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ON EXISTENCE OF THE WEAK SOLUTION FOR NON-LINEAR PARTIAL DIFFERENTIAL EQUATIONS OF ELLIPTIC TYPE

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Introduction. In this paper we shall be concerned with existence and uniqueness of the weak solution for a boundary value problem of equations of the form

$$\sum_{|\ell| \leq \infty} (-1)^{|\ell|} \partial_{\ell} a_{i}(x, D^i u) = f,$$

where the growth of $a_{i}(x, \xi_{i})$ in $\xi_{i}$ is considered in a wide span.

We use well-known methods in reflexive spaces, namely the calculus of variations and the method of monotone operators. These methods are discussed and developed in the works of Browder [5],[6]; Nečas [1],[2]; Vajnberg [8]; Leray-Lions [7] etc.

The mentioned authors consider the growth from below and from above, given by polynomials, having the same degree, e.g.,

$$-c + c_{i} |\xi|^{m} \leq \sum_{|\ell| \leq \infty} \xi_{i} \partial_{\ell} a_{i}(x, \xi_{i}) \leq c_{i} (1 + |\xi|^{m}),$$

$m > 1$ is a real number.

This condition can be weakened for the derivatives $D^i u$ with $|\xi| < \infty$, because of theorems of imbedding.

We shall use the same notations as in [1],[2], as
there are here many references to those works. We shall denote

\[ D^i = \frac{\partial^{\mid i\mid}}{\partial x_{i_1} \cdots \partial x_{i_N}}, \]

where \( i \) is a multiindex, i.e.,

\[ i = (i_1, \ldots, i_N) \]

is a vector, \( i_2 \) for \( l = 1, \ldots, N \) are non-negative integers and \( |i| = \sum_{l=1}^{N} i_2 \leq k \).

In the present paper, the growth \( a_i(x, \xi_j) \) in \( \xi_j \) is described by functions of certain classes.

Let us consider real functions \( g(u) \), for which there exists a positive number \( u_0 \) such that

\[ g(u) \in (C(\infty, \infty) \cap C^1(u_0, \infty)); \]

\( g(u) + u g'(u) \) is non-decreasing and

\[ \lim_{u \to \infty} (g(u) + u g'(u)) = \infty; \]

\( u \cdot g(u) \) is even for \( |u| \geq u_0 \).

II For each \( l \geq 1 \) there exists a constant \( c(l) \) such that

\[ g(lu) \leq c(l) \cdot g(u) \text{ for each } u \geq u_0. \]

III There exists \( l > 1 \) such that

\[ g(u) \leq \frac{1}{l} g(lu) \text{ for each } u \geq u_0. \]

Now, we shall denote \( m_1; m_2; m_3 \) the classes of the functions \( g(u) \) satisfying I, I and II, I, II and III. Let us have \( g_i(u) \in M_1 \) for all \( |i| \leq k \) and suppose \( g_i(u) \geq g_j(u) \) (resp. \( g_i(u) \leq g_j(u) \)) for each \( i, j \) with \( |i|, |j| \leq k \) and \( u \geq u_0 \).

Then the condition for the growth possesses the form

\[ |a_i(x, \xi_j)| \leq C(1 + \sum_{|i| \leq k} g_i(x, \xi_j)) \text{ for } |i| \leq k \]

and \( g_i(u) \leq \min\{g_i(u)|, g_j(u)| \}; \quad (lu \geq u_0) \).
Thus, the growth $a_i(x,\partial^i u)$ in $D^i u$ is limited only by the properties of $q_i(\omega)$ and growth $a_i(x,\partial^i u)$ in $D^i u$ for $i \neq j$ is limited by the functions $q_i(\omega)$ and $q_j(\omega)$ in a very simple form.

In this work we find the weak solution even in such cases, when the degrees of polynomials differ by estimating from above and from below at the same member.

We construct Orlicz spaces $L^\Phi_\Omega(\Omega)$ by means of functions $G_i(\omega) = \omega q_i(\omega)$ - see Krasnosel'skij - Rutickij [4]. Then we construct a space $W^{\Phi}_G(\Omega)$ of Sobolev's type in the following way: $W^{\Phi}_G(\Omega) = \{\omega \in L^\Phi_\Omega(\Omega), \|\omega\| \leq \Phi \}$.

$\Omega$ is a bounded domain of $\mathbb{R}^N$ ($N$-dimensional Euclidean space).

To the given equation we choose $q_i(\omega)$ so close­ly as to obtain even a coerciveness. In special cases, the algebraic condition for coerciveness is of the form

$$\sum_{i,j} c_{i,j} q_i(\omega) \geq c \sum_{i,j} \xi_i q_i(\xi_i) - c$$

When the growth is described by $q_i(\omega) \in M_3$, for $|i| \leq \Phi$, then $W^{\Phi}_G(\Omega)$ is reflexive.

By the class $M_2$, we can describe even a very small growth, e.g.,

$$a_i(x,\xi) = \begin{cases} \frac{\xi_i}{|\xi_i|} \ln (|\xi_i| + 1) & \text{Euler's equation of Example a) in §3, being of this type, stands very near to the equation for minimal surfaces.} \\
\end{cases}$$

By means of the class $M_1$ we can describe a very wide span of growths even very fast, e.g.

$$a_i(x,\xi) = \xi_i e^{(\xi_i)^2}$$. In both the cases $M_1$ and $M_2$
the space $W_{G}^{k}$ need not be reflexive.

In § 1, a preliminary material on Orlicz spaces will be found.

§ 2 deals with existence and uniqueness of the weak solution, if the growth is given by the class $M_{1}$. Generally, we work with a non-reflexive space. In the space $W_{G}^{k}$ we define a convergence which is weaker than the weak one but with respect to which $W_{G}^{k}$ is sequentially compact.

Using Serrin's result [9], we prove lower semi-continuity of the functional with respect to the convergence just defined.

In § 4, existence and uniqueness of the weak solution is studied, when the growth is given by the class $M_{2}$. In this case, too, $W_{G}^{k}$ need not be reflexive.

§ 1.

We begin by presenting some fundamental notions from the theory of Orlicz spaces (see [4]). $G(\omega)$ is called to be a N-function if it is of the form $G(\omega) = \int_{0}^{\omega} \beta(t) dt$, where $\beta(t) > 0$ for $t > 0$ is a continuous on the right, non-decreasing function satisfying $\beta(0) = 0$ and $\lim_{t \to \infty} \beta(t) = +\infty$. When $\beta(t)$ is a continuous increasing function, let us denote by $\rho(t)$ its inverse function and define $P(\nu) =$
$\int_{0}^{\infty} r(t) dt$. $P(\nu)$ is an $N$-function, too, and is called to be conjugate to $G(\mu)$. In the general case $r(t)$ is inverse in some sense to $\alpha(t)$ (see [4]). Further, we shall understand $G(\mu), P(\nu)$ — maybe with indices — to stand for the $N$-functions. There holds the Young's inequality $\mu \cdot \nu \leq G(\mu) + P(\nu)$ for $\mu, \nu \geq 0$. If $Q(\mu)$ is a continuous, convex and even function defined for $|\mu| \geq \mu_0$ and satisfying $\lim_{\mu \to \infty} \frac{Q(\mu)}{\mu} = \infty$, then there exists an $N$-function $G(\mu)$ such that

$$
G(\mu) =
\begin{cases}
  c \cdot |\mu|^\alpha & \text{for } |\mu| \leq \mu_1, \\
  Q(\mu) & \text{for } |\mu| \geq \mu_1
\end{cases}
$$

where $c, \alpha$ and $\mu_1$ are suitable constants (they exist) and $\alpha > 1$. $Q(\mu)$ is called the principal part of $G(\mu)$ and it is denoted $p.p. G(\mu) = Q(\mu)$. $G(\mu)$ satisfies $A_2$-condition, if for arbitrary $k > 1$ there exist constants $c(k)$ and $\mu_0$ such that $G(k \mu) \leq c(k) G(\mu)$ for each $\mu \geq \mu_0$.

Suppose $\Omega$ is a bounded domain of $\mathbb{R}^N$.

The Orlicz class $L^*_G(\Omega)$ is the set of all real functions $\mu(x)$ defined on $\Omega$ and satisfying

$$
\phi(\mu, G) = \int_{\Omega} G(\mu(x)) dx < \infty.
$$

Orlicz space $L^*_G(\Omega)$ is the set of all $\mu(x)$ on $\Omega$, for which $(\mu, \nu) = \int_{\Omega} \mu(x) \nu(x) dx < \infty$ holds for all functions $\nu(x) \in L_p(\Omega)$, where $P(\mu)$ is conjugate to $G(\mu)$, with the norm

$$
\|\mu\|_G = \sup_{\|\nu\|_P \leq 1} |(\mu, \nu)|.
$$
$L^*_G(\Omega)$ is a Banach space. $E_G(\Omega)$ is the closure of bounded functions in the norm of $L^*_G(\Omega)$. If $G(\mu)$ satisfies $\Delta_2$-condition, then $E_G(\Omega) = L^*_G(\Omega)$. In the other case $E_G(\Omega)$ is a nowhere dense set in $L^*_G(\Omega)$ and $E_G(\Omega) \subset L^*_G(\Omega) \subset L^*_p(\Omega)$. For $\mu \in L^*_G$ and $\nu \in L^*_p$ ($G(\mu)$, $P(\nu)$ being conjugate) there holds the Hölder inequality $|\langle \mu, \nu \rangle| \leq \|\mu\|_G \cdot \|\nu\|_p$.

Assertion 1. If $g(\mu) \in M_3$, then there exist $p, q > 1$ and constants $c_1, c_2, \mu_0$ such that
\begin{equation}
\frac{c_1 |\mu|^{p^*} \leq \mu g(\mu) \leq c_2 |\mu|^{p_0}}{\|\mu\|^2}
\end{equation}
for $\mu \geq \mu_0$.

Proof. There exists $G(\mu)$ satisfying $\Delta_2$-condition with p.p. $G(\mu) = \mu g(\mu)$. The existence of $p > 1$ and $c_2$ is a consequence of [4] (Theorem 4.1).

