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REMARKS ON NONLINEAR OPERATORS AND FUNCTIONALS
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Introduction. The purpose of this note is to establish some conditions under which a mapping \( F \) or a subadditive functional \( f \) is additive, linear or positive homogeneous on a linear space \( X \). A typical result (Theorem 4) is as follows. Let \( f \) be a subadditive functional on \( X \). Assume there exist a neighborhood \( V(0) \) of \( 0 \) and a functional \( g \) defined on \( V(0) \) so that \( g(0) = 0 \) and \( f(\mu) \leq g(\mu) \) for each \( \mu \in V(0) \). If \( g \) possesses a linear Gâteaux differential \( Dg(0, h) \) at \( 0 \), then \( f \) is linear on \( X \). Theorem 7 deals with the boundedness property of even subadditive functionals, while Theorem 8 concerns the uniform boundedness of the Gâteaux derivative \( f'(\mu) \) of a convex subadditive functional.

§ 1. Terminology and notations. Let \( X, Y \) be real linear normed spaces, \( X^* \) dual of \( X \), \( E_1 \) 1-dimensional Euclidean space. A mapping \( F : X \to Y \) is said to be
(a) additive on \( X \) if \( F(\mu_1 + \mu_2) = F(\mu_1) + F(\mu_2) \) for every \( \mu_1, \mu_2 \in X \).
(b) homogeneous (positively homogeneous) if \( F(t \mu) = t F(\mu) \) for every \( t \in E_1 \) (for each \( t \geq 0 \)) and every \( \mu \in X \).
(c) linear if \( F \) is additive and homogeneous.

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(d) bounded if for each bounded subset $M \subset X$ $F(M)$ is bounded in $Y$.

A functional $f$ defined on $X$ is called

1) subadditive if $f(\mu_1 + \mu_2) \leq f(\mu_1) + f(\mu_2)$ for every $\mu_1, \mu_2 \in X$;

2) convex on a convex set $M \subset X$ if
$$f(t \mu + (1-t) \nu) \leq tf(\mu) + (1-t)f(\nu)$$
for each $t \in <0, 1>$ and each $\mu, \nu \in M$.

3) odd (even) on $X$ if $f(-\mu) = -f(\mu)$ ($f(-\mu) = f(\mu)$)
for every $\mu \in X$.

We shall say that a mapping $F: X \to Y$ possesses the Baire property in the set $M \subset X$ of the second category in $X$ if there exists a subset $N \subset M$ of the 1st category in $M$ such that the restriction $F/_{M-N}$ of $F$ to $M-N$ is continuous. A set $N \subset X$ is said to be a Baire set in $X$ if there exists an open set $G \subset X$ so that $G-N, N-G$ are both the sets of the 1st category in $X$. For Gâteaux differentials and Gâteaux derivative we use the notions and notations given in the Vajnberg's book [1, chapt.I].

By the one-sided Gâteaux differential $V^+ f(\mu, h)$ of a convex functional $f$ at $\mu_0 \in X$ we mean the limit
$$\lim_{t \to 0^+} \frac{1}{t} [f(\mu_0 + th) - f(\mu_0)] = V^+ f(\mu_0, h), \ h \in X.$$ 
If $f$ is convex and finite on $X$, the one-sided Gâteaux differential $V^+ f(\mu, h)$ exists for every $\mu, h \in X$ and it is subadditive positive homogeneous functional in $h \in X$ for every (but fixed) $\mu \in X$ [2, chapt.10]. Therefore
$$f(\mu + th) - f(\mu) = V^+ f(\mu, th) + o_h(\mu, th), \mu, h \in X,$$
where
$$\lim_{t \to 0^+} \frac{o_h(\mu, th)}{t} = 0.$$
§ 2. We start with the following

**Theorem 1.** Let $X, Y$ be linear normed spaces, $F : X \to Y$ so that $F(tu) = tF(u)$ for every $u \in X$ and $t \in (0, t_0)$, where $t_0 < 1$. Under this assumption the following assertions are valid:

(a) $F$ is positively homogeneous on $X$.

(b) If $F$ possesses a linear Gâteaux differential $DF(0, h)$ at $0$, then $F$ is linear on $X$.

(c) If $F$ has a Gâteaux derivative $F'(0)$ at $0$, then $F$ is linear and continuous on $X$.

