Commentationes Mathematicae Universitatis Carolinae

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*Commentationes Mathematicae Universitatis Carolinae*, Vol. 11 (1970), No. 1, 1--8

Persistent URL: `http://dml.cz/dmlcz/105261`

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11, 1 (1970)

# FACTOR-SPLITTING ABELIAN GROUPS OF RANK TWO

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In this paper we shall give a structural description of factor-splitting torsion free abelian groups of rank two.

Throughout this paper by a group it is always meant an additively written abelian group. A torsion free group $G$ is called factor-splitting if any its factor-group $G/H$ splits (see [3]). We shall use the following notation: If $g$ is an element of infinite order of a mixed group $G$ then $h_p(g)$ denotes the $p$-height of $g$ in the group $G$ (see [2]). $\{H\}_*^G$ denotes the pure closure of a subgroup $H$ in the torsion free group $G$. Instead $\{\{h\}\}_*^G$, $h \in G$ we shall write simply $\{h\}_*^G$. $R_p$ will denote the group of rationals with denominators prime to $p$. Other notation will be essentially that as in [1].

It will be useful to formulate the following statement (see Theorem 2 from [2]):

Let $G$ be a mixed group of torsion free rank one. Two following conditions are necessary and sufficient for $G$ to be split:

($\alpha$) If $T$ is the maximal torsion subgroup of $G$, then to any $g \in G \dot{-} T$ there exists an integer $m \neq 0$ such that $mg$ has in $G$ the same type as $g + T$ in $G/T$ .

($\beta$) To any $g \in G \dot{-} T$ there exists an integer $m \neq 0$ such that for any prime $p$ with $h_p(g+T) = \infty$ there exist the elements $h_0^{(p)} = mg, h_1^{(p)}, h_2^{(p)}, \dots$, such that $p h_{n+1}^{(p)} = h_n^{(p)}$, $n = 0, 1, 2, \dots$ .

Now we are ready to prove the main result:

Theorem 1: A torsion-free group $G$ of rank two is factor-splitting if and only if:

(1) For any two linearly independent elements $g, h \in G$ there is $(\{g\}_*^G \dot{+} \{h\}_*^G) \otimes R_p = G \otimes R_p$ for almost all primes $p$ with $h_p(g) \neq h_p(h)$ .

Proof: Proving the necessity let us suppose that there exist elements $g, h \in G$ which do not satisfy the condition (1). Without loss of generality we can assume that there exists an infinite set $\Pi'$ of primes with $h_p(g) < h_p(h)$ and $(\{g\}_*^G + \{h\}_*^G) \otimes R_p \subsetneqq G \otimes R_p$. For any prime $p \in \Pi'$ there is $h_p(h) < \infty$ (in the other case it is easy to see that $(\{g\}_*^G + \{h\}_*^G) \otimes R_p = G \otimes R_p$ ). Let us denote $l_p = h_p(h) - h_p(g)$ and let $h_p'$ be the solution of the equations $p^{l_p} x = h$.

In view of $(\{g\}_*^G \dot{+} \{h\}_*^G) \otimes R_p \subsetneqq G \otimes R_p$ there exist elements $g_p$ and non-zero integers $\alpha_p$ with $p^{h_p(g)+1} g_p = g + \alpha_p h_p'$. Hence $h_p(g + \{h\}) = h_p(g)$

but $h_p(g+\{h\}^G_*) \geq h_p(g) + 1$ such that $G/\{h\}$ does not satisfy the condition $(\alpha)$ and hence does not split.

Now we shall prove the sufficiency. In view of Lemma 2.6 from [4] it suffices to prove that for any $h \in G$ the factor-group $G/\{h\}$ splits. Let $g \in G \dot{-} \{h\}^G_*$ be an arbitrary element. Let

$\Pi_1 = \{p;\, h_p(g) = h_p(h)\}$ ,

$\Pi_2 = \{p;$ either $h_p(g) > h_p(h)$ or $h_p(g) < h_p(h)$ and $(\{g\}^G_* \dot{+} \{h\}^G_*) \otimes R_p = G \otimes R_p\}$ ,

$\Pi_3 = \{p,\, h_p(g) < h_p(h)$ and $(\{g\}^G_* \dot{+} \{h\}^G_*) \otimes R_p \neq G \otimes R_p\}$.

