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ON PRODUCTS IN GENERALIZED ALGEBRAIC CATEGORIES

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0. Introduction.

Universal algebras of a given type $\Delta = \{ \aleph_{\alpha} \mid \alpha < \beta \}$
($\Delta$ is a family - as a rule increasing - of ordinal numbers
indexed by ordinal numbers) form the category $A(\Delta)$ whose
objects are operational structures, the pairs
$(X; \{ \omega^X_\alpha \mid \alpha < \beta \})$ where $X$ is a set and $\omega^X_\alpha$ are
$\aleph_{\alpha}$-ary operations on $X$, i.e. mappings $\omega^X_\alpha : X^{\aleph_{\alpha}} \rightarrow X$, and
morphisms from $(X; \{ \omega^X_\alpha \})$ to $(Y; \{ \omega^Y_\beta \})$ are mappings $f : X \rightarrow Y$ compatible with operations in
the sense that $\omega^Y_\beta \circ f^{(\aleph_{\alpha})} = f \circ \omega^X_\alpha$ for every
$\alpha < \beta$, where $f^{(\aleph_{\alpha})} : X^{\aleph_{\alpha}} \rightarrow Y^{\aleph_{\alpha}}$ is $f$ acting co-
ordinate-wise on $\aleph_{\alpha}$-tuples from $X^{\aleph_{\alpha}}$.

Here the operations play a role of a "device selecting
suitable mappings" - the morphisms of $A(\Delta)$. Now, we can
let this device work in a more general situation. Take two
functors $F$ and $G$ of the same variance from sets to sets
and define the generalized algebraic category $A(F, G, \Delta)$
as follows: objects are again pairs $(X, \{ \omega^X_\alpha \})$ but op-
ervations $\omega^X_\alpha$ range over $FX$ and take values in
$GX$ (so they are mappings $\omega^X_\alpha : (FX)^{\aleph_{\alpha}} \rightarrow GX$), and,
morphisms are in the covariant case mappings $f : X \rightarrow Y$
such that $\omega^Y_\beta \circ (Ff)^{(\aleph_{\alpha})} = (Gf) \circ \omega^X_\alpha$ for every
\[ \lambda < \beta, \text{ so we have commutative diagrams} \]

\[
\begin{array}{ccc}
(FX)_{\alpha} & \xrightarrow{\omega_{\alpha}^X} & GX \\
(Ff)_{(\alpha)} & \downarrow & \downarrow Gf \\
(FY)_{\alpha} & \xrightarrow{\omega_{\alpha}^Y} & GY
\end{array}
\]

(In the contravariant case the vertical arrows are reversed and compatibility of \( f \) means the fulfilment of the identities \( \omega_{\alpha}^X \circ (Ff)_{(\alpha)} = (Gf) \circ \omega_{\alpha}^Y \) for every \( \lambda < \beta \).)

We shall refer to functors \( F \) involved on the first place in \( A(F, G, \Delta) \), for obvious reasons, as to domain-functors, and to functors \( G \) as to range-functors. Taking \( F = G = I \) — an identical functor, we get clearly \( A(\Delta) \).

It is known that \( A(\Delta) \) always has products (in usual categorical sense). Unfortunately, this pleasant property is very often lost for categories \( A(F, G, \Delta) \) with non-identical domain and range-functors.

It is easily seen that the existence of products in \( A(F, G, \Delta) \) such that the natural forgetful functor preserves them is equivalent to the requirement that \( G \) preserves products. Much less transparent is the general problem of existence of products in categories \( A(F, G, \Delta) \) — the main objective of the present paper. Then the condition that \( G \) preserves products is, of course, far from being necessary and there are many other interesting categories \( A(F, G, \Delta) \) possessing products but with \( G \) not...
preserving products. But generally it is true that the behaviour of the range-functor with regard to products matters here, and, if it does not preserve products, then also the behaviour of the domain-functor with regard to sums (disjoint unions) becomes relevant to the problem.

Presented material is exposed in five sections. The first one brings basic definitions and facts, including conventions about notations used. In the section 2 there are given some necessary conditions for the existence of products in \( A(F, G, \Delta) \). With aid of these it is proved in the section 3 that for \( F, G \) contravariant faithful and \( \Sigma \Delta > 0 \) \( A(F, G, \Delta) \) fails to have products. Section 4 is devoted to more close study of certain properties of covariant functors. The final section 5 gives a number of theorems on products in \( A(F, G, \Delta) \) with covariant functors \( F, G \).

Some problems remain open here, nevertheless, our theorems account for most of familiar functors \( F \) and \( G \).

In final remarks some possible generalizations are indicated.

1. Basic definitions, facts and notation

All functors throughout this paper will be functors from sets to sets (i.e. from the category \( \mathcal{S} \) of all sets and mappings - including void ones - to \( \mathcal{S} \)). Observe that for our purposes we can consider functors only up to the natural equivalence \( \cong \). When systems of functors are discussed, we use the set-theoretic symbols \( \in, \subset, \cup, \cap \).
for shortness sake.

Let $F$ and $G$ be functors of the same variance. $F$ is a sub-functor of $G$ if there exists a monotransformation $\mu : F \to G$; $F$ is a factor functor of $G$ if there exists an epitransformation $\nu : G \to F$; $F$ is a retract of $G$ if there are a monotransformation $\mu : F \to G$ and an epitransformation $\nu : G \to F$ such that $\nu \mu$ is the identical transformation of $F$.

Recall the usual operations over functors (cf.[1]):

(a) The product $F \times G$,
(b) the coproduct (disjoint union) $F \lor G$ defined for functors of the same variance, both can be extended to an arbitrary family $\{ F_\ell \mid \ell \in \mathcal{J} \}$ over a set $\mathcal{J}$ of functors, the results written as $\prod_{\ell \in \mathcal{J}} F_\ell$ and $\lor_{\ell \in \mathcal{J}} F_\ell$, respectively.
(c) The superposition $F \circ G$ of arbitrary functors $G$ and $F$ written (as anywhere else) left-hand, i.e. $(F \circ G)X = FX \circ (G_X)$. If $F$ and $G$ are of different variance, then $F \circ G$ is contravariant, otherwise it is covariant.
(d) The hom-functor $\langle F, G \rangle$ for functors of different variance, its variance being the same as that of $G$. Remember that, writing $H$ for $\langle F, G \rangle$, we have $HX = \{ \phi \mid \phi : FX \to G \times \mathcal{J} \}$ and for $f : X \to Y$ and $H$ covariant $(Hf)(\phi) = (Gf) \circ \phi = (Ff)$. 
Let us last some of the most commonly used functors:

I denotes the identical functor,

\( C_M \) - a constant functor to \( M \); it is both covariant and contravariant;

\( P^+ \) - the covariant power functor;

\( P^+(X) = \{ A : A \subseteq X \}, (P^+(f))(A) = \{ f(x) : x \in A \} \) for \( f : X \to Y \);

\( N \) - a subfunctor of \( P^+ \) assigning to every set \( X \) the set \( N_X \) of all its non-void subsets, evidently \( P^+ \cong N \vee C_1 \);

\( P^- \) - the contravariant power functor, \( P^- \cong \langle I, C_2 \rangle \);

\( \beta \) - a subfunctor of \( (P^-)^2 = P^- \ast P^- \) assigning to every set \( X \) the set \( \beta X \) of all ultrafilters on \( X \);

\( Q_M \) - a cartesian power, \( Q_M \cong \langle C_M, I \rangle \).

We shall often use the next fact from [2]:

**Proposition 1.1.** Every faithful covariant functor has I for its subfunctor. Every faithful contravariant functor has \( P^- \) for its retract.

Let \( \{ X_\alpha : \alpha \in A \}, A \neq \emptyset \), be an arbitrary family of objects of some category \( \mathcal{X} \). Any pair \( \langle X, \{ X_\alpha : \alpha \in A \} \rangle \) - an object \( X \) of \( \mathcal{X} \) together with a family of morphisms \( X_\alpha : X \to X_\alpha, \alpha \in A \) - is called an inverse bound

\(^*\) An alternative description of the functor \( \beta \) is the category of all completely regular topological \( T_1 \) spaces, \( \Phi : \mathcal{T} \to \mathcal{Y} \) the forgetful functor, \( F : \mathcal{Y} \to \mathcal{T} \) the free functor and \( \Psi : \mathcal{T} \to \mathcal{T} \) the functor assigning to each space its \( \beta \)-compactification, then \( \beta = \Phi \circ \Psi \circ F \).
(further "inverse" is often omitted) of the family
\{X_\alpha \mid \alpha \in A\}.

If every other inverse bound \( \langle \gamma, \{\eta_\alpha \mid \alpha \in A\} \rangle \) of \( \{X_\alpha \mid \alpha \in A\} \) factorizes through \( \langle X, \{\sigma_\alpha \} \rangle \), i.e. if there exists a morphism \( h : \gamma \to X \) such that \( \eta_\alpha = \sigma_\alpha \circ h \) for all \( \alpha \in A \), then \( \langle X, \{\sigma_\alpha \} \rangle \) is called a pseudoproduct of the family.

A pseudoproduct is product if the factorization is unique.

A category \( \mathcal{K} \) is said to have (pseudo)products if every family of its objects has a (pseudo)product.

2. Necessary conditions

Let \( \mathcal{K} = A(F, G, \Delta) \) and \( \mathcal{K}_1 = A(F, G_1, \Delta_1) \) be two categories with all the functors \( F, G, F_1, G_1 \) of the same variance and (possibly) of different types \( \Delta = \{\sigma_\alpha \mid \alpha < \beta\} \) and \( \Delta_1 = \{\sigma_\mu \mid \mu < \phi\} \). Denote the objects of \( \mathcal{K} \) by \( X_\sigma = (X, \{\sigma_\alpha \mid \alpha < \beta\}) \) and the objects of \( \mathcal{K}_1 \) by \( X_\omega = (X, \omega \sigma_\mu \mid \mu < \phi) \). If a mapping \( f : X \to Y \) is a morphism in \( \mathcal{K} \) or \( \mathcal{K}_1 \), write simply \( f : X_\sigma \to Y_\sigma \) or \( f : X_\omega \to Y_\omega \), respectively.

Lemma 2.1. Assume that there are assignments \( \Phi \) and \( \Psi \)

\[ \Phi X_\omega = X_\sigma \quad \text{and} \quad \Psi X_\sigma = X_\omega, \]

between the objects of \( \mathcal{K} \) and \( \mathcal{K}_1 \) with the following three properties:

(a) \( f : X_\sigma \to \Phi Z_\omega \Rightarrow f : \Psi X_\sigma \to Z_\omega \),

(b) \( g : Y_\omega \to Z_\omega \Rightarrow g : \Phi Y_\omega \to \Phi Z_\omega \).
Then the existence of pseudoproducts in $\mathcal{K}$ implies the existence of pseudoproducts in $\mathcal{K}_1$.

