{R} (c, t) \), \( \bigwedge_{\tilde{z} \in \omega_\beta} a_{\tilde{z}} = 1 \), i.e.

\[
\operatorname{Int} \bigcap_{\tilde{z} \in \omega_\beta} a_{\tilde{z}} = c.
\]

Evidently \( \bigcup_{\tilde{z} \in \omega_\beta} a_{\tilde{z}} = c \) for any \( \tilde{z} \in \omega_\beta \). Let \( \omega_\alpha < m \). Since \( m \subset \nu (c, t) \), we have \( \bigcap_{\tilde{z} \in \omega_\beta} \bigcup_{\tilde{z} \in \omega_\beta} a_{\tilde{z}} = c \).

Thus

\[
\bigvee_{\tilde{z} \in \omega_\alpha} \bigwedge_{\tilde{z} \in \omega_\beta} a_{\tilde{z} \gamma (\tilde{z})} = \operatorname{Int} \bigcup_{\tilde{z} \in \omega_\alpha} \bigwedge_{\tilde{z} \in \omega_\beta} a_{\tilde{z} \gamma (\tilde{z})} = \operatorname{Int} \bigcap_{\tilde{z} \in \omega_\alpha} \bigcup_{\tilde{z} \in \omega_\beta} a_{\tilde{z}} = c.
\]

Q.E.D.

On the other hand, one can easily prove

Theorem 2.7. Let \( \mathfrak{B} \) be a complete Boolean algebra.
If $B$ is $\prec m \succ$-distributive, then $\forall (\mathcal{F}(B), t(B)) \geq m$.

The essential part of the theorems 2.5 - 2.7 is known (see [4], [6], [8], [11], [10]). Now, we prove two criteria for the Lebesgue distributivity.

Theorem 2.8. If $B$ contains a regular subalgebra isomorphic to $\mathcal{R}(c_0, t_0)$, where $\langle c_0, t_0 \rangle$ is the Cantor set with the obvious topology, the $B$ is not Lebesgue distributive.

Proof. Evidently, it suffices to prove that $\mathcal{R}(c_0, t_0)$ is not Lebesgue distributive.

We may assume $c_0 = \omega_0 \cdot 2$. If $\varphi \in \mathbb{N}^2$, $\eta \in \omega_0$, then $\mu_\varphi = \{ \{ f \in c_0 : \varphi \leq \varphi \}$ is a regular open subset of $c_0$. Let $\kappa_\eta, \kappa = [\frac{\kappa_k}{2^m+1}, \frac{\kappa_{k+1}}{2^m+1}]$, $k = 0, 1, \ldots, 2^{m+2} - 1$; $s_\eta = (n + 1)(n + 4)/2$, $s_{-1} = 0$. Let $\kappa$, $k 2^{m+2}$ be an enumeration of all functions from $(s_\eta - s_{\eta-1})$ into $2$. We define

$$\mathcal{A}^\omega = \bigwedge_{\kappa, \eta} \mu_{\varphi(\kappa)},$$

$$\mathcal{A}^\omega = \emptyset \text{ for other } \kappa \in \mathcal{X}.$$  

For $\varphi \in \mathcal{X}$, we denote $\eta_{\varphi, \kappa} = \bigwedge_{\kappa, \eta} \mu(\kappa, \varphi(\kappa))$. The conditions (1.1) and (1.2) are evidently satisfied (with $z = \{1\}$). By simple computation we obtain

$$\bigwedge_{\kappa, \eta} \bigvee_{\varphi \in \mathcal{X}(\kappa, \eta)} \eta_{\varphi, \kappa} = \emptyset \text{ for any } x \in [0, 1].$$

Q.E.D.
A strictly positive finite measure $m$ on a complete Boolean algebra $B$ (see [10], p. 73) is a $\sigma$-additive measure on $B$ such that $m(x) \neq 0$ for $x \neq 0$ and $m(1) = 1$.

