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Ladislav Procházka

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A NOTE ON COMPLETELY DECOMPOSABLE TORSION FREE ABELIAN GROUPS

Ladislav PROCHÁZKA, Praha

Let G be a torsion free abelian group containing a completely decomposable subgroup H with torsion factor group G/H. In this note we shall find some conditions under which the group G is likewise completely decomposable; all these conditions are related with the notion of p-rank of a torsion free group.

In what follows, by a group we shall understand an additively written abelian group, and the latter p will be reserved for a prime number. If G is a torsion free group then by a basis of G we shall mean any maximal independent set of G; if $M \subseteq G$ then $\{M\}_G^*$ represents the minimal pure subgroup of G containing M. If all non zero elements of G are of the same type $\mathfrak{r}$ then G is said to be homogeneous of the type $\mathfrak{r}$; in general the symbol $\mathcal{T}(G)$ will denote the set of all types of non zero elements in G. For a type $\mathfrak{r}$ the relation $\mathfrak{r}(p) = \infty$ means that in any height belonging to $\mathfrak{r}$ the p-height is ∞. If G is a torsion group then $G_{(p)}$ stands for the p-primary component of G. Other notation and terminology will be essentially that as in [2].

Since many of the following investigations are based on the notion of p-rank of a torsion free group we begin the

Proof. Evidently we can assume $H \neq 0$. Let $A = (x_1, x_2, \dots, x_k)$ be a p^∞-basis of H. The purity of H in G implies the p^∞-independence of A in G. Thus A can be extended to a p^∞-basis $A^* = (x_1, \dots, x_k; x_\iota^* \ (\iota \in I))$ of G. Let B be a basis of H with $A \subseteq B$; therefore, $B = (x_1, x_2, \dots, x_k, \dots, x_n)$. For the set $\overline{B} = (x_1, \dots, x_n; x_\iota^* (\iota \in I))$ we shall show that $\overline{B}$ is independent. In the contrary case we should have a relatior

(1) $$u_1 x_1 + \dots + u_n x_n + v_1 x_{\iota_1}^* + \dots + v_m x_{\iota_m}^* = 0$$

where u_i, v_j are integers and $v_j x_{\iota_j}^* \neq 0$ $(j = 1, \dots, m)$. If $\overline{H} = \{x_1, \dots, x_n, x_{\iota_1}^*, \dots, x_{\iota_m}^*\}_G^*$ then by (1) it is $r = r(H) < m + n$. From the p^∞-independence of the set $(x_1, \dots, x_k, x_{\iota_1}^*, \dots, x_{\iota_m}^*)$ in $\overline{H}$ it follows by [4, Theorem 1] that $r_p(\overline{H}) \leq r - (k+m) < n + m - (k+m) = n - k$. Simultaneously $r_p(H) = n - k$ and $H \subseteq \overline{H}$ which is in contradiction with [5, Theorem 5]. Thus we have established actually the independence of $\overline{B}$. The set $\overline{B}$ may be extended to a basis B^* of G. By [4, Theorem 1] it is $r_p(G) = \operatorname{card}(B^* - A^*)$ and also $r_p(H) = \operatorname{card}(B - A)$. From the inclusion $B - A \subseteq B^* - A^*$ it follows the statement of lemma.

Corollary 2. For a forsion free group G it holds $r_p(G) = 0$ if and only if $r_p(H) = 0$ for each its pure subgroup H of finite rank.

This is an immediate consequence of the previous assertions.

Lemma 3. Let G be a torsion free group and H any of its pure subgroups of finite rank. Then $r_p(G) = r_p(H)$

present note with several assertions concerning this notion. For the definition of p-rank and p^∞-independence see [4].

Lemma 1. Let G be a torsion free group. If $r_p(H) = 0$ for each its pure subgroup H of finite rank then $r_p(G) = 0$ as well.

Proof. Suppose that $r_p(G) > 0$. If A is a p^∞-basis of G (then A is independent) and if B is a basis in G with $A \subseteq B$ then by [4, Theorem 1] it is $card(B - A) = r_p(G) > 0$; therefore, $B - A \neq \emptyset$. Thus B is not p^∞-independent in G, which implies that some finite subset $(x_1, x_2, \dots, x_n)$ of B is p^∞-dependent in G. If we put $H = \{x_1, x_2, \dots, x_n\}^*_G$ then the elements $x_1, x_2, \dots, x_n$ form a basis of H which is p^∞-dependent in H (H is pure in G) ; this means in view of [4, Lemma 3] that $0 < r_p^*(H/\{x_1, \dots, x_n\})$. Since H is of finite rank the Theorem 4 of [7] can be applied. Thus we obtain

$$0 < r_p^*(H/\{x_1, \dots, x_n\}) = r_p(H)$$

which is in contradiction with the hypothesis. Consequently, the validity of $r_p(G) = 0$ is established.

Corollary 1. If G is a p-reduced completely decomposable torsion free group then $r_p(G) = 0$.

Proof. From [5, Lemma 6.1 and Theorem 6] it follows that $r_p(H) = 0$ for each pure subgroup H of finite rank in G.

Lemma 2. Let G be a torsion free group and H a pure subgroup of finite rank in G. Then $r_p(H) \leq r_p(G)$.

if and only if $\varkappa_p(G/H) = 0$.