Proof. Let \( \{ X_\alpha^\omega \mid \alpha \in A \} \) be an arbitrary family of objects in $\mathcal{K}_1$. The family \( \{ \Phi X_\alpha^\omega \} \) has - as any other family in $\mathcal{K}$ - a pseudoproduct, say, \( \langle X_\sigma, \{ f_\alpha \} \rangle \) with \( f_\alpha : X_\sigma \to \Phi X_\alpha^\omega, \alpha \in A \). By (a) it is \( f_\alpha : YX_\sigma \to X_\alpha^\omega \), therefore \( \langle YX_\sigma, \{ f_\alpha \} \rangle \) is a bound of the family \( \{ X_\alpha^\omega \} \).

Let \( \langle \gamma_\omega, \{ q_\alpha \} \rangle \) be an another bound of \( \{ X_\alpha^\omega \} \), i.e. \( q_\alpha : \gamma_\omega \to X_\alpha^\omega \) for \( \alpha \in A \). By (b), \( \{ \Phi \gamma_\omega, \{ q_\alpha \} \} \) is a bound of \( \{ \Phi X_\alpha^\omega \} \), therefore an \( \mathcal{H} : \Phi \gamma_\omega \to X_\sigma \) must exist such that \( q_\alpha = f_\alpha \circ \mathcal{H} \) for all \( \alpha \in A \). By (c) it is \( q_\alpha : \gamma_\omega \to YX_\sigma \), so it is shown that \( \langle YX_\sigma, \{ f_\alpha \} \rangle \) is a pseudoproduct of the family \( \{ X_\alpha^\omega \} \).

**Theorem 2.1.** Let a category $\mathcal{K} = A(F, G, \Delta)$ have (pseudo)products. Then also any category $\mathcal{K}_1 = A(F_1, G_1, \Delta)$ of the same type $\Delta$ but with $F_1, G_1$ being retracts of $F$ and $G$, respectively, has pseudoproducts.

Proof. Let $\Delta = \{ a_\lambda \mid \lambda < \beta \}$.

With aid of natural transformations

\[
F_1 \xrightarrow{\alpha} F \xrightarrow{\gamma} F_1, \quad G_1 \xrightarrow{\epsilon} G \xrightarrow{\pi} G_1
\]

such that $\gamma \circ (\alpha = 1_{F_1})$ and $\pi \circ \epsilon = 1_{G_1}$ define assignments

\[
\phi : \mathcal{K}_1 \to \mathcal{K}_1 \text{ and } \psi : \mathcal{K}_1 \to \mathcal{K}_1
\]

by

\[
\phi(X, \{ \omega_\lambda^X \}) = (X, \{ \sigma_\alpha^X \}) \text{ with } \sigma_\alpha^X = \pi_\lambda \cdot \omega_\lambda^X \cdot \gamma_\alpha
\]

- 55 -
and

\[ \Psi(X, \{ \omega^X_\alpha \}) = (X, \{ \omega^X_\alpha \}) \] with \( \omega^X_\alpha = \pi_\alpha \cdot \omega^X_\alpha \cdot (\omega^\omega_\alpha) \).

It is easy to show that \( \Phi \) and \( \Psi \) thus defined satisfy the conditions (a), (b), (c) of lemma 2.1.

For example, the computation in the covariant case runs as follows:

(a) \((G_t \circ \sigma^X_\alpha = (F_\tau \circ \sigma^X_\alpha)^{(\omega^\omega_\alpha)} \) with \( \sigma^Z_\alpha = \epsilon_\tau \cdot \omega^Z_\alpha \cdot (\omega^\omega_\alpha) \) implies

\[ (G_t \circ \omega^X_\alpha = \omega^X_\alpha \cdot (F_\tau \circ \omega^X_\alpha)^{(\omega^\omega_\alpha)}) \] for \( \omega^X_\alpha = \pi_\alpha \cdot \sigma^X_\alpha \cdot (\omega^\omega_\alpha) = \]

\[ (G_t \circ \omega^X_\alpha = (G_t \circ \pi_\alpha \cdot \sigma^X_\alpha \cdot (\omega^\omega_\alpha)) = \pi_\alpha \circ (G_t \circ \omega^X_\alpha \cdot (\omega^\omega_\alpha) = \]

\[ = \pi_\alpha \circ \sigma^Z_\alpha \circ (F_\tau \circ \omega^X_\alpha \cdot (\omega^\omega_\alpha) = \]

\[ = \pi_\alpha \circ \sigma^Z_\alpha \circ (\omega^Z_\alpha \circ (F_\tau \circ \omega^X_\alpha \cdot (\omega^\omega_\alpha) = \]

\[ = \pi_\alpha \circ \sigma^Z_\alpha \circ (\omega^X_\alpha \cdot (F_\tau \circ \omega^X_\alpha \cdot (\omega^\omega_\alpha) = \]

(b) \((G_t \circ \omega^Y_\alpha = \omega^Y_\alpha \cdot (F_\tau \circ \omega^Y_\alpha)^{(\omega^\omega_\alpha)} \) implies \( G_t \circ \sigma^Y_\alpha = \)

\[ = \omega^Z_\alpha \circ (F_\tau \circ \sigma^Y_\alpha)^{(\omega^\omega_\alpha)} \] for \( \sigma^Y_\alpha = \epsilon_\tau \cdot \omega^Y_\alpha \cdot (\omega^\omega_\alpha) \) and

\[ \omega^Z_\alpha = \epsilon_\tau \cdot \omega^Y_\alpha \cdot (\omega^\omega_\alpha) ; \]

\[ (G_t \circ \omega^Y_\alpha = (G_t \circ \epsilon_\tau \cdot \omega^Y_\alpha \cdot (\omega^\omega_\alpha)) = \epsilon_\tau \circ (G_t \circ \omega^Y_\alpha \cdot (\omega^\omega_\alpha) = \]

\[ = \epsilon_\tau \circ \omega^Z_\alpha \circ (F_\tau \circ \omega^Y_\alpha \cdot (\omega^\omega_\alpha) = \]

\[ = \epsilon_\tau \circ \omega^Z_\alpha \circ (F_\tau \circ \omega^Y_\alpha \cdot (\omega^\omega_\alpha) = \]

\[ = \epsilon_\tau \circ \omega^Z_\alpha \circ \omega^Y_\alpha \cdot (\omega^\omega_\alpha) \cdot (F_\tau \circ \omega^Y_\alpha \cdot (\omega^\omega_\alpha) = \]

\[ = \epsilon_\tau \circ \omega^Z_\alpha \circ \omega^Y_\alpha \cdot (\omega^\omega_\alpha) \cdot (F_\tau \circ \omega^Y_\alpha \cdot (\omega^\omega_\alpha) = \]

\[ (G_\alpha \circ \sigma^Y_\alpha = \sigma^X_\alpha \circ (F_\tau \circ \sigma^Y_\alpha)^{(\omega^\omega_\alpha)} \) with \( \sigma^Y_\alpha = \epsilon_\tau \cdot \omega^Y_\alpha \cdot (\omega^\omega_\alpha) \)

implies \((G_\alpha \circ \omega^Y_\alpha = \omega^X_\alpha \circ (F_\tau \circ \omega^Y_\alpha)^{(\omega^\omega_\alpha)} \) for

\[ \omega^X_\alpha = \pi_\alpha \cdot \sigma^X_\alpha \cdot (\omega^\omega_\alpha) ; \]
The assertion of the theorem follows by lemma 2.1.

There is another way of "collapsing" a category \( A(\mathbb{F}, \mathbb{G}, \Delta) \) so that pseudoproducts are preserved, namely, an essential reduction of the type \( \Delta \) is possible. Before stating the next theorem assume the type \( \Delta = \{ \alpha_\lambda : \lambda < \beta \} \) increasing \( \Sigma \Delta > 0 \) and denote by \( \sigma^\alpha \) the first index with \( \alpha_{\sigma^\alpha} \neq 0 \). Thus, in the case \( \sigma^\alpha > 0 \) it is \( \alpha_\lambda = 0 \) for all \( \lambda < \sigma^\alpha \) and nullary operations enter into consideration.

**Theorem 2.2.** Let a category \( A(\mathbb{F}, \mathbb{G}, \Delta) \) have pseudoproducts. If \( \sigma^\alpha > 0 \), then also the category \( A(\mathbb{F}, \mathbb{G}, \{0, 1\}) \) has pseudoproducts. If \( \sigma^\alpha = 0 \), then \( A(\mathbb{F}, \mathbb{G}, \{1\}) \) has pseudoproducts.

**Proof.** Write the objects of \( \mathcal{K} = A(\mathbb{F}, \mathbb{G}, \Delta) \) in the form \((X, \{ \sigma_\lambda^X \})\) and the objects of \( \mathcal{K}^\gamma = A(\mathbb{F}, \mathbb{G}, \{0, 1\}) \) in the case \( \sigma^\alpha > 0 \) as \((X, \{ \omega_i^X \})\) for \( i = 0, 1 \).

For every \( \lambda, \sigma^\alpha \leq \lambda < \beta \), take natural transformations \( \mu^\lambda : I \rightarrow Q_{\alpha_\lambda} \) and \( \pi^\lambda : \alpha_\lambda \rightarrow I \) such...
that $\pi^\lambda \circ \omega^\lambda = 1_1$, and define assignments

$$\phi : \kappa_i^{\omega^\lambda} \rightarrow \kappa_i^{\sigma^\lambda} \text{ and } \psi : \kappa_i^{\psi^\lambda} \rightarrow \kappa_i^{\sigma^\lambda}$$

by

$$\phi(X,\{\omega_i^X\}) = (X,\{\omega_i^X\}) \text{ with } \sigma_i^X = \omega_i^X \text{ for } \lambda < \sigma^-,$$

$$\sigma_i^X = \omega_i^X \circ \pi_i^{\sigma^-} \text{ for } \lambda \geq \sigma^-.$$

$$\psi(X,\{\sigma_i^X\}) = (X,\{\omega_i^X\}) \text{ with } \omega_i^X = \sigma_i^X, \sigma_i^X = \sigma_i^X \circ (\omega_i^{\sigma^-}).$$

In the case $\sigma^- = 0$ simply discard nullary operations $\omega_i^X$. Again, complete the proof by showing that $\phi$ and $\psi$ satisfy the conditions of lemma 2.1. We shall content ourselves with doing this for the covariant case:

(a) Assuming $(G \circ f)^{\omega_i^X} = (G \circ f)^{\omega_i^X}$ with $\sigma_i^X = \omega_i^X$

for $\lambda < \sigma^-$ and $\sigma_i^X = \omega_i^X \circ \pi_i^{\sigma^-}$ for $\lambda \geq \sigma^-$ we must prove $(G \circ f)^{\omega_i^X} = (G \circ f)^{\omega_i^X}$ for $\omega_i^X = \sigma_i^X$, $

\omega_i^X = \sigma_i^X \circ (\omega_i^{\sigma^-})$,

but

$$(G \circ f)^{\omega_i^X} = (G \circ f)^{\omega_i^X} = \omega_i^X \circ (\omega_i^{\sigma^-}) = \omega_i^X \circ (\omega_i^{\sigma^-}).$$

$$(G \circ f)^{\omega_i^X} = (G \circ f)^{\omega_i^X} = \omega_i^X \circ (\omega_i^{\sigma^-}) = \omega_i^X \circ (\omega_i^{\sigma^-}).$$

(b) Assuming $(G \circ g)^{\omega_i^X} = (G \circ g)^{\omega_i^X}$ we must prove

$(G \circ g)^{\omega_i^X} = \omega_i^X \circ (G \circ g)^{\omega_i^X}$ for $\omega_i^X = \sigma_i^X, \sigma_i^X = \omega_i^X$. 