Iterating the inequality in III, we obtain
\[2^m g(\mu) \leq g(l^n \mu), \quad (\mu \geq \mu_0).
\]
If $0 < \alpha \leq \log_2 2$, then the preceding inequality implies
\begin{equation}
\frac{g(\mu)}{\mu^\alpha} \leq \frac{g(l^n \mu)}{(l^n \mu)^\alpha}, \quad \mu \geq \mu_0.
\end{equation}

We prove the existence of $p > 1$ and $c_1$ by contradiction. Thus, there exists $\{\mu_m\}$ such that $\mu_m \to \infty$ and
\[\lim_{m \to \infty} \frac{g(\mu_m)}{\mu_m^\alpha} = 0.
\]
Let us denote $K = \inf_{\mu \in (\mu_m, \infty)} \frac{g(\mu)}{\mu^\alpha}, \quad K > 0$ (see II, I). Every $\mu_m$ is of the form $\mu_m = l^{m^*} \mu_m'$, where
\[ m \in (\omega_0, \varepsilon \omega_0) \] and \( m \) is a positive integer. According to (1.2) we have a contradiction:

\[ 0 < K \leq \frac{q(\omega_m)}{(\omega_m)^\alpha} \leq \frac{q((ln \omega_m)^{\alpha})}{(ln \omega_m)^\alpha} = \frac{q(\omega_m)}{\omega_m^\alpha} . \]

Example. \( q(\omega) = \omega^\alpha \ln^\alpha \omega \cdot (\ln \ln \omega)^{\alpha} \cdot (\ln \ldots \ln \omega)^{\alpha} \)

where \( \omega \geq \omega_0, \alpha \geq 0 \) and \( \gamma_1, \ldots, \gamma_n \) are real numbers. Let us extend \( q(\omega) \) continuously on \((-\infty, +\infty)\) to obtain an odd function for \( |\omega| \geq \omega_0 \). If \( \alpha > 0 \), then \( q(\omega) \in M_3 \). If \( \alpha = 0 \), \( \gamma_i > 0 \), then \( q(\omega) \in M_2 \).

Let us have \( q(\omega) \in M_4 \) for \( |\omega| \leq \omega \). There exist \( G_4(\omega) \) with p.p. \( G_4(\omega) = \omega q_4(\omega) \). Now, we construct \( W^i_{\mathcal{G}_4}(\Omega) \equiv \{ \omega \in L^i(\Omega) \mid \text{for which } D^i \omega \in \mathcal{E}_{\mathcal{G}_4}(\Omega) \} \), where \( D^i \omega \) is the distribution derivative and \( i \) is multi-index with \( |i| \leq k \). We define

\[ W^k_G(\Omega) = \bigcap_{i=1}^{k} W^i_{\mathcal{G}_4} \] (intersection) with the norm \( \|\omega\|_{W^k_G} = \sum_{i=1}^{k} \|D^i \omega\|_{\mathcal{G}_4} \).

Let \( E(\Omega) \) be the set of all functions defined on \( \Omega \) having derivatives of all orders extendable continuously on \( \overline{\Omega} \). Let \( D(\Omega) \) be a subset of all functions from \( E(\Omega) \) which have support in \( \Omega \).

We define \( W^k_G = \overline{D(\Omega)} \), where the closure is taken in the norm of \( W^k_G \).

Lemma 1. \( W^k_G \) is a Banach space. If \( q_4(\omega) \in M_3 \) for \( |\omega| \leq \omega \), then it is reflexive and separable.

Proof. \( W^k_G \) is a closed subspace of \( \bigcap_{i=1}^{k} L^\infty_{\mathcal{G}_4}(\Omega) \) (topological product of spaces \( L^\infty_{\mathcal{G}_4}(\Omega) \)).
If \( q_\delta (\mu) \in M_3 \), then \( L^*_G \equiv E_G \) is a reflexive and separable space (see [4], Theorems 14.2, 8.2 and 10.1).

Let \( E_\Omega (\mathbb{R}^N) \) be the set of all functions from \( E(\mathbb{R}^N) \) restricted on \( \Omega \).

**Lemma 2.** Suppose \( q_\delta (\mu) \in M_4 \) for \( |i| \geq \delta \).

There holds \( E_\Omega (\mathbb{R}^N) = W^k_{\infty} (\Omega) \), where the closure is taken in the norm of \( W^k_{\infty} \).

Proof is very similar to that made in [3] (Theorem 3.1) and thus we verify only the basic points of this proof.

\( \mu \in E_G (\Omega) \) possesses the following property:

\[ \| \mu \cdot \chi (x, F) \| < \varepsilon, \] if \( \text{mes} \ F < \sigma (\varepsilon) \), where \( \chi (x, F) \) is the characteristic function of the set \( F \subset \Omega \). Using the Lusin's theorem, we conclude that

\[ \| f(x + \varepsilon) - f(x) \|_G < \varepsilon, \] if \( |\varepsilon| < \sigma (\varepsilon) \).

Let us denote \( \mu_\varepsilon (x) \) the mollified function of \( \mu \).

\[ (*) \mu_\varepsilon (x) = \frac{1}{2^N N} \int_{|\xi - \xi| < \varepsilon} \exp \left[ \frac{|\xi - x|^2}{(N-1)|\xi|^2} \right] \mu(\xi) d\xi \], where

\[ \varepsilon > 0, \ \varepsilon = \int_{|\xi| < 1} \exp \left[ \frac{|\xi|^2}{(N-1)|\xi|^2} \right] d\xi \quad \text{and} \quad \mu(\xi) = 0 \] for \( \xi \notin \Omega \).

Suppose \( \nu (x) \in E_B (\Omega) \) and \( \rho (\nu, F) \leq 1 \). We have

\[ \int_{\Omega} \nu (x)(\mu_\delta (x) - \mu (x)) d\nu = \frac{1}{2^N N} \int_{|\xi| < 1} \exp \left[ \frac{|x|^2}{|x|^2 - 1} \right] \int_{\Omega} \nu (x). \]

(1.3)

\[ (\mu (x + \varepsilon x) - \mu (x)) dx \cdot dx \leq \]

\[ \leq \frac{1}{2^N N} \int_{|\xi| < 1} \exp \left[ \frac{|x|^2}{|x|^2 - 1} \right] \| \nu \|_F \cdot \| \mu (x + \varepsilon x) - \mu (x) \|_G \ dx \]
By reason of the previous fact, (1.3) and \( \| v \|_p \leq 2 \), we obtain \( \| u_n - u \|_G \to 0 \) with \( n \to 0 \).

The rest of the proof is the same as that in [3].

**Lemma 3.** Suppose \( g(u) \in M_2 \). Then \( g(u(x)) \) is a bounded mapping from \( L^r_g(\Omega) \) into \( L^s_p(\Omega) \), where \( p.p. \ G(u) = u \circ g(u) \) and \( P(u) \) is its conjugate. This mapping is continuous if and only if \( g(u) \in M_3 \).

**Proof.** For conjugate functions \( G(u), P(u) \) there holds \( P(\frac{G(u)}{u}) < G(u), \ (u > 0) \) (see [4], p.25). (This is easy to see in a geometrical sketch.)

From this inequality we conclude

\[
(1.4) \quad \frac{G(u)}{u} < P^{-1}(G(u)), \quad \text{or} \quad g(u) < P^{-1}(G(u))
\]

\( (u \geq u_\circ) \).

\( P^{-1}(u), G^{-1}(u) \) are inverse functions to \( P(u), G(u) \) for \( u > 0 \). As a consequence of the definition of the norm, (1.4) and the Jensen's inequality we have

\[
(1.5) \quad \| g(u(x)) \|_p \leq \int \frac{P[ g(u(x)) ]}{G(u(x))} \, dx + 1 \leq c + \int G(u(x)) \, dx,
\]

where \( c \) is a constant.

\( G(u) \) satisfies \( \Delta_2 \)-condition and the first part of the lemma is a consequence of (1.5) (see [4] p.95).

If \( g(u) \in M_3 \), then (1.4), (1.5) imply continuity by reason of [4] (Theorem 17.3).

In the case \( g(u) \notin M_3 \) we prove discontinuity of the mapping \( g(u(x)) \). At first, from the Young's inequality we have

\[
(1.6) \quad G^{-1}(u)P^{-1}(v) \leq u + v \quad \text{and hence}
\]
\[ P^{-1}(G(\mu)) \leq 2 \frac{G(\mu)}{\mu}. \]

The fact that \( q(\mu) \) does not possess III, implies \( P(\mu) \) does not satisfy \( \Delta_2 \)-condition. From this we conclude that there exists \( \nu(x) \in L_p(\Omega) \) such that 
\[
\| \nu - \nu_m \|_p \geq d > 0,
\]
where
\[
\nu_m(x) = \begin{cases} 
\nu(x) & \text{if } |\nu(x)| \leq m, \\
0 & \text{if } |\nu(x)| > m.
\end{cases}
\]

We can suppose \( \nu(x) \geq \mu_q \), where \( \mu_q = P^{-1}(G(\mu_0)) \) and \( G(\mu) = \mu q(\mu) \), \( \mu \geq \mu_0 \).

For \( \mu_m(x) = G^{-1}[P(\nu(x) - \nu_m(x))] \) we have
\[
\lim_{m \to \infty} \int_0^1 G(\mu_m(x)) \, dx = 0
\]
and because of \( \Delta_2 \)-condition for \( G(\mu) \), \( \| \mu_m \|_\infty \to 0 \) (see [4], Theorem 9.4).

With respect to (1.6) we have
\[
(1.7) \| q(\mu(x)) \|_p \geq \frac{1}{2} \| \nu_m - \nu \|_p > \frac{d}{2} > 0.
\]

If \( q(0) = 0 \), the proof is finished; otherwise we prove (1.7) with \( q^*(\mu) = q(\mu) - q(0), G^*(\mu), P^*(\mu) \).

However, \( L^*_G \equiv L^*_G, L^*_p \equiv L^*_p \) and, in addition, they have equivalent norms.

