L. K. Bhagchandani; K. N. Mehra
A Saalschutzian theorem for triple series


Persistent URL: http://dml.cz/dmlcz/105236

Terms of use:
© Charles University in Prague, Faculty of Mathematics and Physics, 1969

Institute of Mathematics of the Academy of Sciences of the Czech Republic provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these Terms of use.

This paper has been digitized, optimized for electronic delivery and stamped with digital signature within the project DML-CZ: The Czech Digital Mathematics Library http://project.dml.cz
1. Introduction. The object of the present paper is to obtain a Saalschützian theorem for triple series. The Saalschützian theorems for double series were obtained by Carlitz [1,2] and sum for double hypergeometric series of superior order was recently obtained by Jain [3]. It is interesting to note that results due to Carlitz [l,p.416 (9)], and Jain [3,p.300(1)] are particular cases of our theorem.

2. The following results [5,p.218(8.3.4)], [1,p.417 (12)] and [4] will be required in our investigations:

\[(1) \quad \bar{F}_1(a; b, b'; b + b'; x, y) = \]
\[= (1-x)^{-a} \bar{F}_1(a, b'; b + b'; \frac{y-x}{1-x}) \]
\[= (1-x)^{b-a} (1-y)^{b-a} \times \]
\[\times \bar{F}_1(b + b' - a; b', b'; b' + b'; x, y), \]

- 319 -
(3) \( F_T(a_1, a_2, a_2, b_1, b_1, b_1; c_1, c_1, c_1; x, y, x) \)
\[
= \sum_{m,n,p=0}^{\infty} \frac{(a_1)_m (a_2)_n (b_1)_p}{m! n! p! (c_1)_m+n+p} \ x^m y^n x^p,
\]
where \(|x| < R, |y| < S, |x| < T\) such that \( T = R - RS + S \).

3. The first formula to be established here is

(4) \( F_T(a_1, a_2, a_2, b_1, b_2, b_2; c_1, c_1, c_1; x, y, x) \)
\[
= (1-x)^{a_2-b_2} (1-y)^{b_2-a_2} x^{a_2-b_2} \ x \ F_T(a_2, a_1, a_1, b_2, b_2, b_2; c_1, c_1, c_1; x, y, x),
\]
where \( c_1 = a_1 + a_2 = b_1 + b_2 \).

Proof. Expressing the series (3) for hypergeometric function of three variables \( F_T \) in terms of Appell's function \( F_1 \), we have

(5) \( F_T(a_1, a_2, a_2, b_1, b_2, b_2; c_1, c_1, c_1; x, y, x) \)
\[
= \sum_{m=0}^{\infty} \frac{(a_1)_m (b_1)_m}{m! (c_1)_m} F_1(a_2, b_2, b_2+m; c_1+m; y, x) x^m.
\]

Now employing (1) in (5), we get

(6) \( F_T(a_1, a_2, a_2, b_1, b_2, b_2; c_1, c_1, c_1; x, y, x) \)
\[
= (1-y)^{a_2} F_1(b_1; a_2, a_2; c_1; x, \frac{x-y}{1-y})
\]
provided \( c_1 = b_1 + b_2 \).

Again using (2) in (6), the result (4) follows.
immediately after little simplifications.

4. The main result that is the Saalschützian theorem for triple series obtained here is

\[
\sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-m)_k (-n)_b (-\mu)_t (a_2)_k (a_3)_t (l'_2)_k + (l'_2)_k + t (l'_2)_t}{n! o! t!(d'_1)_k (d'_2)_t (c'_1)_k + s + t}
\]

\[
= \frac{(a_4)_m (a_2)_m+n (l'_1)_m+n (l'_2)_n}{(c'_1)_m+n+p (l'_1-a'_2)_m (a'_2-l'_1)_n (l'_1-a'_1)_p}
\]

provided \(c'_{4} = l'_1 + l'_2 = a'_1 + a'_2, \ l'_1 - a'_2 \) is not an integer, \(1 + a'_2 + l'_2 - m = c'_1 + d'_1, \ 1 + a'_1 + l'_1 - m = c'_1 + d'_2\) and \(1 + a'_1 + l'_2 - n = c'_1 + d'_3\).

Proof. Employing the expansion of (3) in (4), we have

\[
\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(a'_1)_m (a'_2)_n+b (l'_1)_m+n (l'_2)_n}{m! n! (c'_1)_m+n+p} x^n y^m z^p
\]

\[
= \sum_{n=0}^{\infty} \frac{(l'_1-a'_2)_m}{m!} x^m \sum_{n=0}^{\infty} \frac{(a'_2-l'_1)_n}{n!} y^n \sum_{p=0}^{\infty} \frac{(l'_1-a'_1)_p}{p!} z^p \times
\]

\[
\times \sum_{k=0}^{\infty} \sum_{t=0}^{\infty} \frac{(a'_1)_k (a'_3)_t+t (l'_2)_k+t (l'_1)_b}{k! o! t!(c'_1)_k+n+t} x^k y^o z^t
\]

\[
= \sum_{m,n,p=0}^{\infty} \sum_{k=0}^{\infty} \sum_{t=0}^{\infty} \frac{(l'_2-a'_2)_m (a'_2-l'_1)_n (l'_1-a'_2)_p (a'_2)_p}{(m-n)! (m-o)! (n-t)! k! o! t!(c'_1)_k+n+t} \times
\]

\[
\times (a'_1)_{s+t} (l'_2)_k+t (l'_1)_b \times m! n! o! t! (1+a'_2-l'_1-m)_{k} (1+l'_1-a'_2-n)_{o} \times
\]

\[
= \sum_{m,n,p=0}^{\infty} \sum_{k=0}^{\infty} \sum_{t=0}^{\infty} \frac{(l'_2-a'_2)_m (a'_2-l'_1)_n (l'_1-a'_2)_p (-m)_{k} (-n)_{o}}{m! n! o! t! (1+a'_2-l'_1-m)_{k} (1+l'_1-a'_2-n)_{o}} \times
\]

\[
\times (a'_1)_{s+t} (l'_2)_k+t (l'_1)_b \times m! n! o! t! (1+a'_2-l'_1-m)_{k} (1+l'_1-a'_2-n)_{o}
\]

- 321 -
\[\frac{(-n)(a_2)_{\kappa}(a_1)_{\kappa+n}(b_2)_{\kappa+n}(b_1)_{\kappa+\gamma}}{(1+a_{1}-b_1-n)_{\kappa}(c_1)_{\kappa+n+1}} x^m y^n z^p.\]

putting \(1+a_2-b_1-m=d_1, 1+b_1-a_2-m=d_2, 1+b_1-a_1-n=d_3\)

and equating the coefficient of \(x^m y^n z^p\) on both sides of (8), we obtain (7) under the conditions stated.

5. Particular Case. (i) Putting \(m = 0\) or \(n = 0\) in (7) we obtain a result due to Carlitz [1,p.416(9)].

(ii) When we put \(\gamma = 0\) in (7), we get a result due to Jain [3,p.300(1)].

References

(Received March 18, 1969)