<u>Proof.</u> Assume firstly $\varkappa_p(G) = \varkappa_p(H)$ and $\varkappa_p(\overline{G}) > 0$ where $\overline{G} = G/H$. By Lemma 1 there exists a pure subgroup $\overline{K}$ in $\overline{G}$ of finite rank with $\varkappa_p(\overline{K}) > 0$; $\overline{K}$ may be expressed as $\overline{K} = K/H$ where $H \subseteq K$ and K is a pure subgroup of finite rank in G. According to [5, Theorem 6] one can write $\varkappa_p(K) = \varkappa_p(H) + \varkappa_p(\overline{K}) > \varkappa_p(H) = \varkappa_p(G)$ which is a contradiction with Lemma 2. Thus the validity of $\varkappa_p(G/H) = 0$ is proved.

Conversely, let $\varkappa_p(G/H) = 0$ hold. If $\overline{A} = (\overline{x}_\iota ;\ \iota \in I)$ is a p^∞-basis of $\overline{G} = G/H$ then in view of [4, Theorem 1] $\overline{A}$ is a basis of $\overline{G}$ as well. Now we take in each coset $\overline{x}_\iota$ $(\iota \in I)$ an element x_ι and put $A = (x_\iota ;\ \iota \in I)$. It is easy to see that A is p^∞-independent in G; furthermore, if B is any basis of G with $A \subseteq B$ then $\mathit{card}\,(B - A) = \varkappa(H) = n$. Let $A_1 = (y_1, \dots, y_k)$ be a p^∞-basis and $B_1 = (y_1, \dots, y_k, \dots, y_n)$ a basis in H. Clearly, the set $A_2 = A \cup A_1$ is p^∞-independent in G, therefore, A_2 can be extended to a p^∞-basis A^* of G. If B is a basis of G such that $A^* \subseteq B$ then we have by [4, Theorem 1] $\varkappa_p(G) = \mathit{card}\,(B - A^*) \leqq \mathit{card}\,(B - A_2) = n - k = \varkappa_p(H)$. This last inequality with $\varkappa_p(H) \leqq \varkappa_p(G)$ (see Lemma 2) give the desirable relation $\varkappa_p(G) = \varkappa_p(H)$.

In what follows, we shall use the notion of Baer's classes Γ_α of torsion free groups (see [1] and also [2], § 48). We recall that Γ_1 is defined as the class of all countable torsion free groups; for $\alpha > 1$ a torsion free group G belongs to Γ_α if $G \notin \Gamma_\beta$ $(\beta < \alpha)$ and there exists a pure subgroup $S \subseteq G$ of finite rank such that G/S is a direct sum of groups belonging to classes with indices less than α.

If G is a torsion group then by $\Pi(G)$ we shall denote the set of all primes with $G_{(p)} \neq 0$.

<u>Theorem 1.</u> Let G be a torsion free group containing a homogeneous completely decomposable subgroup H with torsion factor group G/H. Let the set $\Pi(S/S \cap H)$ be finite for each pure subgroup S of finite rank in G. Then $G \cong H$ if and only if $\kappa_p(G) = 0$ for each $p \in \Pi(G/H)$ and G belongs to some class Γ_α.

<u>Proof.</u> At first we suppose that $G \cong H$. Thus G is again completely decomposable, therefore, $G \in \Gamma_\alpha$ $(\alpha \leq 2)$. Clearly, for $p \in \Pi(G/H)$ the subgroup H cannot be p-divisible. This fact together with the homogeneity of H imply that H is p-reduced. Now by Corollary 1 we obtain $0 = \kappa_p(H) = \kappa_p(G)$.

For the proof of the sufficiency suppose that $\kappa_p(G) = 0$ whenever $p \in \Pi(G/H)$ and that G belongs to some class Γ_α. Take an arbitrary pure subgroup S in G of finite rank and put $T = S \cap H$; thus T is a pure subgroup in H of finite rank and $\Pi(S/T)$ is finite in view of the hypothesis in theorem. From the relations

$$(2) \qquad S/T = S/(S \cap H) \cong \{S, H\}/H \subseteq G/H$$

it follows that $\Pi(S/T) \subseteq \Pi(G/H)$. In view of $\kappa_p(G) = 0$ for each $p \in \Pi(S/T) \subseteq \Pi(G/H)$ we infer by Lemma 2 that $\kappa_p(S) = 0$ whenever $p \in \Pi(S/T)$. Hence, by [5, Theorem 5] the group S/T is reduced and, therefore, finite. Next T as a pure subgroup of the homogeneous completely decomposable group H is again completely decomposable (see [2,Theorem 46.6]) and homogeneous of the same type as H. Theorem B of [3] gives the relation $S \cong T$. The subgroup S being arbitrary, we have shown that G is homogeneous of the type of H and that each pure subgroup of finite rank in G is completely decomposable. Thus, if $K \subsetneq L$ are two pure subgroups of finite rank in G then by [2, Theorem 46.8 and Theorem 46.6] the group L/K is completely decomposable and homogeneous of the type of G. Consequently, for each pure subgroup S of finite rank in G the group G/S is homogeneous of the same type as G (and also H). According to [2,Theorem 48.2] G is completely decomposable. Finally, the equality $\kappa(G) = \kappa(H)$ implies the desirable relation $G \cong H$.

Corollary 3. Let G be a torsion free group containing a homogeneous completely decomposable subgroup H with reduced torsion group G/H. Let $\Pi(S/S \cap H)$ be finite whenever S is a pure subgroup of finite rank in G. Then $G \cong H$ if and only if G belongs to some class Γ_α.