**Theorem 2.9.** If there exists a strictly positive finite measure on $B$, then $B$ is Lebesgue distributive.

**Proof.** Let $m$ be a strictly positive finite measure on $B$. $B$ is isomorphic to $\mathcal{B}_o(B)/\mathcal{N}_d(B)$. Let $m_o$ be the induced measure on $\mathcal{B}_o(B)$ then $\mathcal{N}_d(B)$ is the set of all sets from $\mathcal{B}_o(B)$ of $m_o$-measure zero.

Let $m_*$ be the product measure of $m_o$ and the Lebesgue measure on $[0,1]$. Thus, $m_*$ is a finite $\sigma$-additive measure on $\mathcal{B}_o(B) \times [0,1]$. If $x \in \mathcal{B}_o(B)$, then $[x]$ denotes the corresponding element of $B$. Let

$$\bigwedge_{n \in \omega_b} \bigvee_{q \in \mathcal{X}_o} a_{nq} = 1 \text{ and } a = 1 - \bigvee_{x \in [0,1]} \bigwedge_{n \in \omega_b} \bigvee_{q \in \mathcal{X}_o} a_{nq} \neq 0.$$

Let $A, A_{nq} \in \mathcal{B}_o(B)$, $a = [A], a_{nq} = [A_{nq}]$.

Evidently $m_o(A) = m(a) > 0$. We denote

$$B_n = \bigcup_{q \in \mathcal{X}_o} \bigcup_{x \in \mathcal{X}_o} (A \cap A_{nq}) \times [0,1].$$

By simple computation we have $m_*(B_n) < \frac{1}{2} m_o(A)$, $B_n \subseteq B_{n+1}$. Therefore $m_*(\bigcup_{n \in \omega_b} B_n) \leq \frac{1}{2} m_o(A)$.

Since $m_*(A \times [0,1]) = m_o(A)$, there exists an $x \in [0,1]$ and a set $C \subseteq A$ such that $m_o(C) > 0$ and $(C \times \{x\}) \cap \bigcup_{n \in \omega_b} B_n = \emptyset$. This fact implies

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§ 3. The main theorem. Let $B_0$ denote the two-elements Boolean algebra consisting of zero and unit elements only. If $z$ is an ultrafilter on $B$, then $\mathcal{Y}_{\infty_0}^{\mathcal{C}(B)}(\mathcal{C}(B))$ is a model class in the sense of the model $\nabla(B,z)$ (see [12]), which is "elementary equivalent to the whole theory", i.e. for an elementary formula (formula without class variables) $\Sigma^* \vdash \varphi(x_1, \ldots, x_n)$ if and only if $\varphi^*(x_1, \ldots, x_n)$ holds true in the model $\mathcal{Y}_{\infty_0}^{\mathcal{C}(B), z}$. Thus, e.g. if the axiom of constructivity $V = L$ is assumed to hold true, then $\mathcal{Y}_{\infty_0}^{\mathcal{C}(B)}$ is the class of all constructive sets in the sense of the model $\nabla(B,z)$.

A condition for the existence of a "new set" of the model $\nabla(B,z)$ (a set which does not belong to $\mathcal{Y}(\mathcal{C}(B))$), is given by the following

Theorem 3.1. Let $\langle x_i, r_i \rangle$ be partially ordered sets, $i = 1, 2$. Let $\mathcal{F} \subseteq x_i \mathcal{P}(x_i)$. Let $B$ be a complete Boolean algebra, $z$ an ultrafilter on $B$. Then $B$ is $(z, \mathcal{F})$-distributive if and only if in the model $\nabla(B,z)$, for every $g \in^* \mathcal{M}an^*(\tilde{x}_1, \tilde{x}_2, \tilde{x}_2)$ there exists an $f \in^* \tilde{x}$ such that $g \subseteq \subseteq^* f$.