**Theorem 1.** If \( q(\mu) \in M_3 \), then
\[
(1.8) \lim_{\| \mu \|_p \to \infty} \int_0^1 \frac{G[\mu(x)]}{\| \mu \|_p} \, dx = \infty \quad (p.p., G(\mu) = \mu q(\mu)).
\]

If \( q(\mu) \in M_3 \) satisfies
\[
(1.9) \text{there exist } \kappa, \beta > 0 \text{ such that } \kappa - \beta < 1
\]
\[
(1.9) \quad c_1 \lambda^{\beta} \cdot q(\mu) \leq q(\lambda \mu) \leq c_2 \lambda^{\beta} \cdot q(\mu)
\]
for \( \lambda \geq \lambda_0, \mu \geq \mu_0 \), then
\[
\lim_{\| \mu \|_p \to \infty} \int_0^1 \frac{G[\mu(x)]}{\| \mu \|_p} \, dx = \infty.
\]
Proof. From (1.2) we conclude
\[ l^m q(u) \leq q(l^m u). \]
For arbitrary \( \lambda > l \) we find an integer \( n > 0 \) such that \( l^m \leq \lambda < l^n \). There holds
\[
q(\lambda u) = q(l^m u \cdot \frac{\lambda}{l^m u}) \geq l^m \cdot \frac{\lambda}{l^m} \cdot q(u) = l^m q(u) \frac{\lambda}{l^m} \geq \frac{1}{l^n} \cdot \lambda^m q(u),
\]
where \( 0 < \alpha \leq \log_2 2, \ l > 1 \) being fixed. Suppose \( G(u) = u q(U) \) for \( |u| \geq u_i > u_0 \) and let us denote
\[ q^*(u) = \begin{cases} q(u), & \text{for } |u| \geq u_i, \\ 0, & \text{for } |u| < u_i. \end{cases} \]
\( q^*(u) \) is an odd, non-decreasing function satisfying (1.10) for all \( \lambda \geq l > 1 \) and \( u \geq 0 \).

Now, we prove (1.8) by contradiction.

Thus, there exists \( \{u_m(x)\} \) satisfying \( \|u_m\|_G \to \infty \) and
\[
\int_\Omega \frac{G[u_m(x)\|_G} \| u_m \|_G dx \leq A \quad \text{for all } m; \ A \text{ is a constant.}
\]
Let us consider \( u_m(x) = \lambda_m v_m(x) \), where \( \|v_m\|_G = R > 2 \) and Hence \( \lambda_m \to \infty \). Evidently,
\[-c + G(u) \leq u q^*(u) \leq G(u) \]
holds for each \( u \), where \( c \) is a suitable constant. There holds
\[
AR \geq \int_\Omega \frac{G[\lambda_m v_m(x)\|_G} \lambda_m dx \geq \int_\Omega |v_m(x)| \cdot |q^*(\lambda_m v_m(x))| dx \geq \]
\[ \geq c_1 \lambda_m^\infty \int_\Omega |v_m(x)| \cdot |q^*(v_m(x))| dx \geq \]
\[ \geq c_1 \lambda_m^{\infty} \int_\Omega G[|v_m(x)| dx - c]. \]
In regard to the known inequality \( \rho(u, G) \geq \frac{1}{2} \|u\|_G \)
for \( \|u\|_G \geq 2 \), it suffices to take \( R = 2(c+4) \) and hence \( \lambda_m^\infty \leq \frac{A}{c_1} 2(c+4) \), which gives us a contra-
diction. The remaining part of the theorem will be proved analogously. We set again \( \mu_m = \lambda_m \nu_m \),

\[
\int_\Omega \frac{G[\mu_m(x)]}{G(\mu_m)} \, dx \leq A \quad \text{and} \quad |\nu_m|_6 = R > 2.
\]

From the first inequality (1.9) we obtain, as in the previous part of the theorem, the following estimate:

\[
(1.11) c_3 \lambda_m^{1+\kappa} \left( \int_\Omega G[\nu_m(x)] \, dx - c \right) \leq \int_\Omega G[\mu_m(x)] \, dx.
\]

From the second inequality in (1.9) we deduce

\[ |\varphi(\mu)| \leq c_2' \cdot |\mu|^\kappa + c \]

for all \( \mu \) and taking

\[ \varphi^{**}(\mu) = \begin{cases} \varphi(\mu), \text{for } |\mu| \geq \mu_1 \\ \text{sgn } \mu \cdot \varphi(\mu_1), \text{for } |\mu| < \mu_1 \end{cases} \]

in account of (1.9) there holds

\[ |\varphi^{**}(\lambda \mu)| \leq c_3 \lambda^{1+\kappa} |\varphi^{**}(\mu)| + c \]

for all \( \mu \) and \( \lambda \geq \lambda_0 \), where \( c_3 \) is a suitable constant.

Thus, considering the inequality \( G(\mu) \leq \mu \varphi^{**}(\mu) \leq G(\mu) + c \) for all \( \mu \), we obtain

\[
\int_\Omega \frac{G[\mu_m(x)]}{\lambda_m |\nu_m(x)|} \, dx \leq \int_\Omega \frac{G[\nu_m(x)]}{\lambda_m |\nu_m(x)|} \, dx \leq c_3 \lambda_m^{1+\kappa} \int_\Omega G[\nu_m(x)] \, dx - c + c
\]

\[
\leq c_3 \lambda_m^{1+\kappa} \left( \int_\Omega G[\nu_m(x)] \, dx - c \right) + c.
\]

Now, considering \( R \) sufficiently large but fixed, we conclude from (1.11) and the last inequality

\[ c \cdot \lambda_m^{1+\kappa} \leq A \]

which gives a contradiction and the theorem is proved.

**Corollary.** If \( \varphi(\mu) \in M \) satisfies (1.9), then

\[
\lim_{\|\varphi(\mu(x))\| \to 0} \int_\Omega \frac{G[\mu(x)]}{\|\varphi(\mu(x))\|} \, dx = \infty \quad (p.p. G(\mu) = \mu \varphi(\mu)).
\]
\[ \text{Proof.} \] There exists a \( c \) such that \( |g(u)| \leq \frac{G(u)}{u} + c \) for each \( u \). We have
\[
\int_{\Omega} \frac{G[u(x)]}{\|u\|_p} \, dx \leq \int_{\Omega} \frac{G[u(x)]}{\|u\|_p} \, dx + c \cdot \int_{\Omega} 1 \, dx.
\]
It suffices to know \( \lim_{\|u\| \to \infty} \frac{G(u)}{\|u\|_p} = \infty \). If
\[ \frac{G(u)}{\|u\|_p} = c, \]
the Hölder inequality implies
\[
\int_{\Omega} \frac{G[u(x)]}{\|u\|_p} \, dx = \int_{\Omega} G[u(x)] \cdot \frac{u(x)}{\|u\|_p} \, dx \leq c
\]
and hence we have a contradiction with (1.8).

**Lemma 4.** Suppose \( g(u) \in M_3 \) and \( \phi \cdot \phi \cdot \phi = \mu \cdot g(u) \). Then there exist constants \( c_1, c_2 \) such that
\[
\| G[D^i u(x)] \|_{L^p} \leq c_1 \sum_{i,j \leq k} \int_{\Omega} G[D^i u(x)] \, dx + c_2
\]
for \( u \in W_2^{k,\infty} \), where \( G(u) = \mu \cdot g(u) \) and \( |i|, |j| \leq k \).

**Proof.** Firstly, we prove (1.12) for \( u \in D(\Omega) \) and to this purpose it suffices to prove
\[
\| G[u(x)] \|_{L^p} \leq c_1 \sum_{i \leq k} \int_{\Omega} G[D^i u(x)] \, dx + c
\]
We imbed \( \Omega \) into the cube \( \sigma \subset \mathbb{R}^N \) with the length \( a \) of the edge and with a center in origin. Putting \( u(x) \equiv 0 \) for \( x \in \mathbb{R}^N - \Omega \) we have
\[
\int_{\Omega} u(x_1, \ldots, x_N) = \int_{-a}^{a} \frac{\partial u}{\partial x_1}(x_1, x_2, \ldots, x_N) \, dx_1 = (x_1 + a) \cdot \frac{\partial u}{\partial x_1} \, dx_1.
\]
There hold
\[
G[u(x)] \leq (x_1 + a) \cdot G[\frac{\partial u}{\partial x_1} \, dx_1], \quad 0 < x_1 + a < 1
\]
from convexity for \( G(\mu) \cdot G(\alpha \mu) \leq \alpha G(\mu), \alpha < 1 \).
Applying the Jensen's inequality in (1.15), we have
\[ G[u(x)] \leq c_1 \int_a^b G[\frac{\partial u}{\partial x_1}] \, dx_1 + c \]
for each \( u \).

The functional \( \int_{\Omega} G[u(x)] \, dx = \frac{G[u(x)]}{u(x)} \, dx \) is continuous from \( L^p_0(\Omega) \) into \( L^q_1(\Omega) \) as a consequence of Lemma 3. If \( u \in W^1_0(\Omega) \), we choose \( u_m \in D(\Omega) \) satisfying \( \| u_m - u \|_{W^1_0} \to 0 \).

Clearly, we may allow \( m \to \infty \) in (1.12) for \( u_m \in D(\Omega) \), and thus we obtain the required assertion.

§ 2.

In this section, we establish two general theorems for existence of a weak solution, where the coerciveness is assumed and then we state algebraic conditions to assure the coerciveness in special cases. We work entirely with the reflexive spaces, except Theorem 4, concerning the compactness of the imbedding.

The boundary \( \partial \Omega \) of the bounded domain \( \Omega \subset \mathbb{R}^N \) is supposed to be Lipschitzian (see [3]).

We shall denote positive constants by \( c \) with or without subscripts and in the same discussion it may denote different constants.

Suppose \( a_\varepsilon(\xi, \xi) \) for \( |\xi| \leq \kappa \) real func-
tions defined for \( x \in \mathbb{R} \) and \(-\infty < \xi_j < \infty\) with \(|j| \leq \kappa\) (\(i, j\) are multiindices). They are continuous in \( \xi_j \) for almost every \( x \in \mathbb{R} \) and measurable in \( x \) by fixed \( \xi_j \). (By this designation we understand \( a_{ij}(x, \xi_j) \) to be a function of \( x \) and a vector \( \xi \equiv (\xi_1, ..., \xi_d) \), where the integer \( d \leq \text{card} \{i, |i| \leq \kappa\} \).

Let us denote \( K = \{i, |i| \leq \kappa\}, L = \{i, |i| = \kappa\} \) and \( M \) some subset of \( K \) with \( K \supset M \supset L \).

We assume \( q_{ij}(\mu) \in m_3, i \in M \) being chosen with respect to an equation given in such a way that

\[
|a_{ij}(x, \xi_j)| \leq c (1 + \sum_{j \in M} q_{ij}(\xi_j))
\]

for all \( i \in M \), where \( q_{ij}(\mu) \in C(-\infty, \infty) \) with \( 0 \leq q_{ij}(\mu) \leq q_j[G_j(G_j(\mu))] \), \( (1 \mu \geq \mu_0) \), p.p. \( G_j(\mu) = \mu q_{ij}(\mu) \) and \( G_j^{-1} \) its inverse function for \( \mu > 0 \).