Proof. Let S be a pure subgroup of finite rank in G and $p \in \Pi(G/H)$. The subgroup H cannot be p-divisible, therefore, it is p-reduced; thus by Corollary 1 we have $\kappa_p(H) = 0$. If we put $T = S \cap H$ then T is pure in H and of finite rank. Thus Lemma 2 implies that $\kappa_p(T) = 0$. For the group S/T we have the relation (2), therefore, S/T is reduced. Hence by [5,Theorem 5] it follows $\kappa_p(S) = \kappa_p(T) = 0$. In view of Lemma 1 this means that $\kappa_p(G) = 0$ for each $p \in \Pi(G/H)$, S being taken arbitrary. Now we may apply Theorem 1.

Corollary 4. Let G be a homogeneous torsion free group such that for almost all primes p it is $pG = G$. Then G is completely decomposable if and only if G belongs to some class Γ_α and $\varkappa_p(G) = 0$ whenever $pG \neq G$.

Proof. Evidently the above mentioned conditions are necessary for the complete decomposability of G.

For the proof of sufficiency take any basis $B = (x_\iota ; \iota \in I)$ of G, set $J_\iota = \{x_\iota\}_G^*$ $(\iota \in I)$ and define $H = \sum_{\iota \in I} J_\iota$. Then G/H is torsion, H is homogeneous of the same type as G and hence $pH = H$ for almost all primes p; it is obvious that $p \in \Pi(G/H)$ implies $pH \neq H$ (and also $pG \neq G$), therefore, the set $\Pi(G/H)$ is finite. Thus we may apply Theorem 1 and we get $G \cong H$.

The following theorem is also a consequence of Theorem 1. For the definition of the groups $H(\tau)$ and $H^*(\tau)$ (if H is a torsion free group and τ a type) see [2], § 42.

Theorem 2. Let G be a torsion free group containing a completely decomposable subgroup H and let G/H be a torsion group with finite set $\Pi(G/H)$. Let $\mathcal{T}(H)$ be inversely well-ordered and put $\overline{G}(\tau) = \{H(\tau)\}_G^*$ and $\overline{G}^*(\tau) = \{H^*(\tau)\}_G^*$ for $\tau \in \mathcal{T}(H)$. If for each $\tau \in \mathcal{T}(H)$ the group $\overline{G}(\tau)/\overline{G}^*(\tau)$ belongs to some class Γ_α and $\varkappa_p(\overline{G}(\tau)/\overline{G}^*(\tau)) = 0$ whenever

$p \in \Pi(G/H)$ then $G \cong H$.

Proof. If $H = \sum_{\iota \in I} J_\iota$ is a complete decomposition of H and if $\mathfrak{r} \in \mathcal{Y}(H)$ then we denote by $H_\mathfrak{r}$ the direct sum of all J_ι $(\iota \in I)$ of the type $\mathfrak{r}$; hence, $H = \sum_\mathfrak{r} H_\mathfrak{r}$ and $H(\mathfrak{r}) = H_\mathfrak{r} \dotplus H^*(\mathfrak{r})$ for $\mathfrak{r} \in \mathcal{Y}(H)$. In view of the definition of $\overline{G}^*(\mathfrak{r})$ we have

$$(3) \qquad \{H(\mathfrak{r}), \overline{G}^*(\mathfrak{r})\} = \{H_\mathfrak{r}, \overline{G}^*(\mathfrak{r})\} = H_\mathfrak{r} \dotplus \overline{G}^*(\mathfrak{r}) .$$

We may also write

$$(4) \quad \widetilde{G}_\mathfrak{r} = [\overline{G}(\mathfrak{r})/\overline{G}^*(\mathfrak{r})] / [\{H(\mathfrak{r}), \overline{G}^*(\mathfrak{r})\}/\overline{G}^*(\mathfrak{r})] \cong$$
$$\cong [\overline{G}(\mathfrak{r})/H(\mathfrak{r})] / [\{H(\mathfrak{r}), \overline{G}^*(\mathfrak{r})\}/H(\mathfrak{r})] .$$

The purity of $H(\mathfrak{r})$ in H implies the equality $H(\mathfrak{r}) = G(\mathfrak{r}) \cap H$ and hence

$$\overline{G}(\mathfrak{r})/H(\mathfrak{r}) = \overline{G}(\mathfrak{r})/[\overline{G}(\mathfrak{r}) \cap H] \cong \{\overline{G}(\mathfrak{r}), H\}/H \subseteq G/H .$$

Thus we have shown (see (4)) that $\Pi(\widetilde{G}_\mathfrak{r}) \subseteq \Pi(G/H)$. From (3) it follows

$$(5) \qquad \{H(\mathfrak{r}), \overline{G}^*(\mathfrak{r})\}/\overline{G}^*(\mathfrak{r}) \cong H_\mathfrak{r} ;$$

this means that $\{H(\mathfrak{r}), \overline{G}^*(\mathfrak{r})\}/\overline{G}^*(\mathfrak{r})$ is a homogeneous completely decomposable subgroup of the group $\overline{G}(\mathfrak{r})/\overline{G}^*(\mathfrak{r})$. It is easy to see that Theorem 1 may be applied to $\overline{G}(\mathfrak{r})/\overline{G}^*(\mathfrak{r})$. From this fact we conclude (see also (5)) the isomorphism relation $\overline{G}(\mathfrak{r})/\overline{G}^*(\mathfrak{r}) \cong H_\mathfrak{r}$; therefore, $\overline{G}(\mathfrak{r})/\overline{G}^*(\mathfrak{r})$ is completely decomposable and homogeneous of the type $\mathfrak{r}$. From $\overline{G}(\mathfrak{r}) = \{H(\mathfrak{r})\}^*_G$ it follows the inequality $type\ x \geq \mathfrak{r}$ whenever