Proof. a) Let $B$ be $(z, \mathcal{F})$-distributive, $g \in^* \mathcal{M}an^*(\tilde{x}_1, \tilde{x}_2)$. We denote $a_{i,j} = F^r(\tilde{x}_i \tilde{x}_j) \in G^7$. The system $a_{i,j}$, $i \in x_1$, $j \in x_2$ satisfies the condi-
tions (1.1) and (1.2). By simple computation we have

\[(3.1) \exists f (\forall x \in \mathcal{F} \& g \leq f \Rightarrow \forall x, \forall i \in \mathcal{F}^{(i)} a_{i,j}^2 \cdot \]

Now the existence of such a function \( f \in \mathcal{F} \) follows from \((z, \mathcal{F})\)-distributivity.

b) Let \( B \) be not \((z, \mathcal{F})\)-distributive, i.e. there is a system \( a_{i,j}, i \in x_1, j \in x_2 \) satisfying (1.1) and (1.2) such that \( \forall A \cup \forall a_{i,j} : k \neq i \in \mathcal{F} \).

We define a function \( g \in \mathcal{C}(B) \) as follows: \( g(\langle j, i \rangle) = a_{i,j} \) for \( i \in x_1, j \in x_2 \). It is easy to see that

\[ F^* g \in \text{Map}^*(\mathcal{F}^* \times \mathcal{F}^* \times \mathcal{F}^* \times \mathcal{F}^* \times \mathcal{F}^* \times \mathcal{F}^* \) \]

The theorem follows by (3.1). Q.E.D.

Corollary 3.2. In the model \( \mathcal{V}(B, z) \), \( m \) and \( n \) are of the same power if and only if \( B \) is not \((z, m \neq n)\)-distributive.

Proof. In the case of \((m \neq n)\)-distributivity, 
\( f \subseteq \subseteq^* g \subseteq \subseteq^* \mathcal{F} \) is equivalent to the condition: \( f \) is not a mapping of \( m \) onto \( n \) in the model \( \mathcal{V}(B, z) \).

Corollary 3.3. In the model \( \mathcal{V}(B, z) \), \( m = \tilde{m} = n = \tilde{n} \) if and only if \( B \) is \((z, m, n)\)-distributive.

Proof. In the case of \((m, n)\)-distributivity, \( f \subseteq \subseteq^{*} \subseteq \subseteq^{*} g \subseteq \subseteq^{*} \mathcal{F} \) is equivalent to the condition \( f \subseteq \subseteq^{*} n \).

Corollary 3.4. \( B \) is not \( z \)-Lebesgue distributive, if and only if the set \([0,1]\) (i.e. the set of real numbers between zero and one, which are in the model class

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\[ \psi(\mathcal{C}(B, z)) \] has the Lebesgue measure \( \leq \frac{1}{2} \) in the model \( \mathcal{V}(B, z) \).

**Proof.** It suffices to note that elements of \( \mathcal{X}_0 \) are finite coverings of measure less than \( \frac{1}{2} \) and an infinite covering belongs to \( \mathcal{P}_x \) if and only if it does not cover the real \( x \).

**Corollary 3.5.** In the model \( \mathcal{V}(B, z) \), for every function \( f \in \mathcal{X}_m \) there are two functions \( h, g \in \mathcal{X}_m \) such that
\[
(\forall \xi)(\xi \in \mathcal{X}_m \rightarrow h(\xi) \leq f(\xi) \leq g(\xi) \land g(\xi) - h(\xi) < \epsilon)
\]
if and only if \( B \) is \((z, n, m, k)\)-distributive.

§ 4. Applications to the Boolean algebras. Using the results of § 3, many theorems concerning complete Boolean algebras may be proved or reproved, e.g. the well known theorem of Pierce-Smith-Tarski for complete Boolean algebras is a direct consequence of the corollary 3.3 (at the Summer Institute in Los Angeles 1967, prof. D. Scott also announced similar proof).

Now we prove some theorems concerning distributive laws in a complete Boolean algebra.

**Theorem 4.1.** Let \( B \) be a complete Boolean algebra. If \( B \) is \((\omega, 2)\)-distributive, then \( B \) is Lebesgue distributive.

**Proof.** Let us suppose that \( B \) is not Lebesgue distributive. Then there is an ultrafilter \( z \) on \( B \) such that the set \([0, 1]\) has the Lebesgue measure less than one in the model \( \mathcal{V}(B, z) \). Therefore there is a real
number which is not in the set \([0,1]\) and, by corollary 3.3, \(B\) is not \((\omega_\alpha,2)\)-distributive.

\[ \text{Q.E.D.} \]

The author knows nothing about the connection between Lebesgue distributivity and the weak \(\omega_\alpha\)-distributivity (may be they are equivalent).