Then $\Pi_1$ , $\Pi_2$ , $\Pi_3$ are disjoint subsets of the set $\Pi$ of all primes whose union is $\Pi$. The set $\Pi_3$ is finite by hypothesis and it was mentioned in the proof of necessity that $h_p(h) < \infty$ . Let us put

(2) $$m = \prod_{p \in \Pi_3} p^{h_p(h) - h_p(g)}$$

Now we are going to prove that

(3) $$h_p(mg + \{h\}) = h_p(mg + \{h\}^G_*)$$

holds for any prime $p$. For $p \in \Pi_1$ we can assume $h_p(g) < \infty$ (if $h_p(g) = h_p(h) = \infty$ then (3) holds evidently). Suppose that the equation $p^{h_p(g)+k}x = g + h'$ is solvable in $G$ where $\rho h = \sigma h'$ for suitable relatively prime integers $\rho, \sigma$. Then $(\sigma, p) = 1$ (in the other case there is $h_p(h') < h_p(g)$ and the given equation has no solution). For suitable inte-

gers $\kappa, s$ there is $\sigma\kappa + p^{h_p(g)+k} s = 1$ and it holds $p^{h_p(g)+k}(\sigma\kappa x + s g) = g + \rho\kappa h$.
Hence

(4) $$h_p(g + \{h\}) = h_p(g + \{h\}_*^G) .$$

In view of $(p, m) = 1$ the formula (3) is valid, too.

Similar calculations show that (3) holds also in the case $p \in \Pi_2$, $h_p(g) > h_p(h)$ and in the case $p \in \Pi_3$. Finally, let $p \in \Pi_2$, $h_p(g) < h_p(h)$ and $(\{g\}_*^G \dotplus \{h\}_*^G) \otimes R_p = G \otimes R_p$. For $h_p(h) = \infty$ it holds (4) and hence (3) evidently. Suppose that $h_p(h) < \infty$. Let the equation $p^k x = g + h'$, $h \in \{h\}_*^G$ have a solution in $G$. In $G$ there exists an element $g'$ with $p^{h_p(g)} g' = g$. It is easy to see that any element of $\{g\}_*^G \otimes R_p$ is of the form $g' \otimes \alpha$, $\alpha \in R_p$. Now we have $p^k(x \otimes 1) = p^k x \otimes 1 = g \otimes 1 + h' \otimes 1$. By hypothesis there exists an element $g' \otimes \alpha$ in $\{g\}_*^G \otimes R_p$ for which $p^k(g' \otimes \alpha) = g \otimes 1 = g' \otimes p^{h_p(g)}$. Hence $g' \otimes (p^k \alpha - p^{h_p(g)}) = 0$ and then $p^k \alpha = p^{h_p(g)}$, which implies $k \leq h_p(g)$. We have shown

(5) $$\begin{aligned} & h_p(g) < h_p(h),\ (\{g\}_*^G \dotplus \{h\}_*^G) \otimes R_p = G \otimes R_p \Longrightarrow \\ & \Longrightarrow h_p(g + \{h\}_*^G) = h_p(g) . \end{aligned}$$

Now it is easy to derive the validity of (3) which shows that the condition $(\alpha)$ is satisfied.

Now we are proceeding to the condition $(\beta)$. Suppose that $h_p(g + \{h\}_*^G) = \infty$. At first, let $p \in \Pi$ be such a prime that $h_p(g) \geq h_p(h)$. Then there exists a $p$-adic integer $\mathfrak{A} = (a^{(k)})$ with $p^k x_k = g + a^{(k)} h$ solvable in $G$ for any $k = 1, 2, \dots$ (see [5]). Hence $p^k(p x_{k+1} - x_k) = (a^{(k+1)} - a^{(k)}) h$ such that $p(x_{k+1} + \{h\}) = x_k + \{h\}$. If $m$ is defined by (2) then clearly the same holds for $mg$ and $mx_k$.

In the case of $h_p(g) < h_p(h)$ and $(\{g\}_*^G + \{h\}_*^G) \otimes R_p = G \otimes R_p$ there is $h_p(g + \{h\}_*^G) = h_p(g) < \infty$ by (5) and hence there is nothing to prove. Finally, for $p \in \Pi_3$ there is $h_p(mg) = h_p(h)$ and it suffices to repeat the first part for $mg$ and $h$. Hence the condition $(\beta)$ is also satisfied which finishes the proof of our Theorem.