(d) If $F$ has a linear Gâteaux differential $DF(u, h)$ on the segment $(0, t_0 h) = \{ u \in X : u = t v, 0 < t < t_0 \}$ for some $0 \neq v \in X$ and $\lim_{t \to t_0} \| DF(t v, v) \| = 0$,

$$\lim_{t \to t_0} \frac{F(t v, v) - F(0, v)}{t} = 0$$

for an arbitrary (but fixed) $h \in X$, then $F$ is linear on $X$.

**Proof.** (a) By our hypothesis there exists

$$\lim_{t \to t_0^+} \frac{F(t v, v) - F(0, v)}{t} = 0$$

for every $h \in X$. As $\nabla F(0, h)$ is positively homogeneous in $h \in X$, $F(h)$ has the same property.

(b) is a strengthening of Th.1 [3]. It can be proved more simply as follows: $\nabla F(0, h)$ exists and $\nabla F(0, h) = F(h), h \in X$. Since $F$ has $DF(0, h)$ at $0$, then $\nabla F(0, h) = DF(0, h) = F(h), h \in X$ and hence $F$ must be linear in $h \in X$.

(c) is clear. (d) is a slight generalization of Th.2 [3].

**Theorem 2.** Let $F : X \to Y$ be a mapping of $X$ into $Y$ so that $F(0) = 0$. Assume $F$ possesses the Gâteaux differential $VF(0, h)$ at $0$. Then $F$ is homogeneous on $X$ if the remainder $\omega(0, h)$ of
VF(0, h) is homogeneous on X. Moreover, let F possess a linear Gâteaux differential DF(0, h) at 0. Then its remainder \( \omega(0, h) \) is homogeneous in \( h \in X \)

Proof. Since \( F(0) = 0 \) and F possesses the Gâteaux differential \( VF(0, h) \) at 0, we have that

\[
F(tu) - VF(0, tu) + \omega(0, tu),
\]

\[
tF(u) - tVF(0, u) + t \cdot \omega(0, u)
\]

for every \( u \in X \) and \( t \in E \). Being \( VF(0, u) \) homogeneous in \( u \in X \),

\[
F(tu) - tF(u) = \omega(0, tu) - t \omega(0, u).
\]

Hence \( F(tu) = tF(u), u \in X, t \in E \) and \( \omega(0, tu) = t \omega(0, u), u \in X, t \in E \). The second assertion follows at once from the first part of Th.2 and from the results (a), (b) of Th.1. Theorem is proved.

**Theorem 3.** Let \( X \) be a linear normed space, \( f \) a convex finite functional on \( X \). Under this assumption the following assertions are valid:

(a) If \( f(tu) = tf(u) \) for every \( u \in X \) and each \( t \in (0, t_0) \), where \( t_0 < 1 \), then \( f \) is subadditive and positive homogeneous on \( X \). Moreover, if \( f \) possesses the Gâteaux differential \( VF(0, h) \) at 0, then \( f \) is linear on \( X \).

(b) If \( f(0) = 0 \) and \( \omega^+(0, h) \) is subadditive in \( h \in X \), then \( f \) is subadditive on \( X \).

(c) If \( f(0) = 0 \), then \( f \) is positive homogeneous on \( X \).

(d) If \( f \) is continuous subadditive functional on \( X \).
and $f(0) = 0$, then $f(tu) \leq tf(u)$ for every $u \in X$ and each $t \geq 0$.

**Proof.** (a) Being $f$ convex, $V_+ f(u, h)$ (for fixed $u \in X$) is subadditive and positive homogeneous on $X$. As $f(0) = 0$, we have for $u, v \in X$, $t \in (0, t_0)$ that

$$f(t(u+v)) = V_+ f(0, t(u+v)) + \omega_+ (0, t(u+v)),$$

(1) $f(tu) = V_+ f(0, tu) + \omega_+ (0, tu)$,

$$f(tv) = V_+ f(0, tv) + \omega_+ (0, tv).$$

Since $f$ is convex, $\omega_+ (0, h) \geq 0$ for every $h \in X$ (Lemma 2 [4]). In view of subadditivity of $V_+ f(u, h)$ and our hypothesis

$$f(u+v) - f(u) - f(v) \leq \frac{1}{t} \omega_+ (0, t(u+v)) - \frac{1}{t} \omega_+ (0, tu) - \frac{1}{t} \omega_+ (0, tv) \leq \frac{1}{t} \omega_+ (0, t(u+v))$$

for every $u, v \in X$, $t \in (0, t_0)$. As

$$\frac{1}{t} \omega_+ (0, t(u+v)) \to 0 \quad \text{whenever } t \to 0_+,$$

$$f(u+v) \leq f(u) + f(v) \quad \text{for every } u, v \in X$$

The second assertion of (a) is an immediate consequence of Theorem 1(a), (b) and Remark 2 [3].