If every pair of \( q_j(\mu), q_{ij}(\mu) \) for \( i, j \in M \) satisfies one of the inequalities \( q_j(\mu) \leq q_{ij}(\mu) \) for \( \mu \geq \mu_0 \), then the condition (2.1) can be rewritten in a slightly stronger but synoptical form:

\[
|a_{ij}(x, \xi_j)| \leq c (1 + \sum_{j \in M} q_{ij}(\xi_j)) \quad i \in M,
\]

where \( q_{ij}(\mu) \in C(-\infty, \infty) \) and \( 0 < q_{ij}(\mu) \leq \min\{1q_j(\mu), q_j(\mu)\} \) for \( |\mu| \geq \mu_0 \).

Condition (2.1) or (2.2) involves \( a_{ij}(x, \xi_j) \equiv 0 \) for \( i \not\in M \) and \( a_{ij}(x, \xi_j) \) are independent on \( \xi_j \) for all \( j \in M \) and \( i \not\in M \).
To the equation given with (2.1) or (2.2) we construct a space \( W^k_{\mathcal{G}} = \bigcap_{i \in M} W^i_{\mathcal{G}_i} \) with the norm
\[
\|\mu\|_{W^k_{\mathcal{G}}} = \sum_{i \in M} \|D_i^k \mu\|_{\mathcal{G}_i}
\]
to which we add \( \|\mu\|_{L^p(\Omega)} \) in the case \((0, \ldots, 0) \notin M\).

From (1.1) (Assertion 1) there exist \( p, q > 1 \) such that \( W^k_p(\Omega) \subset W^k_q(\Omega) \) (algebraically and topologically). Condition (2.1) or (2.2) can be weakened by some information of imbeddings of \( W^k_{\mathcal{G}} \).

**Lemma 1.** Suppose (2.1) or (2.2). Then \( a_i(x, D^i \mu) \), \( i \in M \) is a bounded, continuous mapping from \( W^k_{\mathcal{G}} \) into \( L^p_i(\Omega) \) (\( P_i \) being conjugate to \( G_i(\mu) \)).

**Proof.** From Lemma 3, § 1 and (2.1) we conclude
\[
q_i(G^i(G_i(\mu))) \leq P_i^{-1}(G_i \mathcal{G}_i^{-1}(G_i(\mu))) = P_i^{-1}(G_i(\mu))
\]
for each \( |\mu| \geq \mu_0 \).

Similarly as in Lemma 3, § 1 we obtain from this inequality that \( a_i(x, D^i \mu) \) is a bounded mapping from \( W^k_{\mathcal{G}} \) into \( L^p_i(\Omega) \). The continuity follows from the results [4] (Lemma 17.2, Theorem 17.3).

Condition (2.1) is stronger than (2.2). Indeed, we prove
\[
\min(|q_i(\mu)|, |q_j(\mu)|) \leq 2q_i(G_i^j(G_j(\mu)))
\]
for \( |\mu| \geq \mu_0 \). If \( G_i(\mu) \leq G_j(\mu) \), then \( |\mu| \leq G_i^j(G_j(\mu)) \) and hence \( |q_i(\mu)| \leq q_i(G_i^j(G_j(\mu))) \) in regard to \( l \). If \( G_j(\mu) \leq G_i(\mu) \) (\( |\mu| \geq \mu_0 \)), then \( P_i(\nu) \leq P_j(\nu) \) for \( |\nu| \geq \nu_i(\mu_0) \) (see [4], Theorem 2.1) and hence \( P_i^{-1}(\nu) \leq P_j^{-1}(\nu) \) (\( |\nu| \geq v_i \)).

Using Lemma 3, § 1, we have
\[
q_i(G_i^j(G_j(\mu))) \geq \frac{1}{2} P_i^{-1}(G_i \mathcal{G}_i^{-1}(G_j(\mu))) = -152-
\]
Having Lemma 1, we are able to present the definition of the weak solution of a boundary value problem (see [2],[3]). Let $\mathcal{V}$ be a linear subset of $\mathcal{E}(\Omega)$ with $\mathcal{D}(\Omega) \subset \mathcal{V} \subset \mathcal{E}(\Omega)$. Let us denote $V_\mathcal{V} \equiv \overline{\mathcal{V}}$, where the closure is in the norm of $W_{0,\Omega}^k$. Let $u_0(x) \in W_{0,\Omega}^k$ represent a stable boundary value condition and $g \in (V_\mathcal{V})^\prime$ (dual space), $g(u) = 0$ for $u \in W_{0,\Omega}^k$, the non-stable one. (For the Dirichlet's problem, i.e. $V_\mathcal{V} \equiv W_{0,\Omega}^k$, the functional $g$ is not given.)

$u \in W_{0,\Omega}^k$ is called to be a weak solution of the boundary value problem, if $u - u_0 \in V_\mathcal{V}$ and for all $v \in V_\mathcal{V}$

$$\int_\Omega \sum_{i \in M} D^i v a_i(x,D^2 u) dx - (v, f)_{\Omega} + (v, g)_{\partial \Omega}$$

holds, where $f \in (V_\mathcal{V})^\prime$ and $(v, f)_{\Omega}$, $(v, g)_{\partial \Omega}$ are the values of the functionals at the point $v$.

Using a variational method we shall suppose the symmetry:

$$\frac{\partial a_i(x, \xi)}{\partial \xi_i} = \frac{\partial a_i(x, \xi)}{\partial \xi_i} \quad \text{in the sense of distribution for all } i, l \in M.$$ 

Lemma 2. Suppose (2.1), (2.4) and $f, g \in (V_\mathcal{V})^\prime$.

Then the functional
(2.5)  \( \Phi(\nu) = \int_a^t \int_\Omega \sum_{i \in M} D^i \nu \alpha_i(x, D^i(\omega + t \nu)) \, dx - (\nu, f)_A - (\nu, g \gamma)_{\partial A} \)

is continuous on \( V_\partial \) and has a Gâteaux differential at every point equal to

(2.6)  \( D\Phi(\nu, \tilde{\nu}) = \int_\Omega \sum_{i \in M} D^i \tilde{\nu} \alpha_i(x, D^i(\omega + \nu)) \, dx - (\tilde{\nu}, f)_A - (\tilde{\nu}, g \gamma)_{\partial A} \).

Proof of this lemma is the same as that in [2] Theorem 2.1. We use Lemma 1 and Lemma 2, § 1, only.

Now, monotonicity conditions and a general condition for coerciveness will be written:

\[ \lim_{\nu \to \infty} \frac{1}{\nu} \int_\Omega \sum_{i \in M} D^i \nu \alpha_i(x, D^i(\omega + \nu)) \, dx = \infty \]

(coerciveness).

(2.8)  \( \sum_{i \in M} (\xi_i - \eta_i) [a_i(x, \xi_i) - a_i(x, \eta_i)] \geq 0 \)

(monotonicity).

(2.8 a)  \( \sum_{i \in M} (\xi_i - \eta_i) [a_i(x, \xi_i) - a_i(x, \eta_i)] > 0 \)

for \( \xi \neq \eta \).

\( \xi = (\xi_1, \ldots, \xi_d), \eta = (\eta_1, \ldots, \eta_d) \) are real vectors with \( d = \text{card } M \).

A functional \( \Phi(\omega, \nu) \) defined on \( V_\partial \times V_\partial \) is called to be semi-convex (see Browder [6]), if it is convex and continuous at \( \omega \) by each \( \nu \) fixed and if \( \nu_n \to \nu \) (weak convergence), then \( \Phi(\omega, \nu_n) \to \Phi(\omega, \nu) \) uniformly for \( \omega \) belonging to a bounded set.
Theorem 2. Suppose (2.1), (2.4), (2.7) and one of the following conditions: i) (2.8); ii) there exists a semi-convex \( \phi (u, v) \) where \( \phi (u, u) = \phi (u) \) is from (2.5). Then there exists the solution of (2.3). If (2.8a) is satisfied then the solution of (2.3) is unique.

Proof. Let us define

\[
\inf_{\|v\|_{W^{1,\infty}} = R} \frac{1}{R} \int \sum_{i=1}^{n} D^{i} v a_{i} (x, D^{i} (u_{0} + v)) \, dx = \lambda (R).
\]

\( \lambda (R) \) is measurable and \( \lim_{R \to \infty} \lambda (R) = \infty \) on the ground of (2.7). There holds

\[
\phi (v) \geq \int_{0}^{R} \lambda (tR) \, dt - C R = R \left( \int_{0}^{R} \lambda (t) \, dt - C \right),
\]

where \( \|v\|_{W^{1,\infty}} = R \). But \( \lim_{R \to \infty} \frac{1}{R} \int_{0}^{R} \lambda (t) \, dt = \infty \) and hence \( \lim_{\|v\|_{W^{1,\infty}} \to \infty} \phi (v) = \infty \). \( V_{\phi} \) is reflexive and i) or ii) imply the lower semi-continuity for \( \phi (u) \) and hence there exists a point \( v \in V_{\phi} \) at which \( \phi (u) \) attains its minimum.

If we construct a Gâteaux differential at the point \( v \), we find - with respect to (2.6) - that \( v + u_{0} \) is the solution of (2.3). Uniqueness is clear from (2.8a).

Now we shall apply the theory of monotone operators - see e.g. Browder [5], [6], Leray-Lions [7]. Let us assume \( M = M_{1} \cup M_{2} \) with \( M_{1} \supset L \).

(2.9) The imbedding \( W_{p}^{1} \, g_{0} \to \bigcap_{i} W_{g_{i}}^{1} \) is compact.
(2.10) \( \sum_{i \in M_1} (f_i - \eta_i) [\alpha_i(x, f_i, \eta_i)] \geq 0 \), if \( f \neq \eta \); almost everywhere in \( \Omega \), where \( \alpha \in M_1 \) and \( \beta \in M_2 \).

(2.11) \( \sum_{i \in M_1} \alpha_i(x, f_i) / (\sum_{i \in M_1} |f_i| + \sum_{i \in M_2} \eta_i(x, f_i)) \to \infty \), if \( \sum_{i \in M_1} |f_i| \to \infty \), uniformly for \( f \), \( \beta \in M - L \) from a bounded set and \( x \in \Omega \), where \( \eta_i(x, f) \) are from (2.1).

