Ivan Kolář
Order of holonomy and geometric objects of manifolds with connection

Commentationes Mathematicae Universitatis Carolinae, Vol. 10 (1969), No. 4, 559--565

Persistent URL: http://dml.cz/dmlcz/105251

Terms of use:

© Charles University in Prague, Faculty of Mathematics and Physics, 1969

Institute of Mathematics of the Academy of Sciences of the Czech Republic provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these Terms of use.

This paper has been digitized, optimized for electronic delivery and stamped with digital signature within the project DML-CZ: The Czech Digital Mathematics Library http://project.dml.cz
ORDER OF HOLONOMY AND GEOMETRIC OBJECTS OF MANIFOLDS WITH CONNECTION

Ivan KOLÁŘ, Brno

Our considerations are in the category $\mathcal{C}^\infty$. The standard notations of the theory of jets are used throughout the paper, see [3].

1. Let $P(B, G)$ be a principal fibre bundle with base $B$ and structure Lie group $G$ and let $\Phi = PP^{-1}$ be the groupoid associated to $P$. Let $H$ be a closed subgroup of $G$, let $F = G/H$ be the corresponding homogeneous space and let $E = E(B, F, G, P)$ denote the fibre bundle associated to $P$ with standard fibre $F$; so that $\Phi$ is a groupoid of operators on $E$. Let $\pi$ be the canonical projection $\pi: E \to B$; we shall write $E_x = \pi^{-1}(x), x \in B$.

$\mathcal{Q}_\kappa(\Phi)$ or $\mathcal{Q}_\kappa(\tilde{\Phi})$ or $\mathcal{Q}_\kappa(\bar{\Phi})$ means the fibred manifold of all non-holonomic or semi-holonomic or holonomic elements of connection of order $\kappa$ on $\Phi$ respectively, see [4]. A non-holonomic or semi-holonomic or holonomic connection of order $\kappa$ (shortly: an $\kappa$-connection) on $\Phi$ is a global section $C: B \to \mathcal{Q}_\kappa(\Phi)$ or $C: B \to \mathcal{Q}_\kappa(\tilde{\Phi})$ or $C: B \to \mathcal{Q}_\kappa(\bar{\Phi})$ respectively.
Let $V$ be a manifold, let $Z \in \mathcal{F}(V, E)$ and let $X \in \mathcal{Q}^*(\Phi)$ such that $\alpha X = \mu(\beta Z) = \kappa$. Then the development $X^{-1}(Z)$ of $Z$ into $E_\kappa$ by means of $X$ is defined by

$$X^{-1}(Z) = (X^{-1}\mu Z), Z \in \mathcal{F}(V, E_\kappa),$$

where $\bullet$ means the prolongation of the partial composition law $(\theta, x) \mapsto \theta x$, $\theta \in \Phi$, $x \in E$, see [4]. (We remark that Ehresmann uses the term "the absolute differential of $Z$ with respect to $X$" for $X^{-1}(Z)$.)

Obviously, if $Z \in \mathcal{F}(V, E)$ or $\mathcal{F}(V, E)$ and $X \in \mathcal{Q}(\Phi)$ or $\mathcal{Q}(\Phi)$, then $X^{-1}(Z) \in \mathcal{F}(V, E_\kappa)$ or $\mathcal{F}(V, E_\kappa)$ respectively. Furthermore, if $Z = j_F^\kappa \sigma$, where $\sigma$ is a local section in $E$, then we write $X^{-1}(\sigma)$ instead $X^{-1}(j_F^\kappa \sigma)$ and $X^{-1}(\sigma)$ is called the development of $\sigma$ into $E_\kappa$ by means of $X$.

Let $\mathcal{C}$ be an $\kappa$-connection on $\Phi$, then $C'$ means the prolongation of $C$, which is an $(\kappa+1)$-connection on $\Phi$, see [4]. The $\kappa$-th prolongation of $C$ is defined by iteration $C^{(\kappa)} = C^{(\kappa-1)}$. Every $1$-connection $C$ determines a sequence $C, C', \ldots, C^{(\kappa)}, \ldots$ of semi-holonomic connections. The terms of such a sequence are called simple connections.