- 58 -
if \( \lambda < \sigma \) and \( \omega_2^\lambda = \omega_1^\lambda \circ \pi_2^\lambda \), \( \omega_2^\epsilon = \omega_1^\epsilon \circ \pi_2^\epsilon \) if 
\( \lambda \geq \sigma \) but \((G_g) \circ \omega_2^\lambda = (G_g) \circ \omega_2^\epsilon = \omega_0^\lambda \circ (F_g)^{\alpha \lambda} = \omega_0^\epsilon \circ (F_g)^{\alpha \epsilon}\) for \( \lambda < \sigma \), and, \((G_g) \circ \omega_2^\lambda = (G_g) \circ \omega_1^\gamma \circ \pi_2^\lambda =\)
\[= \omega_1^\gamma \circ (F_g) \circ \pi_2^\lambda = \omega_1^\gamma \circ \pi_2^\lambda \circ (F_g)^{\alpha \lambda} = \omega_1^\gamma \circ (F_g)^{\alpha \lambda} \text{ for } \lambda \geq \sigma.\]

(a) Assuming \((G_h) \circ \omega_2^\lambda = \sigma_2^\lambda = (F_h)^{\alpha \lambda}\) with \(\sigma_2^\lambda = \omega_0^\lambda\) if \(\lambda < \sigma\) and \(\sigma_2^\lambda = \omega_1^\gamma \circ \pi_2^\lambda\) if \(\lambda \geq \sigma\), we are to prove \((G_h) \circ \omega_2^\lambda = \omega_1^\gamma \circ (F_h)^{\alpha \lambda}\) for \(\omega_1^\gamma = \sigma_2^\lambda\) and 
\[\omega_1^\gamma = \sigma_2^\lambda \circ (F_\alpha)^{\epsilon \lambda}\]
but
\[\omega_0^\lambda \circ (F_\alpha)^{\epsilon \lambda} = \sigma_2^\lambda \circ (F_\alpha)^{\epsilon \lambda} = (G_h) \circ \omega_2^\lambda = (G_h) \circ \omega_0^\lambda,\]
\[\omega_1^\gamma \circ (F_\alpha)^{\epsilon \lambda} = \sigma_2^\lambda \circ (F_\alpha)^{\epsilon \lambda} = \omega_1^\gamma \circ \omega_2^\lambda = (G_h) \circ \omega_1^\gamma \circ \pi_2^\lambda = (G_h) \circ \omega_1^\gamma \circ \pi_2^\lambda = (G_h) \circ \omega_1^\gamma.\]

Both retraction of functors and reduction of type in categories \(A(F, G, \Delta)\) by the above theorems can, of course, be made simultaneously and thus obtained categories are then the first ones to be considered when a negative result on products in some \(A(F, G, \Delta)\) is expected.

3. Contravariant case

**Theorem 3.1.** No category \(A(F, G, \Delta)\) with \(\sum \Delta > 0\) and faithful contravariant functors \(F, G\) has products.

**Proof.** Since \(P^-\) is a retract of both \(F\) and \(G\).
(Proposition 1.1), we have, with regard to results of the preceding section, but to show that $A (P^-, P^-, \{0, 1\})$ fails to have pseudoproducts. In fact, unary operations do the whole job, the following proof that $A (P^-, P^-, \{1\})$ has not pseudoproducts shows it:

Suppose that $\langle (S, \omega_x), f_x, f_y \rangle$ is a pseudoproduct of the family consisting of two objects $(X, \omega_X)$, $(Y, \omega_Y)$, where $X = \{a, b, i\}$, $Y = \{c, d\}$, and, $\omega_X$ and $\omega_Y$ are identical unary operations on $P^X$ and $P^Y$, respectively.

Take a well-ordered infinite set $\mathbb{Z} = \{x_\alpha \mid \alpha < \omega \}$ with $\text{card } \mathbb{Z} > \text{card } 2^5$ and define a bound

$\langle (Z, \omega_z), \{g_\alpha, g_\gamma \} \rangle$ by

$g_\alpha (x_\beta) = g_\alpha (x_0) = a$, $g_\alpha (x_2) = g_\alpha (x_3) = \beta$, $g_\alpha (x_0) = \beta$ for $\alpha > 3$,

$g_\gamma (x_\beta) = g_\gamma (x_0) = c$, $g_\gamma (x_1) = g_\gamma (x_0) = d$, $g_\gamma (x_0) = d$ for $\alpha > 3$;

denote $Z_\beta = \{x_\alpha \mid \alpha < \beta \}$ for $\beta < \omega$ the segments of $Z$ and put $\omega_z (x_\beta) = Z_5$, $\omega_z (Z_\beta) = Z_{\beta+1}$ for all $\beta$, $5 < \beta < \omega$, on the remaining part of $P^Z$ take $\omega_z$ identical.

There must exist $h : (Z, \omega_z) \rightarrow (S, \omega_3)$ such that

$g_\alpha = f_x \circ h$, $g_\gamma = f_y \circ h$.

Since $P^h$ is a homomorphism of $(P^S; \omega_3)$ into $(P^Z; \omega_z)$ and at the same time a homomorphism of the complete boolean algebra $(P^S; U, \cap)$ into $(P^Z; U, \cap)$, the image $\mathcal{B}$ of $P^S$ by $P^h$ must be closed under $\omega_z$ and boolean operations.
Clearly \( \{z_0, z_1, \ldots, z_n\} \in \mathcal{L} \), hence
\( \{z_0\} \in \mathcal{L} \) and \( Z_\alpha \in \mathcal{L} \). Assume \( Z_\alpha \in \mathcal{L} \) for all \( \alpha, \beta \leq \alpha < \beta \). If \( \beta \) is isolated, then \( Z_\beta = \omega_2 (Z_{\beta-1}) \in \mathcal{L} \). If \( \beta \) is a limit number, then \( Z_\beta = \bigcup_{\alpha < \beta} Z_\alpha \in \mathcal{L} \). Therefore \( \text{card} \; \mathcal{L} = \text{card} \; Z \), and this, together with \( \text{card} \; 2^S = \text{card} \; \mathcal{L} \), is a contradiction.

4. Covariant functors and their properties

It has been mentioned, that, dealing with categories \( A(F, G, \Delta) \) in the covariant case, it is important to know the behaviour of \( F \) and \( G \) with regard to sums and products, respectively. From this point of view, consider first a following separation property of functors:

Definition 4.1. A covariant functor \( F \) is said to be a separating functor if for any two disjoint subsets \( M, N \) of a set \( X \) it is

\[
(1) \quad \left( P^+ F(i_M)ight)(FM) \cap \left( P^+ F(i_N)ight)(FN) = \emptyset ,
\]

where \( i_M : M \rightarrow X, i_N : N \rightarrow X \) are the corresponding inclusions.

Denote \( 1 = \{0\} \) — a standard one-point set. For every non-void set \( X \) and an element \( \alpha \) in \( X \) define

\[
w^X_\alpha : 1 \rightarrow X \quad \text{by} \quad w^X_\alpha (0) = \alpha , \quad \text{and}, \quad \mu^X : X \rightarrow 1 \quad \text{by} \quad \mu^X _\alpha (x) = 0 \quad \text{for all} \; x \; \text{in} \; X .
\]

Statement 4.1. A functor \( F \) is separating if and only if

\[
(2) \quad w^X_\alpha \neq w^X_\beta \rightarrow \left( P^+ F(w^X_\alpha)ight)(1) \cap \left( P^+ F(w^X_\beta)ight)(1) = \emptyset .
\]
Proof. Condition (2) is equivalent to (1) with $M = \{x\}, N = \{y\}$. Condition (1) reads then as

$$[P^*F(i_{x})](F[x]) \cap [P^*F(i_{y})](F[y]) = \emptyset,$$

but $F[x] = [P^*F(w^{\text{lex}})](F[1])$, therefore

$$[P^*F(i_{x})](F[x]) = [P^*F(i_{y})](F[x]) = [P^*F(w^{\text{lex}})](F[1]) = [P^*F(w_{x}^{\text{lex}})](F[1]),$$

$$[P^*F(i_{y})](F[y]) = [P^*F(w_{y}^{X})](F[1]).$$

So the condition (2) is necessary.

Assume that (2) is fulfilled, but $F$ is not separating, that is, for some set $X$ and two disjoint subsets $M$, $N$ of $X$ we have

$$(3) \quad [P^*F(i_{M})](FM) \cap [P^*F(i_{N})](FN) \neq \emptyset.$$ 

In this case both $FM \neq \emptyset$ and $FN \neq \emptyset$, hence $M = \emptyset$ and $N = \emptyset$, since otherwise it would be $F\emptyset = \emptyset$ and $F$ would have a distinguished point, which contradicts (2).

Choose an element $x$ in $M$ and $y$ in $N$ and define mappings $f : X \rightarrow M$, $g : X \rightarrow N$ by

$$f(t) = \begin{cases} t & \text{for } t \in M, \\ x & \text{for } t \in X\setminus M \end{cases}, \quad g(t) = \begin{cases} t & \text{for } t \in N, \\ y & \text{for } t \in X\setminus N. \end{cases}$$

Note that

$$(4) \quad i_{M} \circ f \circ i_{N} \circ g = w_{x}^{X} \circ u_{X}, \quad i_{N} \circ g \circ i_{M} \circ f = w_{y}^{X} \circ u_{X},$$

$$(5) \quad f \circ i_{M} = i_{N}, \quad g \circ i_{N} = i_{N}.$$
By (3), there exist elements $p$ in $FM$ and $q$ in $FN$ such that

(6) $$(Fi_M)(p) = (Fi_N)(q) = \kappa \in FX.$$ 

It follows by (5) that $$(Ff)(\kappa) = (Ff) \circ (Fi_M)(p) = p$$ and $$\kappa = (Ff)(\kappa) = (Ff) \circ (Fi_M)(q) = q,$$

and, by (6), $$[(Fi_M) \circ (Ff)](\kappa) = \kappa, \quad [(Fi_N) \circ (Fg)](\kappa) = \kappa.$$ 

By (4) it is then $$(Fw_a^X) \circ (\mu_a^X)(\kappa) = (Fw_a^X) \circ (\mu_a^X)(\kappa),$$ 

that is, $$(Fw_a^X)(\kappa) = (Fw_a^X)(\kappa) \quad \text{for } a = (\mu_a^X)(\kappa) \in FA$$

- in contradiction with the fulfillment of (2).

