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SOME NOTES ON VARIOUS ROTUNDITY AND SMOOTHNESS PROPERTIES OF SEPARABLE BANACH SPACES

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1. Notations and definitions. In this paper a space X denotes a real Banach space, X^* the dual space of X. $x_n \to x$ ($x_n \xrightarrow{w} x$) in X and $f_n \xrightarrow{w^*} f$ in X^* mean strong (weak) convergence of a sequence in X and pointwise convergence in X^* respectively. The set of all real numbers is denoted by P and that of all positive integers by N. $K_r^{\|\cdot\|} = \{x \in X; \|x\| \leq r\}$, $S_r^{\|\cdot\|} = \{x \in X; \|x\| = r\}$ for $r > 0$. Analogically $K_r^{*\|\cdot\|}$, $S_r^{*\|\cdot\|}$ in X^*. If no confusion can arise, we write simply K_r, S_r and so on. $C\langle 0, 1\rangle$, $\ell_p(N)$ and so on denote the well-known space with their customary norms (see [9]).

Let X with $\|x\|$ be a space, $\|f\|$ be the dual norm of $\|x\|$ in X^*. We say that X (with $\|x\|$) [X^* (with $\|f\|$)] is (WUR) [(W^*UR)]-space if the following implication is valid respectively:

$$(x_n, y_n \in S_1^{\|\cdot\|},\ \|\frac{x_n + y_n}{2}\| \to 1) \Rightarrow x_n - y_n \xrightarrow{w} 0,$$

$$[(f_n, g_n \in S_1^{*\|\cdot\|},\ \|\frac{f_n + g_n}{2}\| \to 1) \Rightarrow f_n - g_n \xrightarrow{w^*} 0.$$

X is said to be a (R)-space if S_1 contains no open segment and X is called (LUR) [(UR)] if it is true that:

$$x_n, x_0 \in S_1, \left\| \frac{x_n + x_0}{2} \right\| \to 1 \text{ imply } x_n - x_0 \to 0,$$

$$\left[x_n, y_n \in S_1, \left\| \frac{x_n + y_n}{2} \right\| \to 1 \text{ imply } x_n - y_n \to 0 \right],$$

respectively.

X is (G) respective (F), respective (UG), respective (UF) space if the norm of X is Gâteaux differentiable on S_1, respective Fréchet differentiable on S_1, respective uniformly Gâteaux differentiable on S_1, respective uniformly Fréchet differentiable on S_1.

2. Some positive results.

Proposition 1. Let A be an arbitrary countable subset of $\langle 0, 1 \rangle$. Then there exists an equivalent norm $|x|_A$ of $C\langle 0, 1 \rangle$ which has the following property:

$$|x_n|_A \leqq 1, \ |y_n|_A \leqq 1, \ \left| \frac{x_n + y_n}{2} \right|_A \to 1 \text{ imply}$$

a) $(x_n - y_n)(t) \to 0$ for each $t \in A$

and

b) $x_n - y_n \to 0$ in the sense of the space $L_2\langle 0, 1 \rangle$.

Proof. Let $A = \{t_i\}$. Denote

$$I(x) = \sqrt{\sum_{i=1}^{\infty} \frac{1}{2^i} x^2(t_i)}, \quad |||x||| = \|x\|_{L_2\langle 0,1 \rangle} \text{ for } x \in C\langle 0, 1 \rangle.$$

Define

$$|x|_A = \sqrt{\|x\|^2 + I^2(x) + |||x|||^2}$$

where $\|x\|$ is customary supremum - norm of $C\langle 0,1\rangle$. By Minkowski inequality we have that $|x|_A$ is a new equivalent norm of $C\langle 0,1\rangle$.

Suppose now

$$|x_n|_A \leq 1, \quad |y_n|_A \leq 1, \quad \left|\frac{x_n+y_n}{2}\right|_A \to 1 .$$

We have

$$\|x_n+y_n\|^2 + I^2(x_n+y_n) + |||x_n+y_n|||^2 + I^2(x_n-y_n) +$$

$$+ |||x_n-y_n|||^2 =$$

$$= \|x_n+y_n\|^2 + 2(I^2(x_n) + I^2(y_n) + |||x_n|||^2 + |||y_n|||^2) \leq$$

$$\leq 2(\|x_n\|^2 + \|y_n\|^2 + I^2(x_n) + I^2(y_n) + |||x_n|||^2 + |||y_n|||^2) =$$

$$= 2(|x_n|_A^2 + |y_n|_A^2) = 4 .$$

Then

$$0 \leq I^2(x_n-y_n) + |||x_n-y_n|||^2 \leq 4 - (\|x_n+y_n\|^2 +$$

$$+ I^2(x_n+y_n) + |||x_n+y_n|||^2) = 4 - |x_n+y_n|_A^2 .$$

By our assumptions $|x_n+y_n|_A^2 \to 4$. Thus $I^2(x_n-y_n) \to 0$ and $|||x_n-y_n|||^2 \to 0$.