(b) To prove (b) write (1) without $t$ on the left and right sides and use the property that $V_+ f(0, h)$, $\omega_+ (0, h)$ are both subadditive on $X$.

(c) The one-sided differential $V_+ f(0, h)$ of $f$ at 0 is positively homogeneous on $X$. Hence the assertion (c) is a consequence of Theorem 2.

(d) Convexity of $f$ implies ($u \in X$, $t \in <0, 1>$) that
\[ f(tu) = f(tu + (1-t) \cdot 0) \leq tf(u) + (1-t) f(0) = tf(u). \]

Hence \( f(tu) \leq tf(u) \) for each \( u \in X \) and \( t \in (0, 1) \).

Let \( \kappa = \frac{m}{n} \) be a rational number \((m, n \text{ are positive integers})\). In view of subadditivity of \( f \) and the last inequality we have that

\[ f\left(\frac{m}{n} - u\right) \leq m f\left(\frac{1}{n} u\right) \leq \frac{m}{n} f(u). \]

Let \( t \) be a positive irrational number. Then there exists a sequence of rational numbers \( \kappa_n > 0 \) so that \( \kappa_n \to t \). Continuity of \( f \) gives

\[ f(tu) = \lim_{n \to \infty} \kappa_n f(u) \leq \lim_{n \to \infty} \kappa_n f(u) = tf(u), \]

which proves c). This concludes the proof of our theorem.

Remark 1. The assertion (a) of Th.3 one may prove simpler using the properties of \( V f(0, h) \). But we gave preference to the given proof (a) because the proof of the assertion (b) is based on the same arguments as (a).

Theorem 4. Let \( f \) be a subadditive functional on \( X \). Assume there exist a neighborhood \( V(0) \) of \( 0 \) and a functional \( g \) defined on \( V(0) \) so that \( g(0) = 0 \) and \( f(u) \leq g(u) \) for each \( u \in V(0) \). If \( g \) possesses a linear Gâteaux differential \( Dg(0, h) \) at \( 0 \), then \( f \) is linear on \( X \).

Proof. Subadditivity of \( f \) implies that \( f(0) \leq 2 f(0) \). Hence \( f(0) \geq 0 \). But \( 0 \leq f(0) \leq g(0) = 0 \). Suppose \( u \in X, \ h \in X, \ t > 0 \). Then

\[ f(u) = f(u + h - h) \leq f(u + h) + f(-h) \]

and subadditivity of \( f \) implies that

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(2) \(-f(-h) \leq f(u+h) - f(u) \leq f(h)\).

For sufficiently small \(t > 0\), \(th \in V(0)\Rightarrow f(th) = g(th)\). Replace in (2) \(th\) for \(h\) and divide it by \(t > 0\), we have for \(t > 0\) small enough that

\[
\frac{g(-th)}{t} \leq \frac{f(-th)}{t} \leq \frac{f(u+th) - f(u)}{t} \leq \frac{f(th)}{t} \leq \frac{g(th)}{t}.
\]

As \(g(th) = Dg(0,th) + \omega(0,th)\), we obtain that

\[
Dg(0,h) - \frac{1}{t} \omega(0,t(-h)) \leq \frac{1}{t} [f(u+th) - f(u)] \leq Dg(0,h) + \frac{1}{t} \omega(0,th).
\]

Since the limits on the left and the right side of (3) exist and are equal to \(Dg(0,h)\), we conclude that

\[
\lim_{t \to 0^+} \frac{1}{t} [f(u+th) - f(u)] = Vf(u,h) = Dg(0,h)
\]

for each \(h \in X\) and \(u \in X\). Hence \(Vf(u,h) = Vf(u,h) = Dg(0,h)\) for every \(u \in X\) and \(h \in X\). Therefore \(f\) possesses the Gâteaux differential \(Vf(u,h)\) for every \(u \in X\) and \(Vf(u,h) = Dg(0,h)\) for every \(u \in X\), \(h \in X\). As \(f(0) = 0\), by the mean-value theorem

\[
f(u) = Vf(\tau u, u) = Dg(0,u), \quad u \in X, \quad 0 < \tau < 1.
\]

Since \(Dg(0,u)\) is linear on \(X\), our theorem is proved.