For every functor $F$ different from $C_F$, denote by $F^*$ its range-domain restriction to non-void sets and mappings (such a restriction exists, since $F \neq C_F$ implies $FX \neq \emptyset$ for every non-void set $X$). Taking a standard two-point set $2 = \{0, 1\}$, denote $$Q_F = [P^*F(w_0)](F1) \cap [P^*F(w_1)](F1) \subset F2,$$

$$A_F = [P^*F(\mu_0)](Q_F).$$

For a set $X$ let $\nu_X^X : \emptyset \rightarrow X$ be the empty mapping.

**Statement 4.2.** If $A_F = \emptyset$, then $F$ is separating.

If $A_F \neq \emptyset$, then $C_F^*$ is a subfunctor of $F^*$. It is always $$[P^*F(\nu_0^X)](F\emptyset) \subset A_F.$$ 

**Proof.** First show that a non-separating functor $F$
Take a set $X$ with points $x, y, x \neq y$ such that the condition (2) does not hold for $w_x^X$ and $w_y^X$, e.g.

$$(Fw_x^X)(c) = (Fw_y^X)(d) = b \in FX \quad \text{for some } c, d \in F1.$$

Define an injection $d : 2 \to X$ by $d(0) = x$, $d(1) = y$, and, let $\kappa : X \to 2$ be a retraction of $d$, i.e. $\kappa \circ d = 1_2$. Then $w_x^2 = \kappa \circ w_x^X$, $w_y^2 = \kappa \circ w_y^X$, therefore

$$(Fw_x^2)(c) = (F\kappa)(b) = (Fw_y^2)(d) \in Q_F$$

and $A_F \neq \emptyset$.

Assume further $A_F \neq \emptyset$. The mappings $Fw_x^2$ and $Fw_y^2$ coincide on $A_F$; For an element $a$ in $A_F$ there must be elements $q, b$ in $Q_F$ and $c, d$ in $F1$ such that $a = (F\mu_2)(c)$ and $b = (Fw_y^2)(d) = (Fw_x^2)(c)$. Since $\mu_2 \circ w_x^2 = \mu_2 \circ w_y^2 = 1_2$, it is $$(F\mu_2)(c) = b = c = a.$$

Moreover, for every non-void set $X$ all mappings $Fw_x^X$ for $x \in X$ coincide on $A_F$: Take $x, y$ in $X$, $x \neq y$, and the injection $d : 2 \to X$ as above, then $w_x^X = d \circ w_x^2$, $w_y^X = d \circ w_y^2$ and the preceding assertion applies.

Now, define a transformation $\mu : C_{A_F}^* \to F^*$ by $\mu_x^X(a) = (Fw_x^X)(a)$ for $a \in A_F$ and $x \in X$.

Clearly, $\mu_x^X$ does not depend on the choice of $x$ in $X$, it is an injection (for $w_x^X$ is an injection), and
it is a transformation because of $f \circ w^X = w^Y$ for every $f : X \to Y$.

As to the last assertion of the statement 4.2

$$w^2 \circ \varphi^2 = \varphi^0 = w^1 \circ \varphi^1$$

implies $\mathcal{P} \circ f(\varphi^0) \subseteq Q$, and we get the assertion using $\varphi^0 = \mu_2 \circ \varphi^2$

**Statement 4.3.** Every functor $F : F \neq C_f$ can be written as

$$F = F_d \vee F_s,$$

where functors $F_d$ and $F_s$ have following properties:

a) $F_d$ is $C_f$ or $F^*_d$ has a subfunctor $C^*_d$;

b) $F_s$ is the greatest separating subfunctor of $F$ in the sense that every separating subfunctor of $F$ is a subfunctor of $F_s$.

This decomposition of $F$ is unique up to the natural equivalence.

**Proof.** Denote $\tilde{A}_F = (F \setminus A_F)$ - the complement of $A_F$ in $F$ and for every non-void set $X$ put

$$F_d X = [\mathcal{P} \circ f(\mu^X)](A_f), \quad F_s X = [\mathcal{P} \circ f(\mu^X)](\tilde{A}_F).$$

For an arbitrary mapping $f : X \to Y$ it is $\mu^X = \mu^Y \circ f$, therefore $[\mathcal{P} \circ f(\mu^X)](F_d X) \subseteq F_d Y$ and $[\mathcal{P} \circ f(\mu^X)](F_s X) \subseteq F_s Y$.

Define $F_d f$ and $F_s f$, accordingly, as range-domain restrictions of $F f$.

It is proved that for, that $F^* = F_d^* \vee F_s^*$. 
We can now define $F^0 \rightarrow F^0$ and $f^t$ as a domain restriction of $F^t \rightarrow X$, and, $F^0 \rightarrow \emptyset$, $F^t \rightarrow X : \emptyset \rightarrow F^t X$.

It is easily seen that $A_{F^t} = A_F$ and $A_{F^t} - \emptyset$, therefore, by statement 4.2, if $A_F = \emptyset$ then $C_{F^t}$ is a subfunctor of $F^t_X$ and $F^t$ is a separating functor.

Finally, let $\lambda : G \rightarrow F$ be a monomorphism of a separating functor $G$ into $F$. Then necessarily $\lambda^t (t) \in F^t \rightarrow X$ for every $t \in G$, therefore $P^t (\lambda^t) (X) = F^t X$ for every $X \neq \emptyset$, and, of course, $G \emptyset = \emptyset = F^t \emptyset$.

This property of $F^t$ secures uniqueness of the decomposition.

**Corollary** (to Statement 4.1). Every separating functor $F$ is faithful and $F\emptyset = \emptyset$.

**Proof.** Assume $Ff = Fg$ for some mappings $f, g : X \rightarrow Y$. Then $Fw^Y_{f(x)} = F(g \circ w^X_x) = F(g \circ w^X_x) = Fw^Y_{g(x)}$ for all $x$ in $X$, therefore, by (2), $f(x) = g(x)$ for all $x$ in $X$, i.e. $f = g$.

**Definition 4.2.** A functor $F$ is said to be **tight on** $X$, $X \neq \emptyset$, if

$$\bigcup_{x \in X} [P^+ F(w^X_x)] (F 1) = F X.$$  

If this identity does not hold, then $F$ is **loose on** $X$.

If $F$ is tight on every $X$, $X \neq \emptyset$, then it is a **tight functor**, otherwise it is a **loose functor**.
Statement 4.4. If $F$ is loose on $Y$, $Y \neq \emptyset$, and $Y \subset X$, then $F$ is loose on $X$.

Proof. Denote $i_Y: Y \to X$ an inclusion of $Y$ into $X$ and choose some retract $r: X \to Y$ of $i_Y$. Then $r \circ w_x^Y = w_{r(x)}^X$ for every $x$ in $X$. Now, assume that $F$ is tight on $X$, that is, (7) holds. Since $r$ is a surjection, we get

$$F_Y = [P^+ \circ F(r)](FX) = [P^+ \circ F(r)](\bigcup_{x \in X} [P^+ \circ F(w_x^X)](F1)) =$$

$$= \bigcup_{x \in X} [P^+ \circ F(w_x^X)](F1) =$$

$$= \bigcup_{x \in X} [P^+ \circ F(w_{r(x)}^Y)](F1) = \bigcup_{y \in Y} [P^+ \circ F(w_y^Y)](F1)$$

in contradiction with looseness of $F$ on $Y$.

Corollary. If $F$ is loose on a set $X$, $X \neq \emptyset$, then it is loose on every set $Y$ with $\text{card } Y \geq \text{card } X$.

Equivalently, if $F$ is tight on $X$, then it is tight on every $Y$, $Y \neq \emptyset$ with $\text{card } Y \leq \text{card } X$.

Proof. Immediate consequence of Statement 4.4.

Define $w_x^X: I \to X$ by $w_x^X(0) = x$, $w_x^X(1) = y$. For a given functor $F$ denote $W^X_{x,y} = [P^+ \circ F(w_x^X)](F1)$, $w^X_x = [P^+ \circ F(w_x^X)](F1)$.

Statement 4.5. Let a functor $F$ be loose on a given set $X$ with $\text{card } X > 2$, i.e.

$$FX \setminus \bigcup_{x \in X} W^X_{x,a} \neq \emptyset.$$ 

Then

$$FX \setminus \bigcup_{x \in X} W^X_{x,a} \neq \emptyset$$

for arbitrary $a$ in $X$. 

- 67 -
Proof. First note that $W_{x \in X}^x = W_x^x$ for $x$ in $X$ and for any mapping $f : X \rightarrow X$ it is

\[ [P^+ \circ F(f)](W_{x \in X}^x) = W_{x \in X}^x \cdot \]

Assume, now, that $\bigcup_{x \in X} W_x^x = F X$ for some $a$ in $X$.

Choose an element $\mu$ in $F X \setminus \bigcup_{x \in X} W_x^x$. Then for some $x$, $x + a$, $\mu \in W_x^x$. Take an element $b$ in $X$ so that $b \neq a$, $b + x$, and a bijection $f : X \rightarrow X$ such that $f(b) = a$. Then

\[ [P^+ \circ F(f)](\bigcup_{x \in X} W_x^x) = \bigcup_{x \in X} [P^+ \circ F(f)](W_x^x) = \]

\[ = \bigcup_{x \in X} W_{a + x}^x = \bigcup_{x \in X} W_x^x = F X , \]

therefore $\bigcup_{x \in X} W_x^x = F X$, and, for some $y \neq b$, it is

$\mu \in W_b^x$.

It remains to show that $\mu \in W_a^x \cap W_{b+y}^x$ leads to a contradiction: Take a mapping $g : X \rightarrow X$ such that

\[ g(a) = a, \ g(x) = x, \ g(b') = g(y) = \begin{cases} x & \text{if } y \neq a \end{cases} . \]

Then

\[ \nu = (F g)(\mu) \in [P^+ \circ F(g)](W_a^x \cap W_{b+y}^x) \subset \]

\[ \subset W_a^x \cap W_{g(b) \cdot g(y)}^x \subset W_a^x \cup W_{a \cdot x}^x . \]

Statement 4.6. If $F$ is tight, then for every set $X$ and for its arbitrary two subsets $M, N$ it holds

(3) $[P^+ F(i_M^x)(FM) \cup [P^+ F(i_N^x)](FN) = [P^+ F(i_S^x)](FS)$
where $S = M \cup N$, and, $i^X_M : M \rightarrow X$, $i^X_N : N \rightarrow X$, $i^X_S : S \rightarrow X$ are the respective inclusions of $M, N, S$ into $X$.