$0 \neq x \in \overline{G}(\mathfrak{r})$; thus $type\, x = \mathfrak{r}$ for each $x \in$ $\in G(\mathfrak{r}) - \overline{G}^*(\mathfrak{r})$. If we apply the Baer's lemma (see [2, the note following Theorem 46.5]) we can write a direct decomposition

(6) $\overline{G}(\mathfrak{r}) = G_{\mathfrak{r}} \dotplus \overline{G}^*(\mathfrak{r})$ where $G_{\mathfrak{r}} \cong \overline{G}(\mathfrak{r})/\overline{G}^*(\mathfrak{r}) \cong H_{\mathfrak{r}}$.

Now by a transfinite induction on $\mathfrak{r} \in \mathcal{Y}(H)$ we shall show that $\overline{G}(\mathfrak{r}) = \sum_{\mathfrak{r} \leq \mathfrak{b}} G_{\mathfrak{b}}$, for each $\mathfrak{r} \in \mathcal{Y}(H)$. For the greatest element $\mathfrak{r}_0$ of $\mathcal{Y}(H)$ we have $H^*(\mathfrak{r}_0) =$ $= 0 = \overline{G}^*(\mathfrak{r}_0)$, therefore, under (6) $\overline{G}(\mathfrak{r}_0) = G_{\mathfrak{r}_0} = \sum_{\mathfrak{r}_0 \leq \mathfrak{b}} G_{\mathfrak{b}}$.

Let $\mathfrak{r}_1 \in \mathcal{Y}(H)$, $\mathfrak{r}_1 < \mathfrak{r}_0$ and let us suppose that our assertion holds whenever $\mathfrak{r} \in \mathcal{Y}(H)$ and $\mathfrak{r}_1 < \mathfrak{r} \leq \mathfrak{r}_0$. Evidently $H^*(\mathfrak{r}_1) = \bigcup_{\mathfrak{r}_1 < \mathfrak{r}} H(\mathfrak{r})$ and hence $\overline{G}^*(\mathfrak{r}_1) = \bigcup_{\mathfrak{r}_1 < \mathfrak{r}} \overline{G}(\mathfrak{r})$.

From this fact, by the inductive hypothesis we conclude that $\overline{G}^*(\mathfrak{r}_1) = \sum_{\mathfrak{r}_1 < \mathfrak{b}} G_{\mathfrak{b}}$, and in view of (6) we have $G(\mathfrak{r}_1) = \sum_{\mathfrak{r}_1 \leq \mathfrak{b}} G_{\mathfrak{b}}$. Thus the proof by induction is finished. Since $H = \bigcup_{\mathfrak{r} \in \mathcal{Y}(H)} H(\mathfrak{r})$ and $G = \{H\}^*_G$ we get $G = \bigcup_{\mathfrak{r} \in \mathcal{Y}(H)} \overline{G}(\mathfrak{r})$, therefore, $G =$ $= \sum_{\mathfrak{b} \in \mathcal{Y}(H)} G_{\mathfrak{b}}$. This implies (see also (6))

$$G = \sum_{\mathfrak{r} \in \mathcal{Y}(H)} G_{\mathfrak{r}} \cong \sum_{\mathfrak{r} \in \mathcal{Y}(H)} H_{\mathfrak{r}} = H$$

which proves our theorem.

Next we shall prove two elementary statements concerning Baer's classes Γ_α .

Lemma 4. If G_i $(i = 1, 2, \dots, n)$ are torsion free groups such that $G_i \in \Gamma_{\alpha_i}$ $(i = 1, 2, \dots, n)$ then there exists an ordinal $\alpha \leqq max[\alpha_1, \alpha_2, \dots, \alpha_n]$ with $G_1 \dotplus G_2 \dotplus \dots \dotplus G_n = G \in \Gamma_\alpha$.

<u>Proof</u>. If $\alpha_1 = \alpha_2 = \dots = \alpha_n = 1$ then $G \in \Gamma_1$. So let us suppose that $1 < \alpha_i$ for some i ; without loss of generality we may assume that $\alpha_1 = \dots = \alpha_k$ $(k \leqq n)$ and $\alpha_i < \alpha_1 = \alpha_k$ for $k < i$. If $G \in \Gamma_\alpha$ for some $\alpha < \alpha_1 = max[\alpha_1, \alpha_2, \dots, \alpha_n]$ then our lemma is proved. Thus suppose that $G \notin \Gamma_\beta$ whenever $\beta < \alpha_1$. For each i $(1 \leqq i \leqq k)$ there exists a pure subgroup S_i in G_i of finite rank such that G_i / S_i is a direct sum of groups belonging to Baer's classes with indices less than α_1 . Hence $S = S_1 \dotplus S_2 \dotplus \dots \dotplus S_k$ is a pure subgroup in G of finite rank and G/S is a direct sum of groups from classes of indices less than α_1 . Thus $G \in \Gamma_{\alpha_1}$ and lemma is proved.

<u>Lemma 5</u>. Let H be a pure subgroup of finite rank in a torsion free group G . If $G \in \Gamma_\alpha$ then $G/H \in \Gamma_\beta$ for some ordinal $\beta \leqq \alpha$.

<u>Proof</u>. For $\alpha > 1$ the assertion is trivial. Next we shall proceed by induction on α .