Remark 2. The final part of the proof of Th.4 may be done as follows: From (3) it follows that
As \( t \to 0 \), the term on the right side tends to 0. Hence \( f \) possesses a Gateaux differential \( Vf(\mu, h) \) at every point \( \mu \in X \) and \( Vf(\mu, h) = Dg(0, h), h \in X, \mu \in X \). By the mean-value theorem \( f(\mu) = Vf(\tau \mu, u) = V_\mu Dg(0, u), u \in X \).

**Corollary 1.** Let \( f \) be a subadditive functional on \( X \) so that \( f(0) = 0 \). Assume \( f \) possesses a linear Gateaux differential \( Df(0, h) \) at 0. Then \( f \) is linear on \( X \).

**Corollary 2.** Let \( f \) be a convex finite functional on \( X \). Assume there exist a neighborhood \( V(0) \) of \( 0 \in X \) and a functional \( g \) defined on \( V(0) \) so that \( g(0) = 0 \) and \( V_+ f(\mu_0, h) \leq g(h) \) for each \( h \in V(0) \) and for some \( \mu_0 \in X \). If \( g \) has a linear Gateaux differential \( Dg(0, h) \) at 0, then \( f \) possesses a linear Gateaux differential \( Df(\mu_0, h) \) at \( \mu_0 \). Moreover, if \( f \) is continuous, then \( f \) possesses the Gateaux derivative \( f'(\mu_0) \) at \( \mu_0 \).

**Proof.** The one-sided Gateaux differential \( V_+ f(\mu_0, h) \) is subadditive functional on \( X \). From Theorem 4 it follows that \( V_+ f(\mu_0, h) \) is linear in \( h \in X \) on \( X \). Hence \( V_+ f(\mu_0, h) = V_- f(\mu_0, h) \) for every \( h \in X \). This shows that \( f \) has the Gateaux differential \( Vf(\mu_0, h) \) at \( \mu_0 \). But convexity of \( f \) implies that \( Vf(\mu_0, h) = Df(\mu_0, h), h \in X \) (Remark 2 [3]). If \( f \) is continuous, using Proposition
6 [5], we obtain that $D^f(u, h) = f'(u)h$, $h \in X$.

This completes the proof.

**Corollary 3.** Let $f$ be a continuous subadditive functional on $X$. If $f(tu) \leq \varphi(t)f(u)$ for every $u \in X$ and $t \in (0, t_0)$, where $t_0 < 1$, $\varphi$ is a real function on $(0, t_0)$ so that $\lim_{t \to 0^+} \frac{\varphi(t)}{t} = 0$, then $f(u) = 0$ for every $u \in X$.

**Proof.** First of all, $f$ has the Gâteaux derivative $f'(u)$ on $X$ and $f'(u) = 0$ for every $u \in X$.

By Theorem 8.6.1 [6], Chapt. VIII (here we must point out that this theorem is valid even for mappings which have the Gâteaux derivative only) we get that $f(u) = c = \text{const.}$ for every $u \in X$. Since $f(0) \geq 0$, $c \geq 0$. Suppose that $c > 0$. Then we have $c = f(tu) \leq \varphi(t)f(u) = \varphi(t)$ for every $u \in X$ and each $t \in (0, t_0)$. Hence $\frac{1}{t} \leq \frac{\varphi(t)}{t}$ for each $t \in (0, t_0)$ which contradicts with the fact that $\lim_{t \to 0^+} \frac{\varphi(t)}{t} = 0$. Therefore $c = 0$ and $f(u) = 0$ for every $u \in X$.

This completes the proof. Corollaries 1, 3 show that functionals considered in Theorem 1 [7], Th. 1 [8], Th. 4 [3] are linear.