**Proof.** Denote $i^S_M : M \rightarrow S$, $i^S_N : N \rightarrow S$ the inclusions of $M, N$ into $S$, respectively. Then we have

$$(9) \quad i^X_M = i^X_S \circ i^S_M, \quad i^X_N = i^X_S \circ i^S_N.$$ 

It is easy to see that (8) holds, if one of the sets $M, N, S$ is void. Assume further that $M \neq \emptyset$, $N \neq \emptyset$. Then, by tightness of $F$,

$$FM = \bigcup_{x \in M} [P^+ F(w^X_x)](F \mathcal{I}), \quad FN = \bigcup_{x \in N} [P^+ F(w^X_x)](F \mathcal{I}).$$

Using (9), we get

$$[P^+ F(i^X_M)](FM) = [P^+ F(i^X_S)](\bigcup_{x \in M} [P^+ F(w^X_x)](F \mathcal{I})) =$$

$$= \bigcup_{x \in M} [P^+ F(i^X_S \circ w^X_x)](F \mathcal{I}) = \bigcup_{x \in M} [P^+ F(w^X_x)](F \mathcal{I}),$$

and, similarly

$$[P^+ F(i^X_N)](FN) = \bigcup_{x \in N} [P^+ F(w^X_x)](F \mathcal{I}),$$

therefore

$$[P^+ F(i^X_M)](FM) \cup [P^+ F(i^X_N)](FN) = \bigcup_{x \in S} [P^+ F(w^X_x)](F \mathcal{I}),$$

but $w^X_x = i^X_S \circ w^X_x$ for $x$ in $S$, so it is, finally,

$$\bigcup_{x \in S} [P^+ F(w^X_x)](F \mathcal{I}) = [P^+ F(i^X_S)](\bigcup_{x \in S} [P^+ F(w^X_x)](F \mathcal{I})) =$$

$$= [P^+ F(i^X_S)](FS)$$

by tightness of $F$. 

- 69 -
Tight separating functors are exactly the functors preserving sums. Let us formulate this as

**Statement 4.7.** If $F$ does not preserve sums, then $F$ is either loose or it is not separating.

**Remark.** Denote by $\mathcal{P}$, $\mathcal{Y}$, $\mathcal{L}$ the systems of all separating, tight, loose functors, respectively. Each of these systems is closed under $\vee, \times, \circ$ for functors, $\mathcal{P}$ is closed on subfunctors, $\mathcal{Y}$ is closed on subfunctors and factor-functors, $\mathcal{L}$ is closed on extensions ($F \in \mathcal{L}$, $F \xrightarrow{\mu} F' \Rightarrow F' \in \mathcal{L}$). Every $F$ in $\mathcal{Y}$ splits by statement 4.3 into $F_\nu \times F_\mu$ such that $F_\nu \cong C^*_{F_{\alpha}}$ and $F_\mu$ preserves sums.

It is $I \in \mathcal{P} \cap \mathcal{Y}$, constant functors $C^*_M$ are in $\mathcal{Y}, N, \mathcal{P}^h, \beta \in \mathcal{L}$, $\alpha \in \mathcal{L}$ for $\text{card } M \geq 2$.

Turn now to range functors.

**Statement 4.8.** If $G$ does not preserve the product of a family $\{X_\alpha \mid \alpha \in A\}$, then it does not preserve the product of any family $\{Y_\alpha \mid \alpha \in A\}$ with $\text{card } Y_\alpha \geq \text{card } X_\alpha$ for all $\alpha$ in $A$.

**Proof.** Choose for each $\alpha$ in $A$ mappings $i_\alpha : X_\alpha \rightarrow Y_\alpha$, $\pi_\alpha : Y_\alpha \rightarrow X_\alpha$ such that $\pi_\alpha \circ i_\alpha = 1_{X_\alpha}$. Denote $<X, \{X_\alpha\}>$ and $<Y, \{Y_\alpha\}>$ the products of $\{X_\alpha\}$ and $\{Y_\alpha\}$, respectively. Define mappings $i : X \rightarrow Y$ and $\pi : Y \rightarrow X$ by

$$i_\alpha \circ \pi_\alpha = \pi_\alpha \circ i, \quad \pi_\alpha \circ \pi_\alpha = \pi_\alpha \circ \nu.$$

It is then $\nu \circ i = 1_X$.

Assume that $G$ preserves the product of $\{Y_\alpha\}$ and
show that then it preserves the product of \( \{ X_\alpha \} \) too:

For an arbitrary family \( \{ X_\alpha \}, \ x_\alpha \in G \times X_\alpha \) for \( \alpha \in A \), there must exist \( y \in G \times Y \) such that

\[
(G_\pi^Y)(y) = (G_i_\alpha)(x_\alpha),
\]

and, using (10), we get

\[
(G_\pi^X)(x) = x_\alpha \quad \text{for} \quad x = (G_\pi^X)(y)
\]

by easy calculation. The element \( x \) with \( (G_\pi^X)(x) = x_\alpha \) must be unique, since \( (G_\pi^X)(x_1) = (G_\pi^X)(x_2) \) implies

\[
(G_\pi^X)(\psi_1) = (G_\pi^X)(\psi_2) \quad \text{for} \quad \psi_1 = (G_i)(x_1), \ \psi_2 = (G_i)(x_2)
\]

by simple calculation using (10).

Next three definitions reflect certain properties of the functors not preserving products.

Let \( \mathcal{X} = \{ X_\alpha \mid \alpha \in A \} \) be a family of sets. Denote by \( < X, \{ \pi_\alpha^X \} > \) its product \( X = \prod_{\alpha \in A} X_\alpha \) with \( \pi_\alpha^X : X \to X_\alpha \) - the ordinary projections. If a functor \( G \) does not preserve the product of the family \( \mathcal{X} \), then either

(I) there exists a family \( \{ X_\alpha \}, X_\alpha \in G \times X_\alpha \) for \( \alpha \in A \), such that there is no \( x \) in \( G \times X \) with \( (G_\pi^X)(x) = (x_\alpha) \) for all \( \alpha \) in \( A \),

or

(II) there exist two points \( x, y \in G \times X \), \( x \neq y \), such that \( (G_\pi^X)(x) = (G_\pi^X)(y) \) for all \( \alpha \) in \( A \).

**Definition 4.3.** A functor \( G \) not preserving products is said to **blow up** products if for some family of sets the alternative (II) takes place. If, moreover, the alterna-
tive (I) takes place for no family, then \( G \) is said to inflate products.

Definition 4.4. A functor \( G \) not preserving products is said to filtrate products, if for an arbitrary family \( \{ X_\alpha | \alpha \in A \} \) with the product \( \langle X, \{ \pi_\alpha \} \rangle \) the family of mappings \( \{ G\pi_\alpha | \alpha \in A \} \) is separating on \( GX \) in the sense that

\[
(11) \quad \forall \alpha \in A ((G\pi_\alpha)(x) = (G\pi_\alpha)(y)) \Rightarrow x = y
\]

for \( x, y \) in \( GX \).

Remark. The system of all functors with the property (11) is closed under \( \vee, \times, \circ \) and subfunctors. We obtain the system \( \mathcal{F} \) of filtrating functors by removing functors preserving products.

Definition 4.5. A functor \( G \) superinflates products if there exists a family \( \{ X_\alpha | \alpha \in A \} \) of non-void sets with the following property:

There exist \( x_\alpha \) in \( X_\alpha \) and \( y_\alpha \) in \( GX_\alpha \) for all \( \alpha \in A \) such that, denoting \( \langle X, \{ \pi_\alpha \} \rangle \) the product of \( \{ X_\alpha \} \), for an arbitrary set \( S \) and mappings \( \sigma_\alpha : X \times S \to X_\alpha \) such that \( \sigma_\alpha | X = \pi_\alpha \) and

\[
\sigma_\alpha(\beta) = x_\alpha \quad \text{for all } \beta \text{ in } S,
\]

it holds

\[
\text{card}\{ x \in G(X \times S) | (G\sigma_\alpha)(x) = y_\alpha \} \quad \text{for all } \alpha \text{ in } A \}
\]

Statement 4.2. The functors \( N, \beta, \langle P^-, I \rangle \) superflate products. For the system \( \mathcal{H} \) of functors superinflating products it holds:
(α) \( G \) has a subfunctor belonging to \( \mathcal{H} \to G \in \mathcal{H} \),
(β) \( F \times \mathcal{H} \subseteq \mathcal{H} \) for any functor \( F \),
(γ) \( F \) is a covariant faithful functor \( \implies F \circ \mathcal{H} \subseteq \mathcal{H} \),
(δ) \( F, G \) are contravariant faithful \( \implies F \circ G \in \mathcal{H} \),
(ε) \( F \) is contravariant faithful or constant, \( G \in \mathcal{H} \) \( \implies \langle F, G \rangle \in \mathcal{H} \).

**Proof.**
1) \( N \) superinflates products; choose \( X_1 = \{a, b, \bar{a}\}, X_2 = \{c, d, \bar{c}\}, x_1 = a, x_2 = c, \gamma_1 = \{a, b, \bar{a}\}, \gamma_2 = \{c, d, \bar{c}\} \), then the family \( \{X_1, X_2\} \) and points \( x_1, x_2, \gamma_1, \gamma_2 \) meet the requirements of the definition 4.5.

2) \( \beta \) superinflates products; choose \( X_m = \{a_m, b_m\}, \)
\[ n = 1, 2, 3, \ldots, x_m = a_m, \gamma_m = \{a_m, b_m\} \] , then the (countable) system \( \{X_m | m = 1, 2, \ldots\} \) and points \( x_m, \gamma_m \) meet the requirements. (If \( \text{card} \ S < \aleph_0 \) , use the fact that \( \forall x \in \beta X \ \exists (\beta S_m)(x) = \gamma_m \) for \( n = 1, 2, \ldots \geq 2^{\aleph_0} \), if \( \text{card} \ S \geq \aleph_0 \) , then use \( \text{card} \ (\beta S) = 2^{\text{card} S} \).)

3) \( \langle \mathcal{P}^-, 1 \rangle \) superinflates products; again choose the family \( \{X_1, X_2\} \) where \( X_1 = \{a, b, \bar{a}\}, X_2 = \{c, d, \bar{c}\}, x_1 = a, x_2 = c, \gamma_1 : \mathcal{P}^{-}X_1 \to X_1 \) is the constant mapping to \( a \), \( \gamma_2 : \mathcal{P}^{-}X_2 \to X_2 \) is the constant mapping to \( c \).

The assertions (α) - (ε) can be easily proved with aid of the Proposition 1.1.
5. **Covariant case.** We suppose always $F \neq C_0$, $G \neq C_\beta$.

**Theorem 5.1.** Let $A(F, G, \Delta)$ be a category whose type $\Delta = \{ x_\alpha | \alpha < \beta \}$ contains zeros, say, $x_0 = 0$. Then $A(F, G, \Delta)$ has products if and only if $G$ preserves products.

**Proof.** If $G$ preserves products, then, clearly, $A(F, G, \Delta)$ has products, so we have to show the converse implication.

Take an arbitrary family $\{ X_\alpha \mid \alpha \in A \}$ of non-void sets and choose a family $\{ x_\alpha \in GX_\alpha \mid \alpha \in A \}$. Denote $\langle X, \{ \pi^*_\alpha | x \in A \} \rangle$ the product of $\{ X_\alpha \}$ with $\pi^*_\alpha$ — the ordinary projections. We must show that

(a) there exists an element $x$ in $GX_\Delta$ such that $(G\pi^*_\alpha)(x) = x_\alpha$ for all $\alpha$ in $A$,

(b) if for some $x, y$ in $GX_\Delta$ it is $(G\pi^*_\alpha)(x) = (G\pi^*_\alpha)(y) = x_\alpha$ for all $\alpha$ in $A$, then $x = y$.