Assume $\alpha = 1$ and let our lemma hold whenever the corresponding group belongs to a class with index less than α . In G there exists a pure subgroup S of finite rank with $G/S = \sum_{\iota \in I} \overline{G}_\iota$ where $\overline{G}_\iota \in \Gamma_{\beta_\iota}$ for $\beta_\iota < \alpha$ $(\iota \in I)$. Then $S^* = \{S, H\}^*_G$ is likewise of finite rank and we have

(7) $\quad (G/H)/(S^*/H) \cong G/S^* \cong (G/S)/(S^*/S)$,

where S^*/S (S^*/H resp.) is a pure subgroup of finite rank in G/S (in G/H resp.). Thus S^*/S is contained in a direct sum of a finite number of groups $\overline{G}_\iota$ ($\iota \in I$) and in view of Lemma 4 we may suppose that S^*/S lies in some $\overline{G}_{\iota_0}$ ($\iota \in I$).

Hence

(8) $\quad (G/S)/(S^*/S) \cong \overline{G}_{\iota_0}/(S^*/S) \dotplus \sum_{\iota \neq \iota_0} \overline{G}_\iota$

where

(9) $\quad \overline{G}_{\iota_0}/(S^*/S) \in \Gamma_\beta$, $\beta \leqq \beta_{\iota_0} < \alpha$,

following the inductive hypothesis. Now, if $G/H \notin \Gamma_\beta$ for each $\beta < \alpha$ then from (7), (8) and (9) it follows that $G/H \in \Gamma_\alpha$. Thus the proof by induction is finished.

Now we are in position to prove the following theorem.

<u>Theorem 3</u>. Let G be a torsion free group containing a homogeneous completely decomposable subgroup H such that G/H is a torsion group with finite set $\Pi(G/H)$. If $\kappa_p(G) < \aleph_0$ for each $p \in \Pi(G/H)$ and if G belongs to some class Γ_α then $G = G_1 \dotplus G_2$, where G_1 is of finite rank and G_2 is completely decomposable and homogeneous of the same type as H.

<u>Proof</u>. If S is any pure subgroup in G of finite rank then by Lemma 2 it is $\kappa_p(S) \leqq \kappa_p(G) < \aleph_0$ for each prime $p \in \Pi = \Pi(G/H)$. If we put

$$R(S) = \sum_{p \in \Pi} \kappa_p(S)$$

then we have (Π being finite)

$$R(S) \leq \sum_{p \in \Pi} \kappa_p(G) < \aleph_0$$

for each such pure subgroup S. Consequently, among the pure subgroups $S \subseteq G$ of finite rank there exists one with the greatest $R(S)$; we denote it by G_1. Thus $H_1 = H \cap G_1$ is a pure subgroup of finite rank in H and by [2,Theorem 46.8] H_1 is a direct summand in H. We shall write $H = H_1 \dot{+} H_2$ and put $G^* = \{G_1, H_2\}$; since $G_1 \cap H_2 = G_1 \cap H \cap H_2 = H_1 \cap H_2 = 0$, we have $G^* = G_1 \dot{+} H_2$. If we denote $\overline{G} = G/G_1$ then we show that $\kappa_p(\overline{G}) = 0$ for each $p \in \Pi$. On the contrary, assume that $\kappa_{p_0}(\overline{G}) > 0$ for some $p_0 \in \Pi$. Lemma 1 implies the existence of a pure subgroup $\overline{S}$ in $\overline{G}$ of finite rank with $\kappa_{p_0}(\overline{S}) > 0$. Then $\overline{S}$ may be written as $\overline{S} = S/G_1$ where S is pure in G and of finite rank as well. By [5,Theorem 6] it is $\kappa_p(G_1) \leq \kappa_p(S)$ for each $p \in \Pi$ and simultaneously $\kappa_p(G_1) < \kappa_{p_0}(G_1) + \kappa_{p_0}(\overline{S}) = \kappa_{p_0}(S)$ which means that $R(G_1) < R(S)$. The last inequality contradicts the choice of G_1, therefore, $\kappa_p(\overline{G}) = 0$ whenever $p \in \Pi$. Now, by Lemma 5 G/G_1 belongs to some class Γ_α. From the inclusion $H \subseteq G^*$ we conclude $\Pi(G/G^*) \subseteq \Pi(G/H) = \Pi$ and at the same time we have

$$G/G^* \cong (G/G_1)/(G^*/G_1).$$

The group H_2 (as a direct summand of H) is likewise completely decomposable and homogeneous of the type of H. Since $G^*/G_1 \cong H_2$ we can apply Theorem 1 and we get $G/G_1 \cong G^*/G_1 \cong H_2$. Thus we have shown that G/G_1 is completely decomposable and homogeneous of the same type $\mathfrak{n}$ as

H. Since G/H is torsion we have $\varkappa \leq type\ x$ for each $x \in G$, $x \neq 0$; therefore, it is precisely $type\ x = \varkappa$ whenever $x \in G - G_1$. This means that the Baer's lemma may be applied (see [2,Lemma 46.3]) to the group G and its subgroup G_1. Hence, $G = G_1 \dot{+} G_2$, where $G_2 \cong G/G_1 \cong H_2$. This completes the proof of our theorem.

If G is a torsion free group then by $G[p^\infty]$ we shall denote the maximal p-divisible subgroup of G.