**Definition 1.** A space $\mathcal{S}$ with $\kappa$-connection is a quintuple $\mathcal{S} = \mathcal{S}(B, \Phi, E, \sigma, C)$, where $\sigma$ is a global section in $E$ and $C$ is an $\kappa$-connection on $\Phi$.

**Remarks.** For $\kappa = 1$, our definition is equivalent to the definition of a space with connection by A. Svec [7]. The sequence $\mathcal{S}^{(\kappa-1)}(B, \Phi, E, \sigma, C^{(\kappa-1)}), \kappa = 1, 2, \ldots$, of spaces with simple connections is canonically associa-
2. A (holonomic) contact element of dimension $m$ and of order $\kappa$ (shortly: a \textit{contact $m^\kappa$-element}) on a manifold $\mathcal{V}$ at a point $x \in \mathcal{V}$ is the set $X L_x^\kappa$, where $X$ is an $m^\kappa$-velocity on $\mathcal{V}$ at $x$. Such a contact element is called regular, if $m < m = \dim \mathcal{V}$ and if $X$ is a regular velocity. The fibred manifold of all regular contact $m^\kappa$-elements on $\mathcal{V}$ will be denoted by $K_m^\kappa(\mathcal{V})$. Let $\mathcal{U}$ be another manifold and let $Z \in \mathcal{J}_m^\kappa(\mathcal{V} \times \mathcal{U})$, then $Z$ determines a contact $m^\kappa$-element $\kappa(Z)$ on $\mathcal{U}$ at $\beta Z$, $\kappa(Z) = Z L_{\beta Z}^\kappa$, where $L_\beta$ is a (holonomic) $\kappa$-frame on $\mathcal{V}$ at $\alpha Z$.

A manifold $N$ together with a left action of a group $G$ on $N$ is called a $G$-space, see e.g. [1], p.31. A mapping $\varphi$ of $N$ into another $G$-space is called a $G$-mapping, if $\varphi(g \cdot x) = g \cdot \varphi(x)$ for every $x \in N$, $g \in G$. Let $F$ be as above, then the action of $G$ on $F$ is canonically extended to an action on $K_m^\kappa(F)$, so that $K_m^\kappa(F)$ is a $G$-space.

\textbf{Definition 2.} A geometric $m^\kappa$-object $O'$ on $F$ with values in a $G$-space $S$ is a $G$-mapping of $K_m^\kappa(F)$ into $S$. More generally, let $W$ be an invariant subspace of $K_m^\kappa(F)$, then a geometric $m^\kappa$-object on $F$ of type $W$ with values in $S$ is a $G$-mapping of $W$ into $S$.

Let $M$ be an $m$-dimensional submanifold of $F$, then $M$ determines canonically a contact $m^\kappa$-element.
A semi-holonomic contact \( m^\kappa \)-element on a manifold \( V \) is the set \( Y_{m^\kappa} \), where \( Y \) is a semi-holonomic \( m^\kappa \)-velocity on \( V \). Such a contact element is called regular, if \( m < m = \dim V \) and \( Y \) is regular; the fibred manifold of all regular semi-holonomic contact \( m^\kappa \)-elements on \( V \) will be denoted by \( \overline{K}_m^\kappa(V) \). Let \( U \) be another manifold and let \( Z \in \overline{J}_m^\kappa(V, U) \), then \( Z \) determines a semi-holonomic contact \( m^\kappa \)-element \( \kappa_0(Z) \) on \( U \), \( \kappa_0(Z) = Z \overline{h}_m^\kappa L_m^\kappa \), where \( \overline{h}_m^\kappa \) is a semi-holonomic \( \kappa \)-frame on \( V \).

**Definition 3.** Let \( F \) be as above. A semi-holonomic geometric \( m^\kappa \)-object on \( F \) with values in a \( G \)-space \( S \) is a \( G \)-mapping of \( \overline{K}_m^\kappa(F) \) into \( S \).

**Remark.** Analogous definition relates to the non-holonomic case as well.

**Definition 4.** A space \( \mathcal{G}(B, \Phi, E, \mathcal{C}) \) with \( \kappa \)-connection will be called a manifold with \( \kappa \)-connection,
if it holds
a) \( m = \text{dim} \, B \leq n = \text{dim} \, F \); 
b) \( C^{-1}(x)(\sigma) \) is regular for every \( x \in B \).