By theorem 2.2, the category $A(F, G, \{ 0, 1 \})$ has pseudoproducts. To show (a), take the family

$\langle \{ x_\alpha, \{ \sigma^\alpha, \sigma^\alpha_1 \} \mid \alpha \in A \} \rangle$ of objects of $A(F, G, \{ 0, 1 \})$ with operations defined so that for each $\alpha$, $\sigma^\alpha_0$ selects $x_\alpha$ in $GX_\alpha$ and $\sigma^\alpha_1$ carries the whole $FX_\alpha$ into $x_\alpha$.

Let $\langle (S, \{ \sigma^S, \sigma^S_1 \}, \{ \sigma^S_1 \}) \rangle$ be a pseudoproduct of this family. There exists a mapping $h : S \to X$.

Unary operations play no role in our proof and it works in the case $x_\alpha = 0$ for all $\alpha$, $\alpha < \beta$, as well.
such that
\[(1) \quad \mathcal{G}_\alpha = \pi_\alpha \circ \lambda \quad \text{for all } \alpha \text{ in } A.\]

Denote $\rho$ the element in $G \mathcal{S}$ selected by $\mathcal{G}_\alpha^S$. For $\lambda = (G \mathcal{L}) (\rho)$ it is $(G \pi_\alpha) (\lambda) = (G \mathcal{L}) \circ (G \lambda) (\rho) = (G \mathcal{G}_\alpha) (\rho) = \lambda$ for all $\alpha$ in $A$, as required.

To prove (b), assume $(G \pi_\alpha) (\lambda) = (G \mathcal{G}_\alpha) (\mu) = \lambda$ for all $\alpha$ in $A$, and take inverse bounds $\langle (X, \{\sigma_\alpha^X, \sigma_\alpha^X\}) \rangle$, $\langle \pi_\alpha \rangle$ with $\sigma_\alpha^X$ selecting $\lambda$ and $\sigma_\alpha^X$ carrying $FX$ into $\lambda$ and $\langle (X, \{\omega_\alpha^X, \omega_\alpha^X\}) \rangle$, $\langle \pi_\alpha \rangle$ with $\omega_\alpha^X$ carrying $FX$ into $\mu$ selected by $\omega_\alpha^X$.

Let $f, g : X \to S$ be the respective factoring morphisms, that is
\[(2) \quad \pi_\alpha = \mathcal{G}_\alpha \circ f = \mathcal{G}_\alpha \circ g \quad \text{for all } \alpha \text{ in } A,\]
and, in particular,
\[(3) \quad (Gf) (\lambda) = (Gg) (\mu) = \lambda.\]

By (1) and (2) we get $\lambda \circ f = \lambda \circ g = 1_\lambda$, which applied to (3) gives $\lambda = \mu = (G \lambda) (\rho)$.

Consider further only categories $A (F, G, \Delta)$ with a completely positive type $\Delta = \{ \lambda \in A \mid \lambda < \beta \}$, i.e. $\lambda > 0$ for all $\lambda$, $\lambda < \beta$. As a corollary of theorem 5.1 we get

**Theorem 5.2.** If $F \emptyset \neq \emptyset$ and $G$ does not preserve products, then a category $A (F, G, \Delta)$ has not products.

**Proof.** Assume that $A (F, G, \Delta)$ has products. Then $A (C_f, G, \{1\})$ has pseudoproducts, by theorems 2.1 and 2.2, since $F \emptyset \neq \emptyset$ means that $C_f$ is a retract of $F$.

Now, unary operations $\sigma^X : C_f \to GX$ just

- 75 -
select a point in \( G \times X \), therefore \( A(C, G, \{ t \}) \) coincides with \( A(C, G, \{ 0 \}) \) which fails to have pseudoproducts by theorem 5.1, in contradiction with our assumption.

**Theorem 5.1.** Let \( A(F, G, \Delta) \) be a category of a type \( \Delta = \{ \alpha \in \Lambda | \lambda < \beta \} \) with a range-functor \( G \) not preserving products.

If the functor \( Q_{\alpha} \circ F \) is loose for some \( \lambda, \lambda < \beta \), then \( A(F, G, \Delta) \) has not products.

**Proof.** Assume \( Q_{\alpha} \circ F \) loose. Combining statements 4.4 and 4.8 of the preceding section find a set \( X \) such that \( Q_{\alpha} \circ F \) is loose on \( X \) and \( G \) does not preserve a power \( \langle X^a, \{ \pi_\alpha | \alpha \in \Lambda \} \rangle \) for a suitable set \( A \).

(I) Denote \( P = X^a \) and first assume that for some family \{ \( x_\alpha \in G(\times X) \mid \alpha \in \Lambda \} \) there is no point \( v \) in \( G P \) with \( (G \pi_\alpha)(v) = x_\alpha \) for all \( \alpha \) in \( A \).

Using the notation introduced in statement 4.5, define operations \( \sigma_\alpha^\alpha : (FX)^{\pi_\alpha} \to GX, \alpha \in \Lambda, \lambda < \beta \), as follows:

Choose an element \( a \) in \( X \) and an element \( a \) in the part \( \{ P^* G(\times X) \} (G \Delta) \) of \( G2 \), denote \( D_\alpha^a = \bigcup_{x \in X} \{ P^* Q_{\alpha} \circ F(\times X) \} (F \Delta), \sigma_\alpha^\alpha = (G \times X)(a) \), and put

\[
\sigma_\alpha^\alpha(t) = \begin{cases} 
\sigma_\alpha^a \quad \text{for } t \in D_\alpha^a \\
\sigma_\alpha^\alpha \quad \text{for } t \in (FX)^{\pi_\alpha} \setminus D_\alpha^a
\end{cases}
\]

Define \( \sigma_\alpha^\alpha(c) = c \) for all \( t \) in

- 76 -
and note that every \( w_{a,x}^x \), \( x \in X \), is a morphism of \( (\mathcal{L}, \{ \sigma_\alpha^2 \}) \) into \( (X, \{ \sigma_x^X \}) \), since
\[
(G w_{a,x}^x)(d) = (G w_{a,x}^x)(d) \quad \text{for every } x \text{ in } X. \]
Therefore \( \langle (\mathcal{L}, \{ \sigma_\alpha^2 \}), \{ w_{a,x}^x(\alpha) \} \rangle \) with an arbitrary \( \varphi : A \to X \) is an inverse bound of the family
\[
\{ (X, \{ \sigma_x^X \}) | \alpha \in A \} = \mathcal{X}. \]

Suppose that \( \langle (S, \{ \sigma_\alpha^S \}), \{ \sigma_x^S \} \rangle \) is a product of \( \mathcal{X} \) and denote \( \mathbf{h} : S \to P \) the mapping uniquely determined by
\[
(5) \quad \sigma_x^x = \pi_\alpha \circ \mathbf{h} \quad \text{for all } \alpha \text{ in } A.
\]

Denote \( f_x : \mathcal{L} \to S \), \( x \in X \), factoring morphisms of inverse bounds \( \langle (\mathcal{L}, \{ \sigma_\alpha^2 \}), \{ w_{a,x}^x(\alpha) \} \rangle \) with \( \varphi(\alpha) = x \) for all \( \alpha \) in \( A \), i.e.
\[
w_{a,x}^x = \sigma_x^x \circ f_x \quad \text{for all } \alpha \text{ in } A.
\]

Then for a mapping \( \tau : X \to S \) defined by \( \tau(x) = f_x(1) \) it is \( x = w_{a,x}^x(1) = \sigma_x^x \circ f_x(1) = \sigma_x^x \circ \tau(x) \), hence
\[
(6) \quad \sigma_x^x \circ \tau = 1_x \quad \text{for all } \alpha \text{ in } A.
\]

Now, by statement 4.5, choose \( t \) in \( (FX)^{\mathcal{X}} \setminus D^{\mathcal{X}} \), denote \( \eta = (F \tau)(\alpha_x^\alpha)(t) \), \( \nu = (G \mathbf{h})(\alpha_x^\alpha(\lambda)) \), and, using (5) and (6), get
\[
(G \mathbf{h})(\nu) = (G \mathbf{h})(\sigma_x^\alpha(\lambda)) = (G \mathbf{h})(\sigma_x^\alpha(\lambda)) =
\]
\[
\sigma_x^\alpha \circ (F \sigma_x^\alpha)(\sigma_x^\alpha)(\lambda) = \sigma_x^\alpha \circ (F \sigma_x^\alpha)(\sigma_x^\alpha)(\lambda) = \sigma_x^\alpha(t) = \lambda_x \]
for all \( \alpha \) in \( A \), in contradiction with our assumption.

(II) Assume further that (I) happens for no family in \( G \mathcal{X} \),
but for a family \( \{ x_\alpha \in \mathcal{G} X \mid \alpha \in A \} \) there are \( \nu, \nu' \) in \( \mathcal{P} \), \( \nu \neq \nu' \), such that \( (G \pi^\alpha_\alpha)(\nu) = (G \pi^\alpha_\alpha)(\nu') = x_\alpha \) for all \( \alpha \) in \( A \).

Take again the family \( \{(X, \{ \sigma^\alpha_\alpha \}) \mid \alpha \in A \} \) with operations defined by (4) and suppose that it has a product \( \langle (S, \{ \sigma^S_\alpha \}), \{ \pi^S_\alpha \} \rangle \).

Define inverse bounds \( \langle (P, \{ \sigma^P_\alpha \}), \{ \pi^P_\alpha \} \rangle \) and \( \langle (P, \{ \omega^P_\alpha \}), \{ \pi^P_\alpha \} \rangle \) as follows:

Define \( \mu : X \to P \) by \( \pi^\alpha_\alpha \circ \mu = 1_X \) for all \( \alpha \) in \( A \), denote \( \mathcal{D}_P^\alpha = \bigcup_{\mu \in \mathcal{P}} [P^+ \circ \mathcal{Q}_p \circ F(w^{P}_{\omega(\mu)}), 1] (F2) \), \( d_p = (G w^{P}_{\omega(\mu)}) (d) \), and put \( \sigma^P_\mu (\mu) = \omega^P_\mu (\mu) = d_p \) for \( \mu \in \mathcal{D}_P^\alpha, \sigma^P_\mu (\mu) = \nu, \omega^P_\mu (\mu) = \nu' \) for \( \mu \in [P^+ \circ \mathcal{Q}_p \circ F(\mu)](FX) \setminus \mathcal{D}_P^\alpha, \) on the rest of \( (FP)^{a \alpha} \) define \( \sigma^P_\mu \) and \( \omega^P_\mu \) so that all \( \pi^\alpha_\alpha \) become morphisms, which is possible by our assumption.

Note that all \( w^{P}_{\omega(\mu)} \), \( \mu \in \mathcal{P} \), are morphisms of \( (2, \{ \sigma^S_\alpha \}) \) into both \( (P, \{ \sigma^P_\alpha \}) \) and \( (P, \{ \omega^P_\alpha \}) \). Let \( f, f' : P \to S \) be the respective morphisms of \( (P, \{ \sigma^P_\alpha \}) \) and \( (P, \{ \omega^P_\alpha \}) \) into \( (S, \{ \sigma^S_\alpha \}) \) with \( \pi^\alpha_\alpha = \sigma^\alpha_\alpha \circ f = \sigma^\alpha_\alpha \circ f' \) for all \( \alpha \) in \( A \).