(2.11a) \( \sum_{i \in M_1} \alpha_i(x, f_i) / (\sum_{i \in M_1} |f_i| + \sum_{i \in M_2} |\eta_i(x, f_i)|) \to \infty \), if \( \sum_{i \in M_1} |f_i| \to \infty \) uniformly for \( f \), \( \beta \in M - L \) from a bounded set and \( x \in \Omega \).

Let us denote \( (w, A(v, \mu)) = \sum_{i \in M_1} \int \nabla w(x) \cdot \nabla a_i(x, \nabla u, \mu) dx + \sum_{i \in M_2} \int \nabla w(x) \cdot \nabla a_i(x, \nabla u, \mu) dx \), where \( \alpha \in M_1 \) and \( \beta \in M_2 \), \( |f| \leq K \).

Theorem 3. Suppose (2.7) and (2.1) (resp. (2.2)). If one of the following two conditions i) (2.8), ii) (2.9), (2.10), (2.11) (resp. (2.11a)) is satisfied, then there exists the solution of (2.3). If (2.8a) is satisfied then the solution of (2.3) is unique.

Proof. It is sufficient to verify the hypotheses of Leray-Lions Theorem [7].

The operator \( A(\mu, v) \) is continuous and bounded from \( V_{\Omega} \times V_{\Omega} \) into \( (V_{\Omega})' \) because of Lemma 1. If (2.8) holds, then \( M_2 = \emptyset \) and the mentioned
hypotheses are verified.

In the other case we must verify:

1) if \( u_m \to u \) in \( V_\sigma^r \) and \( (u_m - u, A(u_n, u_m)) \to 0 \) then \( A(v, u_m) \to A(v, u) \) in \( (V_\sigma^r)' \) for all \( v \in V_\sigma^r \);

2) if \( u_m \to u \) and \( A(v, u_m) \to v' \) in \( (V_\sigma^r)' \), then \( (u_m, A(v, u_m)) \to (u, v) \), for all \( v \in V_\sigma^r \).

In the case 1) we can prove from (2.1), (2.10) and (2.11), resp. (2.2), (2.10) and (2.11a) similarly as in [7] (see [2] Lemma 3.2) that it is possible to select a subsequence still called \( \{u_{m_i}\} \), satisfying

\[ D^i u_m(x) \to D^i u(x) \]

for \( |i| \leq \mathcal{N} \) almost everywhere in \( \Omega \). For \( i \in M \) we have

\[ a_{i}(x, D^i (u_0 + u_m)) \to a_{i}(x, D^i (u_0 + u)) \]

almost everywhere in \( \Omega \). From Lemma 1 we have

\[ (2.12) \quad \|a_{i}(x, D^i (u_0 + u_m))\|_{p_i} \leq c \quad \text{for all } m. \]

If \( \mathcal{G} \in \mathcal{Y} \), then we have (2.13) for

\[ D^i \mathcal{G} a_{i}(x, D^i (u_0 + u_m)) \]. From the Young's inequality and (2.13) we conclude

\[ \sum_{i \in M_2} \int_{\Omega} p_i \{D^i \mathcal{G} a_{i}(x, D^i (u_0 + u_m))\} \, dx \leq C, \]

for all \( m \).

As a consequence of the Vallée-Poussin's theorem (see [41, p.113]) \( D^i \mathcal{G} a_{i}(x, D^i (u_0 + u_m)) \) have uniformly absolutely continuous integrals and thus from (2.12), (2.9) and Lemma 1 we conclude \( (\mathcal{G}, A(v, u_m)) \to (\mathcal{G}, A(v, u)) \). On account of (2.13), (2.9) and Lemma 1, \( \|A(v, u_m)\|_{(V_\sigma^r)'} \leq c \) holds. \( \mathcal{H} = V_\sigma^r \)

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implies \( A(n_r, u_m) \rightarrow A(n_r, u) \) in the space \((V_0)’\) for all \( u \in V_0^{\ast} \).

In the case 2) we obtain

\[
(2.14) \quad \int_{\Omega} \sum_{i \in M_2} \frac{\partial^i}{\partial x_i^i} (u_m - u) a_i (x, \partial^i (u_m + u)) \, dx \xrightarrow{m \to \infty} 0,
\]

because \( \| \frac{\partial^i}{\partial x_i^i} (u_m - u) \|_{L^2} \xrightarrow{m \to \infty} 0 \) and \( \| a_i (x, \partial^i (u_m + u_m)) \|_{L^2} \leq C \) for \( i \in M_2 \).

\[
\int_{\Omega} \sum_{i \in M_1} \frac{\partial^i}{\partial x_i^i} u \, a_i (x, \partial^i (u_m + u)) \, dx = (u, A_v (u_m)) -
\]

\[
- \int_{\Omega} \sum_{i \in M_1} \frac{\partial^i}{\partial x_i^i} u \, a_i (x, \partial^i (u_m + v), \partial^i (u_m + u_m)) \, dx \xrightarrow{m \to \infty} (u, v') -
\]

\[
- \int_{\Omega} \sum_{i \in M_1} \frac{\partial^i}{\partial x_i^i} u \, a_i (x, \partial^i (u_m + v), \partial^i (u_m + u)) \, dx.
\]

From this and (2.14) we conclude

\[
(2.15) \quad \int_{\Omega} \sum_{i \in M_2} \frac{\partial^i}{\partial x_i^i} u_m a_i (x, \partial^i (u_m + v), \partial^i (u_m + u_m)) \, dx \to (u, v') -
\]

But

\[
\int_{\Omega} \sum_{i \in M_1} \frac{\partial^i}{\partial x_i^i} u_m a_i (x, \partial^i (u_m + v), \partial^i (u_m + u_m)) \, dx \to
\]

\[
\int_{\Omega} \sum_{i \in M_1} \frac{\partial^i}{\partial x_i^i} u_m a_i (x, \partial^i (u_m + v), \partial^i (u_m + u)) \, dx
\]

holds as a consequence of \( u_m \to u \) in \( V_0^{\ast} \) and \( a_i (x, \partial^i (u_m + v), \partial^i (u_m + u_m)) \to a_i (x, \partial^i (u_m + v), \partial^i (u_m + u)) \) in the norm of the space \( L^2 \), because of (2.9) and Lemma 1. Thus, from (2.15) we have

\[
(u_m, A_v (u_m)) \to (u, v').
\]

In the next we shall establish some sufficient conditions for coerciveness, compactness of imbedding and equivalence of norms.
We shall use the following condition for coerciveness:

\[(2.16) \sum_{i \in M} f_i \alpha_i(x, \xi_i) \geq c_i \sum_{i \in M} f_i g_i(\xi_i) - c.\]

\[(2.17) \sum_{i \in M} f_i \alpha_i(x, \eta_i) \geq c_i \sum_{i \in M} f_i g_i(\eta_i) - c.\]

In the case of non-Dirichlet problem we suppose \((0, 0) \in M.\)

Lemma 3. Suppose \((2.2), (2.16)\) and let \(g_i(\mu) \in M^2\) satisfy \((1.9)\) for \(i \in M.\) Then \((2.7)\) holds.

Proof. From every sequence \(v_n \rightharpoonup \infty\) it suffices to select a subsequence \(v_{n_k}\) satisfying \((2.7).\)

According to \((2.16)\) we have

\[
\sum_{i \in M} D^i v_n \alpha_i(x, D^i(u_n + v_n)) dx = \sum_{i \in M} D^i(u_n + v_n) a_i(x, D^i(u_n + v_n)) dx \geq c_i \sum_{i \in M} G_i(D^i(u_n + v_n)) dx - c_2 \sum_{i \in M} \|g_i(D^i(u_n + v_n))\|_{p_i} - c.
\]

Let us divide this inequality by \(\|v_n + u_n\|_{W^{1,p}_0}\).

\[
(2.8) \sum_{i \in M} \frac{\|g_i(D^i(u_n + v_n))\|_{p_i}}{\|v_n + u_n\|_{W^{1,p}_0}}
\]

is bounded, then the assertion for \(v_n\) is true as a consequence of Theorem 1. Otherwise the fraction in \((2.18)\) converges to infinity for a suitable \(v_{n_k}\).

Then, with regard to \((1.9)\), the corollary of Theorem 1 gives

\[
\sum_{i \in M} \frac{\sum_{i \in M} G_i(D^i(u_n + v_{n_k}))}{\|g_i(D^i(u_n + v_{n_k}))\|_{p}} \rightarrow \infty.
\]
Lemma 4. If we substitute (2.7) by Condition (2.17), Theorem 2 remains true.

Proof. From every sequence \( \| u_m \|_{w_0} \to \infty \) it suffices to select a subsequence \( u_{m_k} \) satisfying \( \phi (u_{m_k}) \to \infty \). Let us define \( F(\nu) = \int_0^\nu q(t)dt \), \( q(u) \in M_1 \), \( q(u) \) is increasing to infinity because of I.

The following estimations hold:

\[
F(\nu) = F(\nu_0) + \int_{\nu_0}^\nu q(t)dt \leq F(\nu_0) + b q(\nu) 
\]
for \( \nu \geq \nu_0 \), where \( \nu_0 \) is a suitable positive number and hence \( F(\nu) \leq 2 b q(\nu) \) for \( \nu \geq \nu_1 \).

\( q(u) \) is an odd function and thus \( F(-\nu) \leq 2 b q(\nu) \) for \( \nu \geq \nu_2 \). On the other hand,

\[
F(\nu) = F(\nu_0) + \int_{\nu_0}^\nu q(t)dt \geq F(\nu_0) + \int_0^\nu q(t)dt \geq \frac{1}{2} b \int_{\nu_0}^\nu q(\nu) 
\]
for \( \nu \geq \nu_3 \) and \( F(-\nu) \geq \frac{1}{2} b \int_{\nu_0}^\nu q(\nu) \) for \( \nu \geq \nu_4 \).

( \( \nu_1, \nu_2, \nu_3, \nu_4 \) are suitable positive numbers.)

Thus, there exists a constant \( c \) such that

\[
-c + \frac{1}{2} b \int_{\nu_0}^\nu q(\nu) \leq F(\nu) \leq 2 b q(\nu) + c, \nu \in (-\infty, \infty).
\]

From this estimate and (2.17) we obtain

\[
\phi (\nu) \geq c_1 \sum_{i \in M} \int_0^\nu d x \int_0^{\nu_0} d t D^i v q_i (D^i (u_0 + t v)) dt - c = c_1 \sum_{i \in M} \int_0^\nu d x \int_{D_i u_0 (x)}^{\nu_0 + t v} q_i (x) d t - c_2 \| u \|_{w_0} - c 
\]

where the inner integral has a definite sense for almost all \( x \in \Omega \). By reason of this inequality and
Theorem 1, Lemma 4 follows.