We shall also say that \( m = \text{dim} \, B \) is the dimension of \( \mathcal{F} \).

Remark. A manifold with a \( \mathbf{1} \)-connection is locally equivalent to a submanifold of a space with Cartan connection, cf.[2].

Consider an \( m \)-dimensional manifold with a semi-holonomic \( \mathbf{n} \)-connection and let \( \mathcal{O} \) be a semi-holonomic geometric \( m^\mathbf{r} \)-object on \( E_x, \ x \in B \). The development \( C^{-1}(x)(\sigma) \) of \( \sigma \) into \( E_x \) determines a semi-holonomic contact \( m^\mathbf{n} \)-element \( \mathcal{O}(C^{-1}(x)(\sigma)) \) on \( E_x \) and \( \mathcal{O}(C^{-1}(x)(\sigma)) \in \mathcal{S} \) will be called the value of \( \mathcal{O} \) for \( \mathcal{F} \) at \( x \in B \), so that a semi-holonomic geometric \( m^\mathbf{r} \)-object represents a geometric object for \( m \)-dimensional manifolds with semi-holonomic \( \mathbf{n} \)-connection. Moreover, if \( \mathcal{F}(B, \Phi, E, \sigma, C) \) is a manifold with \( \mathbf{1} \)-connection, then \( \mathcal{O} \) can be applied to the associated manifold \( \mathcal{F}(x^{-1})(B, \Phi, E, \sigma, C(x^{-1})) \) with semi-holonomic \( \mathbf{n} \)-connection; that's why a semi-holonomic geometric \( m^\mathbf{r} \)-object may also be considered as a geometric object of order \( \mathbf{n} \) for \( m \)-dimensional submanifolds of a space with Cartan connection.

4. A semi-holonomic contact \( m^\mathbf{n} \)-element \( \mathcal{Y}_m \mathbf{E}^\mathbf{n} \)
will be said holonomic, if it contains a holonomic \( m^\mathbf{n} \)-velocity.
Definition 5. A manifold $\mathcal{F}(\mathcal{B}, \Phi, \mathcal{E}, \sigma, \mathcal{C})$ with semi-holonomic $\kappa$-connection is called holonomic at $\times \in \mathcal{B}$, if the contact element $\mathcal{K}(C^{-1}(\times)(\sigma))$ is holonomic.

Let $\mathcal{O}$ be a semi-holonomic geometric $m^\kappa$-object on $E_{\times}$, then the restriction of $\mathcal{O}$ to $K_m^\kappa(E_{\times})$ is a holonomic geometric $m^\kappa$-object on $E_{\times}$, since $K_m^\kappa(E_{\times})$ is an invariant subspace of $K_m^\kappa(E_{\times})$. This proves the following

**Theorem.** If a manifold $\mathcal{F}$ with semi-holonomic $\kappa$-connection is holonomic at $\times \in \mathcal{B}$, then the value of every geometric object for $\mathcal{F}$ at $\times$ coincides with the value of a holonomic geometric $m^\kappa$-object on $E_{\times}$.

We can also restate this theorem in the following more intuitive way: if a manifold with semi-holonomic $\kappa$-connection is holonomic at a point, then all its geometric objects at this point coincide with the geometric objects of order $\kappa$ of an $m$-dimensional submanifold of the corresponding homogeneous space.

5. A manifold $\mathcal{F}(\mathcal{B}, \Phi, \mathcal{E}, \sigma, \mathcal{C})$ with a $\lambda$-connection is called $\kappa$-holonomic at $\times \in \mathcal{B}$, if the associated manifold $\mathcal{F}^{(\kappa-1)}(\mathcal{B}, \Phi, \mathcal{E}, \sigma, \mathcal{C}^{(\kappa-1)})$ is holonomic at $\times$. In this case, our theorem gives the conditions that every geometric object of order $\kappa$ of a submanifold of a space with Cartan connection coincides with a geometric object of a submanifold of the corresponding homogeneous space.
In [5], we consider a surface in a 3-dimensional space with projective connection from this point of view and we treat the conditions for \( \tau \)-holonomy geometrically in full details.

References


(Oblatum 15.9.1969)

Mathematical Institute of ČSAV
Janáčkovo nám. 2a, Brno

- 565 -