**Theorem 5.** Let $f$ be an odd subadditive functional on $X$. Suppose $f$ is continuous at $0$. Then $f$ is linear and continuous on $X$.

**Proof.** The inequality $2f(0) \geq f(0)$ implies that $f(0) \geq 0$. On the other hand we have for $u \in X$ that $0 \leq f(0) = f(u - u) \leq f(u) + f(-u)$.
Theorem 6. Let \( f \) be a subadditive functional on \( X \) having a linear Gâteaux differential \( Df(\mu_0, h) \) at some point \( \mu_0 \in X \). If \( f(-\mu_0) = -f(\mu_0) \), then \( f \) is linear on \( X \).

Proof. From \( 0 \leq f(0) = f(\mu_0 - \mu_0) = f(\mu_0) + f(-\mu_0) = 0 \) it follows that \( f(0) = 0 \). If \( \mu_0 = 0 \), then \( f \) is linear by Corollary 1. Suppose that \( \mu_0 \neq 0 \) and that \( h \in X \) is an arbitrary element of \( X \). From \( f(\mu_0) = f(\mu_0 - h + h) \leq f(\mu_0 - h) + f(h) \) and using our hypothesis we have that

\[
\begin{align*}
 f(\mu_0) - f(\mu_0 - h) &\leq f(h) = f((\mu_0 + h) - \mu_0) \\
&\leq f(\mu_0 + h) + f(-\mu_0) = f(\mu_0 + h) - f(\mu_0).
\end{align*}
\]

Consider \( t > 0 \), replace in these inequalities \( h \) by \( th \) and then divide by \( t > 0 \), we get that

\[
\frac{1}{t} [f(\mu_0) - f(\mu_0 - th)] \leq \frac{1}{t} f(th) \leq \frac{1}{t} [f(\mu_0 + th) - f(\mu_0)].
\]

Since \( f \) possesses a linear Gâteaux differential \( Df(\mu_0, h) \) at \( \mu_0 \), we obtain \( (t > 0) \).
\[
Df(u_0, h) - \frac{\omega(u_0, t(-h))}{t} \leq \frac{f(th)}{t} \leq \frac{Df(u_0, h) + \omega(u_0, th)}{t}.
\]

These inequalities imply that there exists

\[
\lim_{t \to 0_+} \frac{f(th)}{t}
\]

and that this limit is equal to \(Df(u_0, h)\) for every \(h \in X\). From this fact we conclude that \(f\) possesses a linear Gâteaux differential \(Df(u, h)\) on \(X\) and that \(Df(u, h) = Df(u_0, h)\) for every \(u, h \in X\). According to the mean-value theorem \(f(u) = Df(tu, u) = Df(u_0, h)\), \(u \in X\), 

\(0 < \varepsilon < 1\) which proves our theorem.

Corollary 4. Let \(f\) be subadditive functional on \(X\). If \(f\) is linear on some open subset \(M \neq 0\) of \(X\), then \(f\) is linear on \(X\).

Theorem 7. Let \(X\) be a linear normed space of the 2nd category in itself, \(f\) a subadditive functional on \(X\). Let one of the following three conditions be fulfilled: (a) \(f\) is even and upper-bounded on a Baire subset of the 2nd category in \(X\); (b) \(f\) is nonnegative on \(X\) and it is upper-bounded on a symmetric Baire subset of the 2nd category in \(X\); (c) \(f\) is even and there exist an open subset \(M \neq 0\) of \(X\), a functional \(q\) defined on \(M\) so that \(q\) possesses a Baire property in \(M\) and \(f(u) \leq q(u)\) for each \(u \in M\).

Then \(f\) is bounded in \(X\).