\( \mathcal{O}, \mathcal{Q} \) will denote the bounded domains of \( \mathbb{R}^N \).

Let \( T \) be a mapping from \( \mathcal{O} \) onto \( \mathcal{Q} \). We shall call \( T \) regular, if it is of the class \( C^1 \) and a 1-1-mapping \( \mathcal{O} \) onto \( \mathcal{Q} \). Let us denote \( D_T \) the Jacobi's determinant of \( T \).

**Lemma 5.** Let \( G(u) \) be an \( N \)-function and \( u \in L^*_G(\Omega) \). If \( T \) is a regular mapping from \( \mathcal{O} \) onto \( \mathcal{Q} \) with \( c_1 \leq |D_T(y)| \leq c_2 \) in \( \mathcal{O} \), then \( v(y) = u(Ty) \) belongs to \( L^*_G(\mathcal{O}) \) and \( \|v\|_{L^*_G(\mathcal{O})} \leq C \|u\|_{L^*_G(\Omega)} \), where \( C \) is independent on \( u \).

**Proof.** Let us assume \( \mathcal{O}(y) \in E_v(\mathcal{O}) \), \( \int P[\mathcal{O}(y)]dy \leq 1 \). We have

\[
\int \mathcal{O}(y)v(y)dy = \int \mathcal{O}(T^{-1}(x))v(T^{-1}(x))|D_{T^{-1}}(x)|dx \leq \frac{1}{c_1} \int \mathcal{O}(T^{-1}(x))u(x)dx.
\]

On the other hand, we have

\[
\int P[\mathcal{O}(T^{-1}(x))]dx = \int P[\mathcal{O}(y)]|D_T(y)|dy \geq \frac{1}{c_2} \int P[\mathcal{O}(y)]dy.
\]

From both inequalities and (4) (Lemma 9.1) we conclude

\[
\|v\|_{L^*_G(\mathcal{O})} \leq C \|u\|_{L^*_G(\Omega)} \text{ for each } u \in L^*_G(\Omega).
\]

**Lemma 6.** Let \( T \) be a regular mapping from \( \mathcal{O} \) onto \( \mathcal{Q} \) with \( c_1 \leq |D_T(y)| \leq c_2 \) in \( \mathcal{O} \) and \( G(u) \) \( N \)-function. Suppose \( u \in W^1_G(\Omega) \). If \( v(y) = u(Ty) \), then \( v \in W^1_G(\mathcal{O}) \) and \( \|v\|_{W^1_G(\mathcal{O})} \leq C \|u\|_{W^1_G(\Omega)} \).

We recall \( W^1_G(\Omega) = W^1_G(\mathcal{Q}) \), where \( G_i(u) = G(u) \), \( \|i\| \leq 1 \).

**Proof is the same as that in [3] (Lemma 3.2).**

We use only Lemma 5 and the fact that

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Lemma 7. Suppose $\partial \Omega \in C^1$. Then for each bounded domain $\Omega^* \subset \Omega$ there exists an extension for functions from $W^1_0(\Omega)$ to functions belonging to $W^1_0(\Omega^*)$ and

$$\|u\|_{W^1_0(\Omega^*)} \leq c \|u\|_{W^1_0(\Omega)},$$

where $c$ is independent on $u$.

Having Lemmas 5 and 6, the proof of Lemma 7 is the same as that in [3] (Theorem 3.9).

Theorem 4. If $\partial \Omega \in C^1$ and $G$ is an $N$-function, then the imbedding $W^1_0(\Omega) \rightarrow E_G(\Omega)$ is compact.

Proof. Let $\{u_m\}$ be a bounded sequence from $W^1_0(\Omega)$, i.e. $\|u_m\|_{W^1_0} \leq c$. Let us take an arbitrary $\Omega^* \subset \Omega$. We extend every $u_m$ to a function belonging to $W^1_0(\Omega^*)$, still called $u_m$ with

$$\|u_m\|_{W^1_0(\Omega^*)} \leq c,$$

because of Lemma 7. We can suppose $\partial \Omega^*$ Lipschitzian. For a smooth function $u \in W^1_0(\Omega^*)$ and $|h| \leq h_0 = \text{dist}(\partial \Omega, \partial \Omega^*)$ we have

$$u(x+h) - u(x) = \int_0^1 \frac{\partial u}{\partial x_i}(x + th) \, dt,$$

where $x \in \Omega$. Supposing $v(x) \in E_p(\Omega)$ and

$$\int_0^1 [v(x)] \, dx = 1$$

we have

$$(2.19) \int_0^1 v(x) [u(x+h) - u(x)] \, dx = \int_0^1 \frac{\partial u}{\partial x_i}(x + th) \, dx.$$

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(2.19) holds also for $u_{m}(x)$ by Lemma 2, § 1. Using the Hölder's inequality in (2.19), we have

$$(2.20) \int_{\Omega} v(x)[u_{m}(x+h)-u(x)] \, dx \leq |h| \sum_{i=1}^{M} \left\| \frac{\partial u}{\partial x_{i}}(x+th) \right\|_{L^{p}} \cdot \left\| \frac{\partial u}{\partial x_{i}}(x+th) \right\|_{L^{q}} \leq C \cdot |h| \cdot \left\| u_{m} \right\|_{L^{q}(\Omega)} \leq C \cdot |h|.$$  

Taking supremum in (2.20) with respect to $v(x)$ we obtain $\left\| u_{m}(x+h)-u_{m}(x) \right\|_{L^{q}} \leq C \cdot |h|$. From [4] (Theorem 11.4 and Lemma 11.1) the compactness in $E(\Omega)$ follows.

**Corollary.** Let us have $N$-functions $G_{i}(u)$ for every $i \in M$ satisfying $G_{i}(u) \geq G_{j}(u)$ for $u \geq u_{0}$ and $|i| > |j|$. Suppose $\partial \Omega \in C^{1}$. Then the imbedding $W_{\Omega}^{k} \rightarrow \bigcap_{i \in M} W_{\Omega}^{k}_{i}$ is compact.

**Assertion 2.** Let $G_{i}(u)$ for each $i \in K$ satisfy $G_{i}(u) \in M_{\omega}$ and

$$G_{i}(u) \geq G_{0}(u) = G_{j}(u) \quad \text{for} \quad (u \geq u_{0}),$$

where $|i| = \kappa$, $|j| < \kappa$.

If there exist numbers $\rho_{0}, \rho_{e}$ from (1.1) corresponding to $G_{e}(u)$ and satisfying

$$(2.22) \quad \rho_{e} < \rho_{0} \cdot \frac{N}{N-\rho_{0}},$$

then $\|u\|_{W_{\Omega}^{k}} \leq c \left( \sum_{|i|=\kappa} \|D^{i}u\|_{G_{i}} + \|u\|_{G_{0}} \right)$ and the imbedding $W_{\Omega}^{k}(\Omega) \rightarrow W_{\Omega}^{k-1}$ is compact.

**Proof.** $W_{G_{j}}^{k} \subset W_{\rho_{0}}^{k}$ (algebraically and topologically).

Using the known imbeddings we have
\[
\|D^i u\|_{L^p} \leq c \|D^i u\|_{L^p} \leq c \cdot \|u\|_{W^{1,p}_0} \quad \text{for } |i| \leq \kappa.
\]

There holds (see [3] § 7)

\[
\|u\|_{W^{1,p}_0} \leq c \left( \sum_{|i| \leq \kappa} \|D^i u\|_{L^p} + \|u\|_{L^p} \right).
\]

From both inequalities we obtain the required inequality.

Finally, \(W^{1}_0 \subset W^{1}_p\) and the imbedding \(W^{1}_p \rightarrow L^{2}_0(\Omega)\) is compact.

**Assertion 3.** Suppose (2.1), (2.7), (2.9) and (2.10), where the equality is admitted. If \(a_\alpha(x, \xi_i)\) is independent on \(\xi_i\) for all \(i \in \mathbb{M}_1, l \in \mathbb{M}_1\), then there exists the solution of (2.3).

Indeed, the hypotheses of Leray-Lions Theorem are evidently satisfied.

**Assertion 4.** If (2.1), (2.4), (2.9), (2.10) (the equality admitted in (2.10)) hold and \(a_\alpha(x, \xi_i)\) is independent on \(\xi_i\) for all \(i \in \mathbb{M}_2, l \in \mathbb{M}_1\), then the functional from (2.5) is semi-convex.

**Proof.** Let us define

\[
\phi(u, v) = \sum_{i \in \mathbb{M}_1} \int_{\Omega} \int_{\mathbb{M}_2} D^i u a_\alpha(x, D^i (u_0 + tv)) dx +
\]

\[
+ \sum_{i \in \mathbb{M}_2} \int_{\Omega} \int_{\mathbb{M}_3} D^i v a_\alpha(x, D^i (u_0 + t\mu)) dx + (f, v) +
\]

\[
+ (\phi, v)_{\partial \Omega} = \phi_1(u) + \phi_2(v).
\]

\(\phi_1(u)\) and \(\phi_2(v)\) are continuous and bounded over the space \(V^{1}_0\) as a consequence of (2.1), Lemma 3§1 and Lemma 1. Regarding the properties of \(a_\alpha(x, \xi_i)\)
and (2.10), the functional $\phi_i(\nu)$ is convex (see [6]). By reason of (2.9) and Lemma 1, $\phi_2(\nu)$ is continuous with respect to the weak convergence in $V_0^r$.

**Assertion 5.** If the stable boundary value condition $u_0(x) \equiv 0$ in the problem (2.3), then (2.7) follows from Conditions (2.16) and (2.1).

Indeed, we use Theorem 1 in the estimate

$$\int_{V_0^r} \sum_{i \in M} \xi_i \alpha_i(x, D^2 \nu) \, dx \geq c_1 \sum_{i \in M} \frac{G_i[D^2 \nu(x)]}{\|D^2 \nu\|_{L^\infty}} \, dx - c.$$ 

In many cases the weaker conditions than (2.16) and (2.17) will be sufficient.