In a similar way as the theorem 4.1, one can prove

**Theorem 4.2.** Let \(B\) be a complete Boolean algebra.

a) \(B\) is \((m \neq n)\)-distributive if and only if \(B\) is \((n \neq m)\)-distributive.

b) If \(B\) is \((m,2)\)-distributive, then \(B\) is \((m \neq n)\)-distributive for any \(n \neq m\).

Using properties of model classes we can prove

**Theorem 4.1.** Let \(B\) be a complete Boolean algebra.

a) Let \(\omega_\beta > \omega_\alpha \geq \text{cf} (\omega_\beta)\). If \(B\) is \((\omega_\beta \neq \omega_{\xi + 1})\)-distributive for every \(\xi : \alpha \leq \xi < \beta\), then \(B\) is \((\omega_\alpha \neq \omega_\beta)\)-distributive.

b) Let \(\omega_\beta > \omega_\alpha\) be the first cardinal which is confinal with a cardinal \(\leq \omega_\alpha\). Let \(\xi < \beta\). If \(B\) is \((\omega_\alpha,2)\)-distributive, \((m \neq n)\)-distributive for every \(m,n, \omega_\alpha \leq m,n \leq \omega_\xi\), then \(B\) is \((\omega_\alpha, \omega_\xi)\)-distributive.

**Proof.** We prove the part b) only. Let \(\omega_\xi\) be the smallest cardinal for which b) does not hold true. Thus, there is a complete Boolean algebra which is not \((\omega_\alpha, \omega_\xi)\)-distributive. It is easy to see that there is an ultrafilter \(z\) on \(B\) such that, in the model \(\mathcal{V}(B,z)\), \(\omega_\alpha \omega_\xi \not\equiv \omega_\alpha \omega_\xi\). Since \(\omega_\xi\) is not confinal with \(\omega_\alpha\), then also

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\[ \omega_\gamma \neq \omega_\eta \text{ for some } \eta < \xi \]. That contradicts to the assumption.

Q.E.D.

§ 5. A consistency result. Using the corollaries 3.3 and 3.4, we can prove a consistency result concerning the Lebesgue measure. The first part of this theorem was announced by the author in [11], the second part was presented on the Prague Set Theory Seminarium in October 1967. As the author was informed (January 1968), R. Solovay has announced similar results.

**Theorem 5.1.** Let the axiom of constructivity hold true.

a) Let \( D \) be the Boolean algebra of regular open subsets of the Cantor set. Let \( z \) be an ultrafilter on \( D \). In the model \( \mathcal{V}(D,z_\omega) \) the following hold true:
   (i) there is a nonconstructive real number,
   (ii) the set of constructive real numbers is of power \( \aleph_1 \) and has the Lebesgue measure zero.

b) Let \( C \) be the Boolean algebra of Borel subsets of the unit segment factorized by the ideal of all sets of the Lebesgue measure zero. Let \( z_1 \) be an ultrafilter on \( C \). In the model \( \mathcal{V}(C,z_1) \) the following holds true:
   (i) there is a nonconstructive real number,
   (ii) the set of all constructive real numbers is of power \( \aleph_1 \) and is not Lebesgue measurable.

**Proof** Both \( D \) and \( C \) are not \( (\omega_\omega,2) \)-distributive and fulfil the countable chain condition. Thus, by theo-

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rem 2.1, corollaries 3.2 and 3.3, a) (i), b) (i) hold true and the set $[0,1]$ is of power $\aleph_1$ in both models.

If there is a nonconstructive real number and the set of constructive real numbers is of power $\aleph_1$, then this set is either of Lebesgue measure zero or is not Lebesgue measurable (see [1]). Using this fact, the theorem follows by the theorems 2.8 and 2.9 and the Corollary 3.4.

Q.E.D.

References


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