Let t_i be an arbitrary but fixed element of A. Then for every $\frac{\varepsilon}{2^i} > 0$ there exists an index $n_0 \in N$ such that for each $n \in N$, $n \geq n_0$ we have $I^2(x_n-y_n) \leq \frac{\varepsilon}{2^i}$. But $\frac{1}{2^i}(x_n-y_n)^2(t_i) \leq I^2(x_n-y_n)$.

Thus $(x_n - y_n)^2(t_i) \leqq \varepsilon$ for $n \geqslant n_0$.

Remark. The method of the proof of Proposition 1 is similar to that of M.I. Kadec ([10]).

For a bounded $A \subset X$ we denote the diameter of A by $\sigma(A)$. A point $x \in A$ is a diametral point of A provided $\sup\{\|x-y\|, y \in A\} = \sigma(A)$. A convex set $K \subset X$ is said to have normal structure (cf.[3]) if for each bounded convex subset H of K which contains more than one point, there is some point $x \in H$ which is not a diametral point of H.

Proposition 2. Let X with $\|x\|$ be a separable space. Then there exists an equivalent norm $|||x|||$ which has the property that each convex subset of X has normal structure with respect to $|||x|||$.

Proof. If X is a separable Banach space, then there exists a total countable subset $M \subset S_1^{*\|f\|}$ (see [9;chapt.II,§ 1,4 d]). Let $M = \{f_i\}$. Denote

$$|||x||| = \sqrt{\|x\|^2 + I^2(x)} \quad ,$$

where

$$I(x) = \sqrt{\sum_{i=1}^{\infty} \frac{1}{2^i} f_i^2(x)} \; .$$

Then $|||x|||$ is an equivalent norm of X to $\|x\|$. Let $|||x_n||| \leqq 1$, $|||y_n||| \leqq 1$, $|||\frac{x_n+y_n}{2}||| \to 1$. Analogically as in Theorem 5 of [23] we obtain

$f_i(x_n - y_n) \to 0$ as $n \to \infty$ for every $i \in N$.
Now, let K be a convex bounded subset of X which

contains at least two distinct points u, v. Then $\frac{u+v}{2}$ is not a diametral point of K. Suppose on the contrary $\frac{u+v}{2}$ is a diametral point of K. Then there exists a sequence $x_n \in K$ such that $|||\frac{u+v}{2} - x_n||| \to \delta(K)$ (= diameter of K with respect to $|||x|||$). Then $|||\frac{u-x_n}{\delta(K)}||| \leqq 1$, $|||\frac{v-x_n}{\delta(K)}||| \leqq 1$,

$$\frac{1}{\delta(K)} \cdot |||\frac{u-x_n+v-x_n}{2}||| \to 1 .$$ Thus $f_i(u-x_n-(v-x_n)) \to$ $\to 0$ as $n \to \infty$ for each $i \in N$. Then $f_i(u-v) =$ $= 0$ and thus $u = v$ - a contradiction. V.L. Klee ([15]) has proved that if X is separable, then there exists an equivalent norm of X which is (G) and (R) jointly and whose dual norm is (R).

Since X is (UG) iff X^* is (W^*UR) ([19], a short proof [6]), we have the following generalization of this result of V.L.Klee:

<u>Proposition 3</u>. Let X be a separable Banach space. Then there exists an equivalent norm of X which is (UG) and (LUR) and whose dual norm is then (W^*UR).

<u>Proof</u>. M.I. Kadec [10] has constructed an equivalent norm $\|x\|_1$ which is (LUR) and in the paper [23] we constructed an equivalent (UG) -norm $\|x\|_2$.

Then the dual norm of $\|x\|_2$ is (W^*UR). A method of E. Asplund [1] gives (see [24]) an equivalent norm which has the desired properties.

Remark 1. It follows from a result of R. Výborný [22] that a new equivalent norm of X constructed in Proposition 3 has the following property, too:

$x_n \xrightarrow{w} x_0$, $\|x_n\| \to \|x_0\|$ imply $x_n \to x_0$.

Proposition 4. Let X^* be a separable space. Then there exists an equivalent norm $\|x\|'$ of X which is (LUR) and (WUR) and whose dual norm is (LUR) and (W^*UR). Thus $\|x\|'$ is (F) and (UG).

Proof. In this case we have: M.I. Kadec ([11]) has constructed an equivalent norm $\|x\|_1$ of X whose dual norm is (LUR).