(2.16a) $\sum_{i \in M} \xi_i \alpha_i(x, \eta_i) \geq c_1 \sum_{i \in M} \xi_i \phi_i(\xi_i) - c$ for almost all $x \in \Omega$.

(2.17a) $\sum_{i \in M} \xi_i \alpha_i(x, \eta_i) \geq c_1 \sum_{i \in M} \xi_i \phi_i(\eta_i) + c_2 \xi_i \phi_i(\eta_i) - c$ for almost all $x \in \Omega$. For the Dirichlet's problem there is $c_2 \geq 0$.

One can see easily that one of the conditions (2.16), (2.17), (2.16a), (2.17a) implies Condition (2.11a).

**Assertion 6.** Let us have $\phi_i(\nu) \in M_3$ for all $i \in K$ and suppose $\phi_i(\nu) \geq \phi_i(\mu) = \phi_i(\nu)$ for $\mu \geq \mu_0$, where $l \in L$, $j \in K - L$. Suppose $\phi_i(\nu)$ satisfies (1.9) for all $i \in K$. If (2.2), (2.16a), (2.10) and (2.22) hold, where $M_1 = L$, $M_2 = K - L$, then there exists the solution of the Dirichlet's problem (2.3).

**Proof.** We define $A(\nu, \mu)$ as in Theorem 3, putting only $M_1 = L$, $M_2 = K - L$. 

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Condition (2.9) is a consequence of Assertion 2. If we have \( \partial \Omega \in C^1 \), we do not need Condition (2.22), because (2.9) is a consequence of Theorem 4. One can see easily that Condition (2.16a) implies (2.11a). Thus, it suffices to prove coerciveness of the operator.

From (2.16a) and Lemma 4, §1 we conclude:

\[
(v, A(v)) \geq c_1 \sum_{i \in \mathcal{K}} \int_{\mathcal{D}^i} [D^i(u_0 + v)] dx -
\]

\[
- c_2 \sum_{i \in \mathcal{K}} \|g_i^2(D^i(u_0 + v))\|_{p_i} - c \geq c'_1 \sum_{i \in \mathcal{K}} \int_{\mathcal{D}^i} [D^i(u_0 + v)] dx -
\]

\[
- c_2 \sum_{i \in \mathcal{K}} \|g_i^2(D^i(u_0 + v))\|_{p_i} - c'.
\]

Similarly as in Lemma 3, we deduce from the last inequality:

\[
\frac{(v, A(v))}{\|v\|_{W^{1,p}}} \rightarrow \infty, \text{if } \|v\|_{W^{1,p}} \rightarrow \infty.
\]

The rest of the proof is the same as that in Theorem 3.

**Remark.** If the stable boundary value condition \( u_0(x) \equiv 0 \), then in Assertion 6 we do not need Condition (1.9) for \( q_i^2(u) \), \( i \in \mathcal{K} \), by the same argument as in Assertion 5.

**Assertion 7.** Let us have \( q_i^2(u) \in M_3 \) for all \( i \in \mathcal{K} \) and suppose \( q_i^2(u) \geq q_j^2(u) = q^2(u) \) for \( u \geq u_0 \), where \( l \in \mathcal{L}, j \in \mathcal{K} - \mathcal{L} \). Suppose (2.1), (2.4), (2.17a) and (2.8). For non-Dirichlet problem suppose, in addition, (2.22). Then there exists the solution of (2.3).

**Proof.** It suffices to prove \( \phi(v) \rightarrow \infty \), if \( \|v\|_{W^{1,p}} \rightarrow \infty \), where \( \phi(v) \) comes from (2.5). Similarly as in Lemma 4 we have

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\( \phi(\nu) \geq c_1 \sum_{i \in M} I_G \left[ D^i(u_0 + \nu) \right] dx + 
+ c_2 \int G_0 [u_0 + \nu] dx - c \| \nu \|_{W_0^1} - c. \)

If (2.22) holds, then the required result will be obtained from Assertion 2 and Theorem 1. We use Lemma 4,§1 for the Dirichlet's problem, and

\( \phi(\nu) \geq c'_1 \sum_{i \in M} I_G \left[ D^i(u_0 + \nu) \right] dx - c \| \nu \|_{W_0^1} - c' \)

holds. The assertion follows from Theorem 1.

**Examples.** Suppose \( q^i(u) \in M_2 \) for all \( i \in M \) and let \( f(x) \), \( g(x) \) be measurable bounded functions defined on \( \Omega, \partial \Omega \). Let us consider an equation of the form

(2.23) \[ \sum_{i \in M} (-1)^{\text{dim} D^i} [l_i(x) + q^i(D^i u)] = f, \]

where \( l_i(x) \geq c > 0 \) are measurable bounded functions.

\( u_0 \in W_0^1 \) and \( q(\cdot) \) give the stable and non-stable boundary value conditions.

a) If \( q^i(u) \geq 0 \) for \( u \in (-\infty, \infty) \) and for all \( i \in M \) and if the imbedding \( W_0^1 \subseteq \bigcap_{i \in M} W_0^i \) is compact, then there exists a weak solution of the equation (2.23).

b) Let \( q^i(u) \geq 0 \) for \( u \in (-\infty, \infty) \) and \( |i| = k \).

For \( |i| < k \) we assume \( q^i(u) \geq 0 \) for \( u > 0 \) and \( q^i(u) \leq 0 \) for \( u < 0 \). Then there exists a weak solution of (2.23).

c) If \( q^i(u) > 0 \) for \( u \in (-\infty, \infty) \) and for all \( i \in M \), then there exists the unique weak solution of (2.23).
The cases a), c) are evident from Theorem 2 and Lemma 4. In the case b), if \( v_h \rightharpoonup v \) in \( W^{0,\infty} \), then for
\[
\Phi_2(v) = \sum_{i=1}^{N} \int_0^t dt \sum_{j} l_i(x) D^i v_{x,i}(D^i(u_0 + t v)) \, dx
\]
holds.

\[
\Phi_2(v) \leq \lim_{k \to \infty} \inf \Phi_2(v_{m_k}) , \text{ where } v_{m_k} \text{ is a suitable subsequence of } \{v_{m} \} \text{. Indeed, it is possible to select } \{v_{m} \} \text{ from } \{v_{m} \} \text{ satisfying } D^i v_{m_k}(x) \rightharpoonup D^i v(x) \text{ for all } i \in M - L, \text{ almost everywhere in } \Omega \text{ and } 0 \leq F(s) = \int_0^s q(t) \, dt \text{ for } s \in (-\infty, \infty).\]

Thus, Assertion b) is a consequence of the Fatou's lemma.

The concrete examples of this type are
\[
- \Delta u + q(u) = f ,
\]
\[
\frac{N}{\ell_i} - \frac{\partial}{\partial x_i} [l_i(x) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_i} | \ln m_i (| \frac{\partial u}{\partial x_i} | + 1)] = f ,
\]
where \( m_i > -1 \) and \( m_i \leq 0 \); \( m_i \), \( m_i \) real numbers.

§ 3.

Now, let us consider a wide span of the growths (2.1) given by the class \( M_\ell \). If \( q(u) \) does not possess Condition II, then \( q(u(x)) \) is not a mapping from \( L^\infty_0(\Omega) \) into the dual space \( L^p_\infty(\Omega) \), where p.p. \( G(u) = u q(u) \). Indeed, in such a case there exists \( u(x) \in L^\infty_0(\Omega) \) such that \( \infty = \int_\Omega G[u(x)] \, dx = c + \int_\Omega u(x) q(u(x)) \, dx \), \( c \) being a finite constant.
$g(v(x)) \notin L^p_p(\Omega)$ because of the Hölder inequality. Thus, the method of monotone operators is not directly applicable as in § 2. In addition, we must admit the values $+\infty$ for a functional from (2.5) if we intend to use the calculus of variations. Finally, if the functional from (2.5) is finite at the point $v$, it need not be finite at the point $v + v'$ and thus there are difficulties with the Gateaux differential.

The weak solution for special cases of this direction was obtained by M.I. Višik [10], by means of the Galerkin's method.

We shall solve the Dirichlet's boundary value problem for the minimum of the functional

(3.1) $\phi(v) = \int_{\Omega} f(x, D^2v)dx + \int_{\Sigma} g(x, D^2v)dx + F(v),$ 

\[
\frac{\partial^2 v}{\partial y^2} = \frac{\partial^2 \mu_p}{\partial y^2} \text{ on } \partial \Omega, \text{ for } l = 0, 1, \ldots, k - 1,
\]

where $v$ is an exterior normal, $i, j$ are multi-indices with $|i| \leq k, |j| < k$. $\mu_p \in W^{k,1}(\Omega)$ satisfying $\phi(\mu_p) < \infty$ gives us the boundary values. $F(v)$ is some linear functional.

Let $M, M_1, M_2, K$ and $L$ be from § 2.

1) $f(x, \xi_i) \geq 0$ is continuous in all variables

(3.2) $x \in \mathbb{N}, |\xi_i| < \infty$ for $i \in M_1$ and $(0, \ldots, 0) \notin M$.

2) $f(x, \xi_i)$ is convex in $\xi$.

3) $|f(x, \xi_i) - f(y, \xi_i)| \leq k |x - y|^1 [1 + f(x, \xi_i)],$
where $\lambda(\sigma)$ is a positive function with $\lim_{\sigma \to 0} \lambda(\sigma) = 0$.

(3.3) $g(x, \xi_j)$ is a real-valued function for $x \in \Omega$, $|\xi_j|^2 < \infty$ with $j \in M_2$. It is continuous in $\xi_j$ for almost every $x \in \Omega$ and measurable in $x$ by $\xi_j$ fixed.

(3.4) $|g(x, \xi_j)| \leq c(1 + \sum_{j \in M_2} G_d(\xi_j))$.

(3.5) $f(x, \xi_i) + g(x, \xi_j) \geq c_i \sum_{j \in M} G_d(\xi_j) - c$,

where $\kappa > 1$ is constant.

(3.6) $G_{i}(\mu) = m_i$ for all $i \in M_1$ and $\frac{G_{i}(\mu)}{\mu} \in M_3$

for all $j \in M_2$.

Let us construct a space $W^i(G^i_x) = \{ \mu \in L_i(\Omega); D^i \mu \in L^\infty(\Omega) \}$, where $D^i \mu$ is the distribution derivative, $i \in M$. Let us denote $W^{i}_{G_x} = \bigcap_{i \in M} W^i(G^i_x)(\Omega)$ with the norm $\| \mu \|_{W^{i}_{G_x}} = \sum_{i \in M} \| D^i \mu \|_{G^i_x}$, to which we add $\| \mu \|_{L^i(\Omega)}$ in the case $(0, ..., 0) \notin M$.