<u>Theorem 4.</u> Let G be a torsion free group of finite rank containing a homogeneous completely decomposable subgroup H such that G/H is a torsion group with finite set $\Pi(G/H)$. If the type set $\mathcal{T}(G)$ is ordered then G is completely decomposable just if $\kappa_p(G) = \kappa(G[p^\infty])$ for each $p \in \Pi(G/H)$.

<u>Proof.</u> If G is completely decomposable then for every prime number p it is $\kappa_p(G) = \kappa(G[p^\infty])$ (see [5,Theorem 6 and Lemma 6.1]).

Conversely, assume that $\kappa_p(G) = \kappa(G[p^\infty])$ whenever $p \in \Pi(G/H)$ and show that G is completely decomposable. We shall proceed by induction on the cardinality of $\mathcal{T}(G)$. If $\mathcal{T}(G) = \{\varkappa_1\}$ then G is a homogeneous group of the type $\varkappa_1$. Let $H = \sum_{i=1}^{n} J_i$ be a complete decomposition of H and put $J_i^* = \{J_i\}_G^*$ $(i = 1, 2, \dots, n)$; thus we have $G^* = \{J_1^*, \dots, J_n^*\} = \sum_{i=1}^{n} J_i^*$ and $type\ J_i^* = \varkappa_1$ $(i = 1, \dots, n)$. Since $H \subseteq G^*$, G/G^* is a torsion group with $\Pi(G/G^*) \subseteq \Pi(G/H)$. We shall show that the group G/G^* is reduced. On the contrary, assume that G/G^* contains a subgroup $C(p_0^\infty)$ for some $p_0 \in \Pi(G/G^*)$. By [5,Theorem 5] this implies the inequality $0 \leqq \varkappa_{p_0}(G^*) < \varkappa_{p_0}(G)$. The group G is homogeneous, therefore, $G[p^\infty] = 0$ or $G[p^\infty] = G$ for every prime p. By hypothesis it is $\varkappa_{p_0}(G) = \varkappa(G[p_0^\infty])$ and hence in view of the inequality $0 < \varkappa_{p_0}(G)$ we conclude that $G[p_0^\infty] = G$. From the purity of J_i^* in G it follows $J_i^*[p_0^\infty] = J_i^*$, hence $\varkappa_{p_0}(J_i^*) = 1$ $(i = 1, \dots, n)$ (see [5, Lemma 6.1]), therefore

$$\varkappa_{p_0}(G^*) = n = \varkappa(G) = \varkappa_{p_0}(G) ;$$

thus we get a contradiction with $\varkappa_{p_0}(G^*) < \varkappa_{p_0}(G)$. This already proves that G/G^* is reduced, as stated, Since $\Pi(G/G^*)$ is finite, we have shown that the group G/G^* is finite as well. By Theorem B of [3] we have $G \cong G^*$, therefore, G is completely decomposable.

Next suppose that $\operatorname{card} \mathcal{T}(G) = k \geq 2$ and the theorem holds whenever the corresponding type set contains less than k elements. Let $\mathfrak{t}_0 < \mathfrak{t}_1 < \dots < \mathfrak{t}_{k-1}$ be the sequence of all elements of $\mathcal{T}(G)$. If we set $G_1 = G(\mathfrak{t}_1)$ then G_1 is pure in G and $\mathcal{T}(G_1) = \{\mathfrak{t}_1, \dots, \mathfrak{t}_{k-1}\}$. The subgroup $H_1 = G_1 \cap H$ is pure in H, therefore, H_1 is a direct summand of H (see [2, Theorem 46.8]); thus we may write $H = H_1 \dotplus H_2$. Let $H_2 = \sum_{i=1}^{m} J_i$ be a complete decomposition of H_2 and put $J_i^* = \{J_i\}_G^*$ $(i = 1, \dots, m)$. Evidently $\operatorname{type} J_i^* = \mathfrak{t}_0$ $(i = 1, \dots, m)$ and $H_2^* = \{J_1^*, \dots, J_m^*\} = \sum_{i=1}^{m} J_i^*$. Since $H_2^* \cap G_1 = 0$ we may define a group G^* by setting $G^* = G_1 \dotplus H_2^*$; therefore $G^*/G_1 \cong H_2^* =$

$= \sum_{i=1}^{m} J_i^*$. We have also

(10) $$(G/G_1)/(G^*/G_1) \cong G/G^*$$

and $\Pi(G/G^*) \subseteq \Pi(G/H)$ as a consequence of $H \subseteq G^*$. Next we shall prove the following assertion:

(A) If for a prime number p there exists an index j $(0 \leqq j \leqq k-1)$ with $t_j(p) = \infty$ and if i is the smallest of such j's then $G[p^\infty] = G(t_i)$.

Indeed, $t_i(p) = \infty$ implies the inclusion $G(t_i) \subseteq G[p^\infty]$. On the other hand, if $0 \neq g \in G[p^\infty]$ and *type* $g = t_j$ then $t_j(p) = \infty$ and hence $i \leqq j$. Thus we have $t_i \leqq t_j$, therefore, $g \in G(t_j) \subseteq G(t_i)$. This means that the inclusion $G[p^\infty] \subseteq G(t_i)$ likewise holds, and the proof of (A) is complete.

