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ONE THEOREM ON ROTUNDITY AND SMOOTHNESS OF SEPARABLE BANACH SPACES

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In this paper X denotes a real Banach space, X^* the dual space of X . We denote strong (weak) convergence in X by $x_n \rightarrow x$ ($x_n \xrightarrow{w} x$) , the pointwise convergence in X^* by $f_n \xrightarrow{w^*} f$. $S_1 = \{x \in X ; \|x\| = 1\}$, $S_1^* = \{f \in X^* ; \|f\| = 1\}$.

Definition 1 (V.L. Šmuljan [5,6], D. Cudia [3,4]). Banach space X is called (WUR)-space if the following implication is valid:

$$(x_n, y_n \in S_1, \|\frac{x_n + y_n}{2}\| \rightarrow 1) \Rightarrow x_n - y_n \xrightarrow{w} 0 .$$

Definition 2 (V.L. Šmuljan [5,6] , D.Cudia [3,4]). Banach space X^* is said to be (W^*UR)-space if the following condition is satisfied:

$$(f_n, g_n \in S_1^*, \|\frac{f_n + g_n}{2}\| \rightarrow 1) \Rightarrow f_n - g_n \xrightarrow{w^*} 0 .$$

Definition 3 (V.L. Šmuljan [5,6]). Banach space X is said to be (UG)-space if the norm of X is uniformly Gâteaux differentiable on S_1 .

Theorem 1 (V.L. Šmuljan [5,6]). Banach space X is (UG) iff X^* is (W^*UR) . X^* is (UG) iff X is (WUR).

Suppose now $\|x\|_1$ and $\|x\|_2$ are two equivalent norms in a Banach space X . Denote

$f_0(x) = \frac{1}{2} \|x\|_1^2$, $g_0(x) = \frac{1}{2} \|x\|_2^2$. Using the results of A. Brøndsted ([2]), E. Asplund ([1]) has constructed the sequences of the functions:

$$\left.\begin{aligned} f_{n+1}(x) &= \frac{1}{2}(f_n(x) + g_n(x)) \\ g_{n+1}(x) &= \inf_{y \in X} \{\frac{1}{2}(f_n(x+y) + g_n(x-y))\} \end{aligned}\right\} \text{ for } n \geq 0 .$$

These sequences converge to a common finite-valued convex homogeneous of second order function h .

Further we denote f_n^* , g_n^* , h^* the dual functions of f_n , g_n , h respectively, for example: $f_0^*(x) = \sup_{y \in X} (\langle x, y \rangle - f_0(y))$ where $\langle x, y \rangle$ denotes the duality of X, X^* .

Then $f_n^*(x) = \frac{1}{2} \|x\|_{f_n^*}^2$, $g_n^*(x) = \frac{1}{2} \|x\|_{g_n^*}^2$, $h^*(x) = \frac{1}{2} \|x\|_{h^*}^2$, where $\|x\|_{f_n^*}$, $\|x\|_{g_n^*}$, $\|x\|_{h^*}$ are the dual norms of $\|x\|_{f_n} = (2f_n(x))^{\frac{1}{2}}$, $\|x\|_{g_n} = (2g_n(x))^{\frac{1}{2}}$, $\|x\|_h = (2h(x))^{\frac{1}{2}}$. These norms are equivalent on X^* and X respectively. Further we have:

$$f_{n+1}^*(x) = \inf_{y \in X^*} \{\frac{1}{2}(f_n^*(x+y) + g_n^*(x-y))\} ,$$

$$g_{n+1}^*(x) = \frac{1}{2}(f_n^*(x) + g_n^*(x))$$

for every $n \geq 0$.

These facts follow from the results of A. Brøndsted ([2]) and E. Asplund ([1]).

<u>Definition 4</u>. Let $\|x\|$ be some norm of a Banach space X , $\|f\|$ be a norm of X^*. Denote $\tilde{f}(x) = \frac{1}{2} \|x\|^2$. Then $\tilde{f}$ is said to be (WUR)-function if the following relation is true: For every $\varepsilon > 0$ and

each $g \in S_1^*$

$$\inf_{\substack{\|x\|=1 \\ |g(x-y)| \geq \varepsilon}} \{ \tilde{f}(x) - 2\tilde{f}(\tfrac{x+y}{2}) + \tilde{f}(y) \} > 0 .$$

Denote $\tilde{g}(x) = \frac{1}{2} \|f\|^2$. $\tilde{g}$ is said to be (W^*UR)-function if the following condition is satisfied: For every $\varepsilon > 0$ and each $x \in S_1$

$$\inf_{\substack{\|f\|=1 \\ |(f-g)(x)| \geq \varepsilon}} \{ \tilde{g}(f) - 2\tilde{g}(\tfrac{f+g}{2}) + \tilde{g}(g) \} > 0 .$$

Proposition 1. If $\tilde{f}(x) = \frac{1}{2} \|x\|^2$ then $\tilde{f}$ is (WUR) iff $\|x\|$ is (WUR). Similarly for the case of (W^*UR).

Proof. It is a slight modification of that of E. Asplund for the case of local uniform rotundity.

Proposition 2. If f_0 (or g_0) is (WUR), then h is (WUR). Analogically for the case of (W^*UR).

Proof. It is analogous to that of E. Asplund for the case of local uniform rotundity.

Theorem 2. Suppose X^* is separable Banach space. Then there exists an equivalent norm of X which is (UG) and (WUR) jointly.

Proof. In the paper [7] we have proved that there exists in this case an equivalent norm $\|x\|_1$ of X which is (WUR) and an equivalent norm $\|x\|_2$ of X which is (UG). Denote $f_0(x) = \frac{1}{2} \|x\|_1^2$, $g_0(x) = \frac{1}{2} \|x\|_2^2$.

Then h constructed as above is (WUR) (Proposition 2).

$g_0^*(f) = \frac{1}{2} \| f \|_2^2$, where $\| f \|_2$ is the dual norm of $\| x \|_2$. $\| f \|_2$ is (W*UR) (Theorem 1). Then g_0^* is (W*UR) (Proposition 1). h^* is then (W*UR) (Proposition 2). Define $h(x) = \frac{1}{2} ||| x |||^2$, $h^*(x) = \frac{1}{2} ||| f |||^2$ where $||| f |||$ is the dual norm of $||| x |||$. Then $||| x |||$ is an equivalent norm of X which is (WUR) (Proposition 1) and its dual norm $||| f |||$ is (W*UR). Thus $||| x |||$ is (WUR) and (UG) jointly (Theorem 1).

R e f e r e n c e s

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