Together with (5) we get \( \mu \circ f = \mu \circ f' = 1_{P} \) so \( f \) and \( f' \) are injections, and it cannot be \( f = f' \), since then it would be \( \sigma^S_\mu \circ (Ff)(\mu) = (Gf)(\nu) = (Gf)(\nu') \) for any \( \mu \) in \( [P^+ \circ \mathcal{Q}_p \circ F(\mu)](FX) \setminus \mathcal{D}_P^\alpha \).
Therefore it is $f(\mu^*) = f'(\mu^*)$ for some $\mu^*$ in $P$.

Now, $\langle (2, \{\sigma_2^2\}), \{\pi_\alpha \circ w_{\mu(a)p_*}\} \rangle$ is an inverse bound of $\mathcal{X}$ with two different factoring morphisms through $\langle (S, \{\sigma_2^2\}), \{\sigma_\alpha\} \rangle$, namely, $f \circ w_{\mu(a)p_*}$ and $f' \circ w_{\mu(a)p_*}$.

As a simple corollary we have

**Theorem 5.4.** If $F$ is faithful, $G$ not preserving products, and, $\Delta$ contains a number $\alpha_\lambda$ different from 1, then $A(F, G, \Delta)$ has not products.

**Proof.** $Q_{\alpha_\lambda} \circ F$ has a subfunctor $Q_{\alpha_\lambda}$ which is loose for $\alpha_\lambda > 1$.

**Theorem 5.5.** If $F$ is not separating and $G$ blows up products, then $A(F, G, \Delta)$ has not products.

**Proof.** Assume that for a family $\{X_\alpha | \alpha \in A\}$ with the product $\langle P, \{\pi_\alpha\} \rangle$ there are $\nu$, $\nu'$ in $GP$, $\nu \neq \nu'$, such that $(G\pi_\alpha)(\nu) = (G\pi_\alpha)(\nu')$ for all $\alpha$ in $A$.

Take a family $\{(X_\alpha, \sigma_\alpha) | \alpha \in A\}$ of objects of $A(F, G, \{i\})$ with $\sigma_\alpha(t) = (G\sigma_\alpha)(\nu)$ for all $t$ in $FX_\alpha$, $\alpha \in A$, and, suppose that the family has a pseudoproduct $\langle (S, \sigma_\alpha), \{\sigma_\alpha\} \rangle$.

Define inverse bounds $\langle (P, \sigma_\alpha), \{\sigma_\alpha\} \rangle$ and $\langle (P, \sigma'_\alpha), \{\sigma_\alpha\} \rangle$ by $\sigma_\alpha(t) = \nu$, $\sigma'_\alpha(t) = \nu'$.
for all \( t \) in \( FP \), denote \( f, f' : P \to S \) the corresponding morphisms such that \( \pi_{\alpha} = \sigma_{\alpha} \cdot f = \sigma_{\alpha} \cdot f' \) for all \( \alpha \) in \( A \).

If \( F \) is not separating, then there exists an element \( t \) in \( FP \) such that \( (Ff)(t) = (Ff')(t) = \mu \).

It is then
\[
(Gf)(\nu) = (Gf')(\nu') = \sigma_S(\mu).
\]

Now, \( f \) and \( f' \) have a common retraction \( h : S \to P \) defined by \( \sigma_{\alpha} = \pi_{\alpha} \cdot h, \alpha \in A \), that is, \( h \circ f = h \circ f' = 1_P \). Applying to the identity (7) we get \( \nu = \nu' \) - a contradiction.

Let us call a type \( \Delta = \{ \alpha | \lambda < \beta \} \) with \( \alpha_\lambda = 1 \) for all \( \lambda, \lambda < \beta \), a unary type.

**Theorem 5.6.** A category \( A(F, G, \Delta) \) with \( G \) not preserving products and whose type is not unary has products if and only if \( F\emptyset = \emptyset, F^* \cong C^*_m \) and \( G \) filtrates products (\( F^* \) is a range-domain restriction to non-void sets and mappings).

**Proof.** If \( A(F, G, \Delta) \) has products, then \( F \) is neither loose nor faithful. Therefore \( Fw_x^y = Fw_y^x \) for arbitrary \( x, y \) in \( X \) and \( Fw_x^x \) is - by tightness - a bijection between \( FA \) and \( FX \) independent of choice of \( x \) in \( X \) . Putting \( \varepsilon^x = Fw_x^x \) we obtain a nat. equivalence \( \varepsilon : C_{F^*} \to F^* \).

Since \( F \) is not separating, \( G \) must then, by theorem 5.5, filtrate products.
The condition $F\emptyset = \emptyset$ has been established by theorem 5.2.

Assume, conversely, that the conditions imposed on $F$ and $G$ are fulfilled. Let

$\mathcal{X} = \{(X_\alpha, \{\sigma_\alpha^x | \lambda < \beta \} | \alpha \in A) | \alpha \in A \}$ be an arbitrary family of objects of $A(F, G, \Delta)$. Let $\langle P, \{\pi_\alpha \} \rangle$ be the product $P = \prod_{\alpha \in A} X_\alpha$ with ordinary projections.

If, for some $m$ in $M^A$, there is no $\mu$ in $GP$ such that $(G\pi_\alpha)(\mu) = \sigma_\alpha^x(m)$ for all $\alpha$ in $A$, then every inverse bound $\langle (Y, \{\sigma_\alpha^y \}^x, \{\eta_\alpha \} \rangle$ of $\mathcal{X}$ must be void and is, in fact, a product of $\mathcal{X}$.

If, for every $m$ in $M^A$, $\lambda < \beta$, there exists some $\mu$ in $GP$ such that $(G\pi_\alpha)(\mu) = \sigma_\alpha^x(m)$ for all $\alpha$ in $A$, then $\langle (P, \{\sigma_\alpha^p \}^x, \{\pi_\alpha \} \rangle$ with $\sigma_\alpha^p$ defined by

$$(G\pi_\alpha)\sigma_\alpha^p = \sigma_\alpha^x$$

for all $\alpha$ in $A$ is a product of $\mathcal{X}$.

Theorem 5.7. A category $A(F, G, \Delta)$ with a unary type $\Delta$ and $G$ filtrating products has products if and only if $F$ is a tight functor with $F\emptyset = \emptyset$, in particular if $F$ preserves sums.

Proof. The condition is necessary by theorem 5.3 and 5.2.

Let $\mathcal{X} = \{(X_\alpha, \{\sigma_\alpha^x \} | \alpha \in A) | \alpha \in A \}$ be an arbitrary family of objects of $A(F, G, \Delta)$, let $\langle X, \{\pi_\alpha \} | \alpha \in A \rangle$ be the product $X = \prod_{\alpha \in A} X_\alpha$ with ordinary projections $\pi_\alpha$, $\alpha \in A$.

Define a system $\mathcal{U}$ of admissible subsets of $X$. 
by the condition that \( M \in \mathcal{C} \) if and only if for every \( t \) in \( FM \), there exists a family \( \{ \mu_\alpha \in GM \mid \alpha \prec \beta \} \) such that

\[
(1) \quad \sigma_\alpha^\alpha \cdot [F(\tau_\alpha \cdot i_M^X)](t) = [G(\tau_\alpha \cdot i_M^X)](\mu_\alpha) \quad \text{for all } \alpha \text{ in } A,
\]

where \( i_M^X : M \to X \) is the inclusion of \( M \) into \( X \).

Since \( G \) filtrates products, the family \( \{ \mu_\alpha \} \) is uniquely determined by \( t \) and \( \langle (M, \{ \sigma_\alpha^M \}), \{ \tau_\alpha \cdot i_M^X \} \rangle \) with \( \sigma_\alpha^M(t) = \mu_\alpha \) for \( t \) in \( FM \). It becomes an inverse bound of \( \mathcal{X} \).

Denote \( S = \bigcup_{M \in \mathcal{C}} \) — the union of all admissible subsets of \( X \), \( i_M^S : M \to S \), \( M \in \mathcal{C} \), — the inclusion of \( M \) into \( S \). Since \( F \) is tight, we have by statement 4.6

\[
\bigcup_{M \in \mathcal{C}} [P^+(F(i_M^S))(FM)] = FS,
\]

therefore, for every \( \phi \) in \( FS \), we have \((F i_S^X)(\phi) = (F i_M^X)(t) \) for some admissible set \( M \) and \( t \) in \( FM \).

Putting \( \nu_\alpha = (Gi_M^S)(\sigma_\alpha^M(t)) \) we get

\[
\sigma_\alpha^\alpha \cdot [F(\tau_\alpha \cdot i_M^X)](\nu_\alpha) = \sigma_\alpha^\alpha \cdot [F(\tau_\alpha \cdot i_M^X)](t) =
\]

\[
= [G(\tau_\alpha \cdot i_M^X) \cdot \sigma_\alpha^M(t)] = [G(\tau_\alpha \cdot i_M^X) \cdot (Gi_M^S) \cdot \sigma_\alpha^M(t) = [G(\tau_\alpha \cdot i_M^X)](\nu_\alpha),
\]

therefore \( S \) is admissible. Moreover, it is easily seen that \( i_M^S \) is a morphism of \( (M, \{ \sigma_\alpha^M \}) \) into \( (S, \{ \sigma_\alpha^S \}) \).

It remains to show that \( \langle (S, \{ \sigma_\alpha^S \}), \{ \tau_\alpha \cdot i_M^S \} \rangle \) is a product of \( \mathcal{X} \).

Let \( \langle (Y, \{ \sigma_\alpha^Y \}), \{ \eta_\alpha \} \rangle \) be an in-
verse bound of $x$, i.e.
\[ \sigma_x^\times \cdot (F \eta_x) = (G \eta_x) \cdot \sigma_x^Y \]
for all $x$ in $A$, and let $h : Y \to X$ be the mapping
uniquely determined by $\tau_x \cdot h = \eta_x$, $x \in A$.
Denote $M = (P^+ \mathcal{A})(Y)$ and let $\hat{h} : Y \to M$ be
the range restriction of $h$. Then we have $h = i^X_M \circ \hat{h}$
and $\eta_x = \tau_x \cdot i^X_M \circ \hat{h}$, therefore
\[ (2) \quad \sigma_x^\times \cdot [F(\tau_x \cdot i^X_M)] \cdot (F \hat{h}) = [G(\tau_x \cdot i^X_M)] \cdot (G \hat{h}) \cdot \sigma_x^Y. \]

Now, for every $t$ in $F^M$ there exists an $y$ in $F^Y$ such that $(F \hat{h})(y) = t$. By (1) and (2) it must be
\[ (G \hat{h}) \cdot \sigma_x^Y(y) = \sigma_x^M(t) = \sigma_x^M \cdot (F \hat{h})(y), \]
therefore $\hat{h}$ is a morphism of $(Y, \{ \sigma_x^Y \})$ onto $(M, \{ \sigma_x^M \})$, $M$ is admissible, and $f : i^M_M \circ \hat{h}$ is the unique
factoring morphism of $(Y, \{ \sigma_x^Y \})$ into $(S, \{ \sigma_x^S \})$
such that $\eta_x = (\tau_x \cdot i^X_S) \cdot f$ for all $x$ in $A$.