Now we shall show that the group G/G^* is reduced. On the contrary, suppose that $C(p^\infty)$ is a subgroup of G/G^*. By [5, Theorem 5] we have

(11) $$0 \leqq r_p(G^*) < r_p(G) .$$

Since $r_p(G) = r(G[p^\infty])$, from (11) we conclude that there exists an element g, $0 \neq g \in G[p^\infty]$. If $t_j =$ *type* g then $t_j(p) = \infty$. Let i be the smallest among the indices j's with $t_j(p) = \infty$; then by (A) it is $G[p^\infty] = G(t_i)$. If $i = 0$ then $G = G(t_0) = G[p^\infty]$ and the group G is p-divisible. Hence, the group $G^* = G_1 \dot{+} \sum_{j=1}^{m} J_j^*$ as a direct sum of pure subgroups of

G is likewise p-divisible; therefore,

$$\kappa_p(G^*) = \kappa(G^*) = \kappa(G) = \kappa_p(G)$$

which is in contradiction with (11). For $i \geqq 1$ we have under (A) $G[p^\infty] = G(\tau_i) \subseteq G(\tau_1) = G_1$, and hence

$$\kappa_p(G) = \kappa(G[p^\infty]) = \kappa_p(G[p^\infty]) \leqq \kappa_p(G_1) \leqq \kappa_p(G^*)$$

which again contradicts to (11). Thus we have shown that G/G^* is really reduced. This fact together with the finiteness of $\Pi(G/G^*)$ imply that the group G/G^* itself is finite, G being of finite rank. Since G^*/G_1 is homogeneous and completely decomposable, in view of (10) we may apply Corollary 3 and we get

$$G/G_1 \cong G^*/G_1 \cong H_2^* = \sum_{i=1}^{m} J_i^* \ .$$

Thus G/G_1 is homogeneous of the type τ_0, *type* $g = \tau_0$ for each $g \in G - G_1$, therefore, $G = G_1 \dotplus G_2$ and $G_2 \cong$ $\cong G/G_1 \cong \sum_{i=1}^{m} J_i^*$ which is a consequence of Baer's lemma ([2, Lemma 46.3]).

For the complete proof of our theorem it remains to prove that G_1 is completely decomposable. We have already remarked that $H_1 = G_1 \cap H$ is a homogeneous and completely decomposable subgroup of G_1. Since

$$G_1/H_1 = G_1/(G_1 \cap H) \cong \{G_1, H\}/H \subseteq G/H \ ,$$

G_1/H_1 is a torsion group with $\Pi(G_1/H_1) \subseteq \Pi(G/H)$. Because G_1 is of finite rank and the set $\Pi(G_1/H_1)$ is finite, under [5, Theorem 1] G_1/H_1 is a direct sum of a finite group and of a divisible group. Thus, there exists in G_1 a subgroup K_1 such that $H_1 \subseteq K_1$, K_1/H_1 is finite and G_1/K_1 is divisible; evidently $\Pi(G_1/K_1) \subseteq$

$\subseteq \Pi(G_1/H_1) \subseteq \Pi(G/H)$. As a consequence of Corollary 3 we get $K_1 \cong H_1$ and hence K_1 is a completely decomposable homogeneous group. For $\Pi(G_1/K_1) = \emptyset$ it is $G_1 = K_1$ and G_1 is really completely decomposable. Thus, suppose $\Pi(G_1/K_1) \neq \emptyset$ and take $p \in \Pi(G_1/K_1)$; we shall verify that $\kappa_p(G_1) = \kappa(G_1[p^\infty])$. Since $C(p^\infty)$ is a subgroup of G_1/K_1, in view of [5,Theorem 5 and 6] we have

$$0 \leqq \kappa_p(K_1) < \kappa_p(G_1) \leqq \kappa_p(G) = \kappa(G[p^\infty]) .$$

Hence $G[p^\infty] \neq 0$ and there exists an index j $(0 \leqq j \leqq k-1)$ with $\kappa_j(p) = \infty$; again denote by i the smallest of such j's. By the statement (A) it must be $G[p^\infty] = G(\kappa_i)$. If $i = 0$ then $G[p^\infty] = G(\kappa_0) = G$, therefore, $G_1[p^\infty] = G_1$ and in this case $\kappa_p(G_1) = \kappa(G_1) = \kappa(G[p^\infty])$. If $i \geqq 1$ then $G[p^\infty] = G(\kappa_i) \subseteq G(\kappa_1) = G_1$ and we conclude $G[p^\infty] = G_1[p^\infty]$. Since $G[p^\infty] \subseteq G_1$, we have also (see [5,Theorem 6])

$$\kappa_p(G) = \kappa(G[p^\infty]) = \kappa_p(G[p^\infty]) \leqq \kappa_p(G_1) \leqq \kappa_p(G)$$

which implies that $\kappa_p(G_1) = \kappa(G[p^\infty]) = \kappa(G_1[p^\infty])$. Thus we have shown that $\kappa_p(G_1) = \kappa(G_1[p^\infty])$ for each $p \in \Pi(G_1/K_1)$. Because $\mathcal{T}(G_1) = \{\kappa_1, \ldots, \kappa_{k-1}\}$, by inductive hypothesis the group G_1 is completely decomposable. The proof by induction is thus finished.

From Theorems 3 and 4 we may conclude the following statement.

Theorem 5. Let G be a torsion free group containing a homogeneous completely decomposable subgroup H such that G/H is a torsion group with finite set $\Pi(G/H)$. Suppose that the type set $\Upsilon(G)$ is ordered and that $r_p(G) < \aleph_0$ for each prime $p \in \Pi(G/H)$. Then the group G is completely decomposable if and only if G belongs to some class Γ_α and $r_p(G) = r(G[p^\infty])$ for each $p \in \Pi(G/H)$.