<u>Theorem 2</u>: Any homogeneous torsion free group of rank two is factor-splitting.

<u>Proof</u>: The condition (1) is clearly satisfied in this case.

The following Theorem shows that there is a great variety of factor-splitting torsion free groups of rank two. For any subset $\Pi' \subset \Pi$ we shall define the group $R_{\Pi'}$ as the group of all rationals with denominators relatively prime to any $p \in \Pi'$.

<u>Theorem 3</u>: Let $\Pi_1$, $\Pi_2$ be disjoint subsets of $\Pi$ such that $\Pi \doteq (\Pi_1 \cup \Pi_2)$ is finite and let $G$ be a torsion free group of rank two.

If $G \otimes R_{\Pi_1}$ is completely decomposable and

$G \otimes R_{\Pi_2}$ homogeneous then $G$ is factor-splitting.

<u>Proof</u>: If $\Pi'$ is any set of primes, then

(6) $h_p(g) = h_p(g \otimes 1)$, $p \in \Pi'$ and the second height is meant in $G \otimes R_{\Pi'}$.

Clearly, $h_p(g) \leq h_p(g \otimes 1)$. On the other hand let $p^{\ell}(\sum_{i=1}^{m} g_i \otimes \frac{k_i s}{s_i}) = g \otimes 1$, $(s_i, p) = 1$. If we put $s = s_1 \cdot s_2 \cdot \ldots \cdot s_m$ we have $(s, p) = 1$ and

$s \cdot p^{\ell}(\sum_{i=1}^{m} g_i \otimes \frac{k_i}{s_i}) = p^{\ell}(\sum_{i=1}^{m} \frac{k_i s}{s_i} g_i) \otimes 1 = sg \otimes 1$,

therefore $p^{\ell} \sum_{i=1}^{m} \frac{k_i s}{s_i} g_i = sg$ and hence the equation $p^{\ell}x = g$ is solvable in $G$.

Now let $g, h$ be any two linearly independent elements from $G$. Then in view of homogeneity of $G \otimes R_{\Pi_2}$ and (6) it holds $h_p(g) = h_p(h)$ for almost all primes $p \in \Pi_2$. Suppose that $p \in \Pi_1$, $h_p(g) \neq h_p(h)$ and $(\{g\}_*^G + \{h\}_*^G) \otimes R_p \subsetneq G \otimes R_p$. It may be easily shown that there exists an element

$u \otimes 1 \in G \otimes R_p \dot{-} (\{g\}_*^G \dot{+} \{h\}_*^G) \otimes R_p$ with

$p(u \otimes 1) \in (\{g\}_*^G \dot{+} \{h\}_*^G) \otimes R_p$. Hence $p(u \otimes 1 \otimes 1)$

lies in $(\{g\}_*^G \dot{+} \{h\}_*^G) \otimes R_{\Pi_1} \otimes R_p$ and in view of (6) $u \otimes 1 \otimes 1$ does not lie in this group. But this can occur for a finite number of $p \in \Pi_1$ only in view of the complete decomposability of $G \otimes R_{\Pi_1}$, Theorem 3 from [4] and Theorem 1. Hence $G$ satisfies the condition (1) and our proof is finished.

Let $\pi'$ be any set of primes. We call a torsion free group $G$ homogeneous with respect to $\pi'$ if the types of any two non-zero elements from $G$ restricted on $\pi'$ are equal. Now it is easy to see that Theorem 3 can be formulated in the following way:

<u>Theorem 3'</u>: Let $G$ be a torsion free group of rank two and $x_1$, $x_2$ any its basis. Let us denote by $\pi_1$ the set of those primes $p$ for which the $p$-primary component of $G/(\{x_1\}_*^G \dot{+} \{x_2\}_*^G)$ vanishes. If $G$ is homogeneous with respect to $\pi_2 \dot{-} \pi'$ where $\pi'$ is finite and $\pi_2 = \pi \dot{-} \pi_1$ then $G$ is factor-splitting.

<u>Remark</u>: The special cases of Theorem 4 are the following: 1) If $\pi_1$ is finite and $G$ is divisible with respect to $\pi_2 \dot{-} \pi'$ then $G$ is almost divisible (see [3], Theorem 5). If $\pi_2 = \pi \dot{-} \pi_1$ is finite then $G$ is primitive (see [3], Theorem 2).

R e f e r e n c e s

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(Oblatum 7.11.1969)