He has also constructed in X an equivalent norm $\|x\|_2$ which is (LUR) ([10]). The method of E. Asplund mentioned above gives an equivalent norm $\|x\|_3$ of X which is (LUR) and whose dual norm is also (LUR). In the paper [24] we have constructed an equivalent norm $\|x\|_4$ of X which is (WUR) and whose dual norm is (W^*UR). The method of E. Asplund used for $\|x\|_3$ and $\|x\|_4$ gives an equivalent norm $\|x\|'$ of X which is (LUR) and (WUR) and whose dual norm is (LUR) and (W^*UR). It follows from the result of A.R. Lovaglia ([16]) that $\|x\|'$

is (F). That it is also (UG) it follows immediately from the duality between (UG) of X and (W^*UR) of X^* mentioned above.

Corollary. Let X be a reflexive separable Banach space. Then there exists an equivalent norm of X which is (WUR), (LUR), (F), and (UG) and whose dual norm has the same properties.

3. Some counterexamples.

Remark 2. Let X be a (WUR)-space, Y be a closed linear subspace of X. Then Y is (WUR)-space.

Proof. It follows immediately from the Hahn-Banach theorem.

Remark 3. A space X has an equivalent norm which is (WUR) iff X is isomorphic to a (WUR)-space Y.

Proof. One part of this assertion is obvious. Suppose X is isomorphic to a (WUR)-space Y. Introduce an equivalent norm of X by

$$|x| = \| Tx \|_Y .$$

Let $|x_n| = |y_n| = 1$, $\left| \frac{x_n + y_n}{2} \right| \to 1$. This means

$\| Tx_n \|_Y = \| Ty_n \|_Y = 1$, $\left\| \frac{Tx_n + Ty_n}{2} \right\|_Y \to 1$. As Y is (WUR)-space we have $Tx_n - Ty_n \xrightarrow{w} 0$ in the space Y. As T^{-1} is continuous and linear, we have

$$x_n - y_n = T^{-1}(Tx_n - Ty_n) \xrightarrow{w} 0 \text{ in } X .$$

Remark 4. $\ell_1(N)$ has no equivalent (WUR) -norm.

Proof. It follows immediately from the fact that the weak and norm convergence of sequences coincide in $\ell_1(N)$ and from the fact that $\ell_1(N)$ has no equivalent (UR) -norm as it is not reflexive.

Remark 5. $C\langle 0, 1\rangle$ has no equivalent (WUR)-norm.

Proof. It follows immediately from Remarks 2,3,4 and from Banach-Mazur Theorem concerning the universality of the space $C\langle 0, 1\rangle$.

Remark 6. If we introduce in the space $C\langle 0, 1\rangle$ an equivalent (LUR) -norm by a method of M.I. Kadec ([10]), we obtain an example of (LUR) -space which has no equivalent (WUR) -norm.

M.M. Day ([9,p.191]) has proved that X^* is (R) iff each two-dimensional $X/_L$ is (G) .

V.L. Klee ([15]) has proved the following assertion: If B is a separable normed linear space and L is a non-reflexive closed subspace of B such that the dimension of $B/_L$ is not less than two, then there exists an equivalent norm $\|x\|'$ of B which is (G) and its corresponding norm of $B/_L$ is not (G) .

Remark 7. Let X be a separable Banach space such that X^* is not separable. Take L -any nonreflexive closed subspace of X such that $X/_L$ is two-dimensional. By the result of V.L. Klee there exists an equivalent norm $\| x \|'$ of X which is (G) but its corresponding norm of X/L is not (G). Thus (as it was pointed by D. Cudia [6]) we have an example of a (G) space Y such that Y^* is not (R). Thus this space is not (UG). As X^* is not separable, Y has no equivalent (F)-norm ([11],[17]).

Remark 8. Let X be a separable Banach space such that X^* is not separable. If we introduce an equivalent (UG)-norm in X ([23]), we have an example of a space Y which is (UG) but has no equivalent (F)-norm.

Let S be an index set, X be a Banach space of real-valued functions on S. If for each $s \in S$ a normed space N_s is given, let $P_X N_s$ be the space of all those functions x_s on S such that

(i) x_s is an element of N_s for every s in S, and

(ii) if ξ is the real-valued function defined by $\xi(s) = \|x_s\|_{N_s}$ for each s in S, then ξ is in X. We define the norm of $x \, (\equiv x_s)$ in $P_X N_s$ by

$$\| x \| = \| \xi \|_X .$$

If X satisfies the condition that whenever ξ is in X and

$$|\eta(s)| \leqq |\xi(s)|$$

for all s, then η is in X and

$$\| \eta \| \leqq \| \xi \| ,$$

then $P_X N_s$ is a Banach space if each of the N_s are Banach spaces. For the references see [9].

Remark 9. It follows from the result of A.R.Lovaglia ([16]) that $Y = P_{\ell_2(N)} \ell_{m+1}(N)$ is (LUR). But from the results of M.M. Day ([7]) we have that this space is reflexive and has no equivalent (UR)-norm. The dual space of Y is then (F)-space ([16]) and has no equivalent (UF)-norm ([19]).

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