Proof. Evidently if G is completely decomposable then $G \in \Gamma_\alpha$ $(\alpha \leqq 2)$ and $r_p(G) = r(G[p^\infty])$ for every prime p.

Next assume that $G \in \Gamma_\alpha$ and $r_p(G) = r(G[p^\infty])$ for each $p \in \Pi(G/H)$, and show that G is completely decomposable. If G is of finite rank then it suffices to apply Theorem 4. For $r(G) \geqq \aleph_0$, by Theorem 3 we have $G = G_1 \dotplus G_2$ where $r(G_1) < \aleph_0$ and G_2 is completely decomposable and homogeneous of the same type as H; evidently $G_2 \neq 0$. If we put $H_1 = G_1 \cap H$ then H_1 is pure in H and in view of [2, Theorem 46.6] H_1 is likewise homogeneous and completely decomposable. Since

$$G_1/H_1 = G_1/(G_1 \cap H) \cong \{G_1, H\}/H \subseteq G/H$$

it is $\Pi(G_1/H_1) \subseteq \Pi(G/H)$ and hence $\Pi(G_1/H_1)$ is finite. Clearly, for any $p \in \Pi(G_1/H_1) \subseteq \Pi(G/H)$ the groups H and G_2 are p-reduced. This means that $G[p^\infty] \subseteq G_1$, therefore, $G[p^\infty] = G_1[p^\infty]$. From the complete reducibility of G_2 it follows (see Corollary 1)

$$0 = r_p(G_2) = r_p(G/G_1);$$

Thus, by Lemma 3 (see also the hypothesis of our theorem), we have $r_p(G_1) = r_p(G) = r(G[p^\infty]) = r(G_1[p^\infty])$. In view of the inclusion $\mathcal{T}(G_1) \subseteq \mathcal{T}(G)$, Theorem 4 may be applied to the group G_1 and its subgroup H_1. Hence G_1 is completely decomposable which completes the proof of the theorem.

Corollary 5. Let G be a torsion free group with ordered type set $\mathcal{T}(G)$, let $pG = G$ be for almost all primes p and let $r_p(G) < \aleph_0$ whenever $pG \neq G$. Then G is completely decomposable if and only if G belongs to some class Γ_α and $r_p(G) = r(G[p^\infty])$ for every prime p with $pG \neq G$.

Proof. Remark at first that the conditions of theorem are necessary for the complete decomposability of G. To verify their sufficiency we shall construct a suitable subgroup H in G. Let u denote the type satisfying $u(p) = \infty$ whenever $pG = G$ and $u(p) \neq \infty$ for every p with $pG \neq G$; thus if $0 \neq x \in G$ then $u \leqq \text{type } x$. Consider a basis $B = [x_\iota \ (\iota \in I)]$ of G and take the subgroups $J_\iota \subseteq G \ (\iota \in I)$ of rank 1 such that $\text{type } J_\iota = u$ and $x_\iota \in J_\iota \ (\iota \in I)$. If we define $H = \sum_{\iota \in I} J_\iota$ then the factor group G/H is torsion,

$\Pi(G/H)$ is finite and $p \in \Pi(G/H)$ implies $pG \neq G$. Thus for $p \in \Pi(G/H)$ we have $r_p(G) = r(G[p^\infty])$. Now in view of Theorem 5 we may state that G is completely decomposable.

Now we give another formulation of the preceding theorem.

__Theorem 5*.__ Let G be a torsion free group satisfying all conditions of Theorem 5. Then the group G is completely decomposable if and only if G belongs to some Baer's class Γ_α and $r_p(G/G[p^\infty]) = 0$ for each $p \in \Pi(G/H)$.

__Proof.__ By hypothesis, we have $r_p(G) < \aleph_0$ whenever $p \in \Pi(G/H)$. Since $r(G[p^\infty]) \leq r_p(G)$ we conclude that $r(G[p^\infty]) < \aleph_0$ for $p \in \Pi(G/H)$. Thus, in view of Lemma 3, the condition $r_p(G/G[p^\infty]) = 0$ is equivalent to $r_p(G) = r_p(G[p^\infty])$, and Theorem 5 may be applied.

To conclude this note we mention one simple example.

__Example.__ If p is a fixed prime then by $R_{(p)}$ we denote the additive group of all rationals with denominators prime to p. Let U_n $(n = 1, 2, \dots)$ be an infinite sequence of groups satisfying $U_n \cong R_{(p)}$ $(n = 1, 2, \dots)$ and set $G = \sum_{n=1}^{\infty *} U_n$; thus G is a p-reduced torsion free group that is q-divisible for every prime $q \neq p$. This means that G is homogeneous of the same type as $R_{(p)}$. By [1, Theorem 12.6] the group G is separable, therefore, every its pure non zero subgroup of finite rank is a direct sum of finitely many groups $R_{(p)}$. According to Corollary 3 and Lemma 1 we have $\varkappa_p(G) = 0$. If $[x_\iota\ (\iota \in I)]$ is a basis of G and if we put $J_\iota = \{x_\iota\}_G^*$ $(\iota \in I)$ and $H = \sum_{\iota \in I} J_\iota$, then H is a homogeneous completely decomposable subgroup of G with torsion p-primary factor group G/H. Nevertheless, G is not completely decomposable (see [1,Theorem 12.4]), therefore, in view of Theorem 1 G belongs to no Baer's class Γ_α'. But first of all this example shows that the Theorems 1, 5 and 9 in [6] do not hold if the hypothesis on countability is omitted.

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