Proof. Assume (a). Then \(0 \leq f(0) = f(u - u) \leq f(u) + f(-u) = 2f(u), u \in X \Rightarrow f(u) \geq 0\) for
every $\mu \in X$. By our hypothesis there exist a Baire subset $B$ of the 2nd category in $X$ and a constant $C > 0$ so that $f(\mu) \leq C$ for each $\mu \in B$. Then the set $W$ of all differences $w = \mu - \nu$, where $\mu, \nu \in B$, is a neighborhood of $0$ in $X$. Hence there exists $\sigma_0 > 0$ such that $\|w\| < \sigma_0 \Rightarrow w \in W$. For any $w \in W$ with $\|w\| < \sigma_0$ we have $(w = \mu - \nu, \mu, \nu \in B) 0 \leq f(w) = f(\mu - \nu) = f(\mu) + f(\nu) \leq 2C$. Let $\mu$ be an arbitrary point of the ball $\|\mu\| \leq R$. Then there exists an integer $n_0$ so that $R n_0 \leq \sigma_0$. We obtain that $0 \leq f(\mu) = f(\frac{\mu}{n_0} n_0) = n_0 f(\frac{\mu}{n_0}) \leq 2C n_0$. This shows that $f$ is bounded in $X$. The proof of (b) is similar to that of (a).

Assuming (c) we see that $M$ is a set of the 2nd category in $X$. By our hypothesis there exists $\mu_0 \in M - A$, where $A$ is a set of the 1st category in $M$, so that the restriction $g|_{M - A}$ of $g$ to $M - A$ is continuous at $\mu_0$. Hence there exists a non-empty open subset $N \subset M$ so that $\mu_0 \in N$ and $\mu \in N - A \Rightarrow g(\mu) \leq g(\mu_0) + 1$. The set $B = N - A$ is a Baire set of the 2nd category in $X$. Hence $\mu \in B \Rightarrow f(\mu) \leq g(\mu_0) + 1$. The rest results at once from (a) of our theorem. Theorem is proved.

**Corollary 5.** Let $X$ be a linear normed space of the second category in itself, $f$ a subadditive even functional on $X$. If $f$ is upper semicontinuous at
some point \( u_0 \in X \), then \( f \) is bounded in \( X \).

In the sequel we shall use the so-called Banach-Steinhaus uniform-boundedness principle: Let \( X, X_1 \) be linear normed spaces, \( A \) be a set of the 2nd category in \( X \), \( M \) a set of linear continuous operators of \( X \) into \( X_1 \). If \( x \in A \implies \sup_{u \in M} \| u(x) \| < \infty \), then \( \sup_{u \in M} \| u \| < \infty \).

We prove the following

**Theorem 8.** Let \( X \) be a linear normed space of the second category in itself, \( f \) a convex continuous subadditive and finite functional on \( X \). Assume \( f \) possesses the Gâteaux differential \( Vf(\mu, h) \) on the set \( N \subset X \), \( N \neq \emptyset \).

Then there exists a constant \( C > 0 \) so that
\[
\| f'(\mu) \| \leq C
\]
for each \( \mu \in N \), where \( f'(\mu) \) denotes the Gâteaux derivative \( f'(\mu) \) of \( f \) at \( \mu \). In particular, if \( N \) is convex, then \( f \) is Lipschitzian on \( N \) with constant \( C \).

**Proof.** If \( 0 \in N \), then \( f \) is additive on \( X \) according to Theorem 3 a). Being \( f \) continuous it is homogeneous and hence linear on \( X \). Therefore our conclusions are trivially fulfilled.

Assume that \( 0 \notin N \). By Proposition 6 [5] \( Vf(\mu, h) = f'(\mu)h \) for each \( \mu \in N \) and every \( h \in X \). Using Lemma 2 [4] and subadditivity of \( f \) we obtain
\[
-f(-h) \leq f(\mu) - f(\mu - h) \leq f'(\mu)h \leq f(\mu + h) - f(\mu) \leq f(h)
\]
for each \( \mu \in N \) and \( h \in X \). Hence
for each $u \in N$ and $h \in X$. By theorem 2.5.3 \[ |f'(u)h| \leq \max (|f(h)|, |f(-h)|) \]

for each $u \in N$ and $h \in X$. By theorem 2.5.3 [9]

$|f(h)| \leq M_f(\|h\| + 1)$ for every $h \in X$, where

$0 \leq M_f = \sup_{\|h\| \leq 1} f(h) < + \infty$. Hence $h \in X \Rightarrow \sup_{u \in N} |f'(u)h| \leq M_f(\|h\| + 1)$. According to Banach-Steinhaus principle there exists a constant $C > 0$ so

$\sup_{u \in N} \|f'(u)\| \leq C$. Thus the first part of our theorem is proved. To prove the second assertion it is sufficient to use the above fact and the mean-value theorem. This concludes the proof.

References


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