As a corollary we have

**Theorem 5.8.** A category $A(F, \mathcal{G}, \Delta)$ of a unary
type and with $F$ not preserving sums has products if and
only if $F$ is tight with $F\emptyset = \emptyset$ and $G$ filtrates
or preserves products.

**Proof.** If $A(F, \mathcal{G}, \Delta)$ has products, then $F$
must be tight by theorem 5.3, $F\emptyset = \emptyset$ by theorem 5.2,
therefore it cannot be separating and $G$ then cannot blow
up products by theorem 5.5.
The converse has been asserted in theorem 5.7.

**Theorem 5.9.** If $G$ superinflates products, then $A(F, G, \Delta)$ has not products.

**Proof.** Having in view the theorems 5.1, 5.6, 5.8, we shall have only to prove that $A(F, G, \Delta)$ has not products in the case of a unary type $\Delta$ and the functor $F$ preserving sums. Then it is $F \cong I \times \mathcal{C}_M$ and thus $A(F, G, \Delta)$ is isomorphic to some $A(I, G, \Delta')$ with a suitable unary type $\Delta'$. Therefore to prove the theorem, it will do to show that $A(I, G, \{ I \})$ has not pseudoproducts. The proof then runs as follows.

Let $\{ X_\alpha \mid \alpha \in A \}$, $x_\alpha \in X_\alpha$, $\psi_\alpha \in G X_\alpha$ enjoy the properties stated in the definition 4.5. Let $\sigma_\alpha : X_\alpha \to G X_\alpha$, $\alpha \in A$, be the constant mapping assigning to every $x$ from $X_\alpha$ the element $\psi_\alpha$. We shall show that the family $\mathcal{X} = \{ (X_\alpha, \sigma_\alpha) \mid \alpha \in A \}$ of objects of $A(I, G, \{ I \})$ fails to have a pseudoproduct in this category.

Assume that the family $\mathcal{X}$ has a pseudoproduct, say, $\langle (P, \sigma_P) ; \{ P_\alpha \mid \alpha \in A \} \rangle$. Let $\mathcal{M}$ be an arbitrary infinite cardinal number. It will be shown that $\text{card } P \geq \mathcal{M}$.

Let $\langle X, \{ T_\alpha \mid \alpha \in A \} \rangle$ be the cartesian product of the family $\{ X_\alpha \mid \alpha \in A \}$. Let $S$ be a set with $\text{card } S \geq \mathcal{M}$. Define an inverse bound $\mathcal{Z} = \langle (Z, \sigma_Z) ; \{ \sigma_\alpha \mid \alpha \in A \} \rangle$ of the family $\mathcal{X}$ as follows:

$Z = X \vee S$, $\sigma_\alpha : Z \to X_\alpha$ is a mapping such that $\sigma_\alpha | X = T_\alpha$, $\sigma_\alpha (\lambda) = x_\alpha$ for all $\lambda$ in $S$. 

- 84 -
To define the operation $\sigma^*_Z$ denote

$$M = \{ x \in G(X \cup S) \mid (G \sigma^*_Z)(x) = y \} \quad \text{for all } x \text{ in } A \}$$

Let $< \triangleleft$ be a well-ordering of $S$, for a given $\beta$ in $S$, denote

$$S_{\beta} = \{ t \in S \mid t < \beta \} \quad \text{For an } \beta \text{ in } S$$

denote further $M_{\beta} = M \cap [(P \circ G)(i_\beta)](G(X \cup S_\beta))$,

where $i_\beta : X \cup S_\beta \rightarrow Z$ is the inclusion. Since $G$ super-inflates products we have $\text{card } M_\beta > 1 + \text{card } S_\beta$.

Therefore, we can now define $\sigma^*_Z : Z \rightarrow GZ$ by the transfinite induction in such a way that $\sigma^*_Z(X) \cap \sigma^*_Z(S) = 0$, $\sigma^*_Z$ is one-to-one on $S$ and for every $\beta$ in $S$ it is $\sigma^*_Z(X \cup S_\beta) \subset M_{\beta}$.

Then, clearly,

$(G \sigma^*_\infty) \circ \sigma^*_Z = \sigma^*_\infty \circ \sigma^*_Z$ so $Z$ really is an inverse bound.

Let $f : Z \rightarrow P$ be a factoring morphism, i.e.

(1) $f_{\infty} \circ f = \sigma^*_\infty$ for all $\alpha$ in $A$,

(2) $\sigma^*_P \circ f = (Gf) \circ \sigma^*_Z$.

We shall show that $f$ is one-one. On $X$ it follows immediately by (1), further proceed by transfinite induction.

Let $\alpha \in S$ and let $f \circ i_\alpha$ be one-to-one, $i_\alpha : X \cup S_\alpha \rightarrow Z$ being the inclusion. Then also $G(f \circ i_\alpha)$ is one-to-one, therefore $Gf$ is one-to-one on $M_{\alpha}$.

It remains to show $f$ to be one-to-one on $X \cup S_\alpha$ \cup \{ $\beta$ \}. But it would be, otherwise, $f(\alpha) = f(\beta')$ for some $\beta'$ in $X \cup S_\alpha$, and, by (2), $(Gf) \circ \sigma^*_Z(\beta) = (Gf) \circ \sigma^*_Z(\beta')$, in contradiction with $\sigma^*_Z(\alpha) \neq \sigma^*_Z(\beta'), \sigma^*_Z(\beta), \sigma^*_Z(\beta') \in M_{\alpha}$ and $Gf$ being one-to-one on $M_{\alpha}$.
Appendix

A. Although the problem of products in $A(F, G, \Delta)$ is not solved completely in the present paper, we can nevertheless show that the theorems proved here clear up many situations. Let $\mathcal{D}$ denote the least system of functors containing $I, N, \beta, \mathcal{C}_M$ with $M \neq \beta$, closed with regard to operations $\lor, \times$ (over sets), $\circ, <-, ->$ (whenever defined) and to natural equivalence. From this recursive definition of $\mathcal{D}$ and with aid of the results of the section 4 we can prove easily:

If $F$ in $\mathcal{D}$ is covariant, then either $F \not= \emptyset$ or $F \cong I \times \mathcal{C}_M$ or $F$ is loose;

if $G$ in $\mathcal{D}$ is covariant, then either $G$ preserves products and $G \cong \mathcal{Q}_M$, or $G$ filtrates products and $G \cong \bigvee_{\epsilon \neq \beta} \mathcal{Q}_M$, or $G$ has for a subfunctor one of the functors $\beta, \beta \times I, N, N \times I, < P^-, I >$ and hence superinflates products.

Therefore, from the theorems stated in the paper it follows that:

If $F, G$ are covariant functors belonging to the system $\mathcal{D}$, then $A(F, G, \Delta)$ has products exactly in the following two distinct cases:

1) $G \cong \mathcal{Q}_M$;

2) $\Delta$ is unary, $F \cong I \times \mathcal{C}_M$, $G \cong \bigvee_{\epsilon \neq \beta} \mathcal{Q}_M$.

B. Beside categories $A(F, G, \Delta)$ treated in the text it is but natural to study also the categories
whose objects are all pairs \((X, \mathcal{O})\) with \(X\) - a set and \(\mathcal{O}\) - a system of partial operations of the type \(\Delta\) from the set \(FX\) into \(GX\), or, the categories \(R(F, G, \Delta)\) with objects \((X, \mathcal{O})\) - a set with a relational system, i.e. the system of multivalued partial operations of the type \(\Delta\) from \(FX\) into \(GX\) (see also [3]).

The authors have chosen for study the categories \(A(F, G, \Delta)\) since the behaviour of categories \(P(F, G, \Delta)\) and \(R(F, G, \Delta)\) with regard to products is essentially simpler. The theorem 3.1 is valid - after some quite formal modifications - for categories \(P(F, G, \Delta)\) and \(R(F, G, \Delta)\). Therefore, for faithful contravariant \(F, G\) and \(\Sigma \Delta > 0\) the categories \(P(F, G, \Delta)\) and \(R(F, G, \Delta)\) have not products. If \(F, G\) are covariant, then \(R(F, G, \Delta)\) always has products and the forgetful functor preserves them.

In situations treated in the paper, the behaviour of \(P(F, G, \Delta)\) differs from that of \(A(F, G, \Delta)\) only in the following case: If \(G\) filtrates products then \(P(F, G, \Delta)\) always has products and the forgetful functor preserves them. All other results and their proofs brought in the text can be with just formal changes transformed to \(P(F, G, \Delta)\).

C. It is, of course, possible to regard a system of structures simultaneously. If \(\mathcal{I}\) is a set, then categories
A(\{F_\lambda, G_\lambda, A_\lambda \mid \lambda \in \mathcal{I}\}), P(\{F_\lambda, G_\lambda, \Delta_\lambda \mid \lambda \in \mathcal{I}\),

R(\{F_\lambda, G_\lambda, \Delta_\lambda \mid \lambda \in \mathcal{I}\}) are defined in an obvious way.

It is clear that all proofs of non-existence of products are of that kind that, as soon as for some \( \lambda_0 \) in \( \mathcal{I} \) the category \( A(\{F_{\lambda_0}, G_{\lambda_0}, \Delta_{\lambda_0} \}) \) has not products by some of the stated theorems, then \( A(\{F_\lambda, G_\lambda, \Delta_\lambda \mid \lambda \in \mathcal{I}\) has not products either.

Further, we can assert the following: Let for every \( \lambda \) in \( \mathcal{I} \) \( G \) preserves products, or, for every \( \lambda \in \mathcal{I} \), \( \Delta \) be unary, \( G \) filtrate products and \( F_\lambda \) be tight with \( F_\lambda \emptyset = \emptyset \). Then \( A(\{F_\lambda, G_\lambda, \Delta_\lambda \mid \lambda \in \mathcal{I}\) has products.

We do not bring explicitly the results for categories \( P(\ldots) \) and \( R(\ldots) \).

D. Let \( A^*(F, G, \Delta) \) be a full subcategory of the category \( A(F, G, \Delta) \) whose objects are exactly the objects of \( A(F, G, \Delta) \) with a non-void underlying set. All the results in the text claiming the non-existence of products in \( A(F, G, \Delta) \) are without any changes valid in \( A^*(F, G, \Delta) \) as well. The positive results on the existence of products are slightly different. Completing in a simple way the proof of the theorem 5.6 we can for example prove: If the type \( \Delta \) is not unary, then \( A^*(F, G, \Delta) \) has products if and only if \( G \) preserves products.

If \( G \) filtrates or superinflates products, then
$A^*(F, G, \Delta)$ has not products even for a unary type $\Delta$.

The same problems on products as in $A(F, G, \Delta)$ remain open for categories $A^*(F, G, \Delta)$.

References


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