(3.7) Let the imbedding $W^{i}_{G_x} \to \bigcap_{i \in M_2} G^i_x(\Omega)$ be compact.

Let us choose $i \leq n_i$ for all $i \in L$ such that $\| \mu \|_{G^i_x} \leq G_i(\mu)$ for $\mu \geq \mu_o$ (the case "$n_i$ are larger" is of more interest) and denote $p = \min \{ n_i, i \in L \}$.

(3.8) $F(\nu) \in (W^p_n)^*$ (dual space);

(3.9) $f(x, \xi_j)$ is strictly convex and $g(x, \xi_j)$.
is convex in $\xi$.

**Theorem 5.** Suppose $\partial \Omega$ Lipschitzian for $p > 1$ and $\partial \Omega \in C^1$ for $p = 1$. If (3.2) to (3.8) are satisfied, then there exists a minimum of (3.1) in $W^0_{G^k}$.

If, in addition, (3.9) holds, the minimum is unique.

At first we prove two lemmas.

Let us define $W^0_{G^k} = \{ u \in W^0_{G^k} \}$, for which

$$
\frac{\partial^l u}{\partial y^l} = 0 \quad \text{on} \quad \partial \Omega \quad \text{for} \quad l = 0, 1, \ldots, k - 1.
$$

In this case $D(\Omega) = W^0_{G^k}$ need not be true. $W^0_{G^k}$ is evidently a closed subspace of $W^0_{G^k}$.

We introduce $*_X$ convergence in the space $W^0_{G^k}$ by the following way: $u_m \rightarrow^*_X u$, if $\sum D^i u_m(x) v^{(i)}(x) \to \sum D^i u(x) v^{(i)}(x)$, for all $v^{(i)}(x) \in E_{P_i}(\Omega)$ and for each $i \in M$, $P_i$ being conjugate to $G_i$.

In general, $W^0_{G^k}$ need not be reflexive and $*_X$ convergence can be weaker than the weak convergence.

**Lemma 1.** $W^0_{G^k}$ is compact with respect to $*_X$ convergence; more exactly, from any bounded subset $B \subset W^0_{G^k}$ it is possible to select $\{ u_m \} \subset B$ and $u \in W^0_{G^k}$ such that $u_m \rightarrow^*_X u$.

If $\partial \Omega \in C^1$ for $p = 1$, then $W^0_{G^k}$ is closed with respect to $*_X$ convergence.

**Proof.** The space $L^2_{G}(\Omega)$, $G$ being an $N$-function, possesses the properties (see [4], Theorems 14.3 and 14.4):
1) From any bounded subset $A \subset L^*_g(\Omega)$ it is possible to select \( \{\mu_n\}_n \subset A \) such that

\[
\|\mu_n\|_g \leq c.
\]

\( W^{k-1}_g(\Omega) \) is a closed linear subset of \( \prod_{i \in M} L^*_g(\Omega) \) (cartesian product). By a successive selection we find \( \mu_n \in B \) and \( \mu^{(i)} \in L^*_g(\Omega) \) for all \( i \in M \) satisfying

\[
\int_\Omega D^i \mu_n(x) v_i(x) \, dx \to \int_\Omega D^i \mu(x) v_i(x) \, dx
\]

for all \( v_i \in E_{p_i}(\Omega) \) and for each \( i \in M \). There exists \( \mu(x) \in L^1(\Omega) \) such that \( \mu_n \xrightarrow{L^1(\Omega)} \mu \).

We find easily that \( D^i \mu(x) = \mu^{(i)}(x) \) for all \( i \in M \) and thus the first part of the lemma is proved.

Now, suppose \( \mu_n \xrightarrow{\mathcal{D}x} \mu \) for \( \mu_n \in W^{k-1}_g(\Omega) \) and \( \mu \in W^{k-1}_g(\Omega) \). In accordance with (3.10) there exists \( c \) such that \( \|\mu_n\|_{W^{k-1}_g(\Omega)} \leq c \) and hence \( \mu_n \to \mu \) in the norm of the space \( W^{k-1}_g(\Omega) \). Thus, we have

\[
\frac{\partial^l \mu}{\partial \nu^l} = 0 \quad \text{on} \quad \partial \Omega \quad \text{for} \quad l = 0, 1, \ldots, k-2.
\]

Now, let us suppose \( p = 1 \) and \( \partial \Omega \in C^1 \). Using the Green's theorem, we obtain for each \( \mathcal{F} \in \mathcal{E}(\Omega) \)

\[
\int_{\Omega} D^i \mu_n \varphi \, dx = \int_{\Omega} D^i \mu \varphi \, dx + \int_{\partial \Omega} D^i \mu_n \frac{\partial \varphi}{\partial x_i} \, ds,
\]

where \( \varphi \) is \( j \)-th component of the exterior normal and \( i + 1 \equiv i + (0, \ldots, 1, \ldots, 0) \), \( \|i\| = k-1 \).
From \( \times \) convergence we conclude

\[(3.11) \quad \int_{\partial \Omega} \mathcal{D}^i u_m \psi \, d\mathbf{s} \to \int_{\partial \Omega} \mathcal{D}^i u \psi \, d\mathbf{s} .\]

Let us denote \( f_n(\varphi) = \int_{\partial \Omega} \mathcal{D}^i u_m \psi \, d\mathbf{s} \) and similarly \( f(\varphi) \). There holds

\[(3.12) |f_n(\varphi)| \leq c \|u_m\|_{\mathcal{F}_k}. \quad \varphi \in \mathcal{C}(\Omega). \]

Restrictions of functions from \( \mathcal{E}(\overline{\Omega}) \) on \( \partial \Omega \) form a dense subset in \( \mathcal{C}(\partial \Omega) \). (3.12) holds for \( f(\varphi) \), too. We can uniquely extend \( f_n, f \) on \( \mathcal{C}(\partial \Omega) \) and thus \( f_n(\varphi) \to f(\varphi) \) for each \( \varphi \in \mathcal{C}(\partial \Omega) \). In (3.11) we substitute \( \varphi = \varphi_j \psi, \psi \in \mathcal{C}(\partial \Omega) \) and then we sum up (3.11) through \( j = 1, 2, \ldots \), \( N \). And hence

\[(3.13) \quad \int_{\partial \Omega} \mathcal{D}^i u_m \psi \, d\mathbf{s} \to \int_{\partial \Omega} \mathcal{D}^i u \psi \, d\mathbf{s} \]

for \( |i| = \kappa - 1 \).

From (3.13) we deduce

\[\int_{\Omega} \frac{\partial^k-1 u}{\partial y^k} \psi \, d\mathbf{s} = 0 \quad \text{for all } \psi \in \mathcal{C}(\partial \Omega) \]

and thus \( u \in \mathcal{W}_{\kappa}^{k,x} \).

In the case \( \kappa > 1 \), \( \partial \Omega \) is Lipschitzian. Suppose \( 1 < \sigma < \frac{N}{N-1}, \sigma \leq \kappa \). For \( u \in \mathcal{W}_{\kappa}^{k,x} \) we have at least \( \mathcal{D}^i u \psi \in L_\sigma(\partial \Omega) \), where \( \frac{1}{\sigma} = \frac{1}{\nu} - \frac{1}{(N-1)\nu} \) and \( |i| = \kappa - 1 \). For \( \varphi \in \mathcal{E}(\overline{\Omega}) \) there holds (3.13) and

\[\int_{\partial \Omega} \mathcal{D}^i u_m \psi \, d\mathbf{s} \leq c \|u_m\|_{\mathcal{F}_k}. \quad \varphi \in \mathcal{C}(\overline{\Omega}) \to c \cdot \|\mathcal{F}_k\|_{\mathcal{C}(\overline{\Omega})} \].

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where \( \varphi^{-1} + \varphi^{-1} = 1 \). Restrictions of functions from \( E(\Omega) \) are dense in \( L_{\text{TL}}(\partial \Omega) \). We can uniquely extend \( f_n, f \) on \( L_{\text{TL}}(\partial \Omega) \) and \( f_n(\varphi) \to f(\varphi) \) for each \( \varphi \in L_{\text{TL}}(\partial \Omega) \). From this we deduce

\[
\frac{\partial \varphi^{-1}}{\partial \nu} = 0 \quad \text{on} \quad \partial \Omega \quad \text{again.}
\]

**Lemma 2.** Let us assume (3.2) to (3.4) and (3.6) to (3.8). If \( \mu_m \xrightarrow{s} \mu \), \( \mu_m, \mu \in W^{k,p}_{\text{TL}} \), then

\[
\phi(\mu_o + \mu) \leq \lim \inf_{n \to \infty} \phi(\mu_o + \mu_n)
\]

**Proof.** \( \mu_m \xrightarrow{s} \mu \), where \( \mu_m, \mu \in W^{k,p}_{\text{TL}} \) implies \( \mu_m \to \mu \) in the norm of the space \( W^{k,p}_{\text{TL}} \). The results of J. Serrin [9] can be extended to the higher derivatives and hence

\[
\int_\Omega f(x,D^k(\mu_o + \mu))\,dx \leq \lim \inf_{n \to \infty} \int_\Omega f(x,D^k(\mu_o + \mu_n))\,dx.
\]

The functional \( \int_\Omega q(x,D^k\mu)\,dx \) is continuous from \( \bigcap_{k=2}^\infty W^{k,p}_{\text{TL}} \) into \( L_{\text{TL}}(\Omega) \) as a consequence of (3.3), (3.4), (3.6) and (3.7) and with respect to [4] (Lemma 17.2 and Theorem 17.3 where we set \( M^1_{\text{TL}}(\mu) = \mu \)). The functional \( F(\nu) \) is continuous because of (3.8). Thus, we have

(3.14) \( \phi(\mu_o + \mu) \leq \lim \inf_{n \to \infty} \int_\Omega f(x,D^k(\mu_o + \mu_n))\,dx + \)

\[
+ \lim_{n \to \infty} \int_\Omega q(x,D^k(\mu_o + \mu_n))\,dx + \lim_{n \to \infty} F(\mu_o + \mu_n) \leq \lim \inf_{n \to \infty} \phi(\mu_o + \mu_n).
\]

**Proof of Theorem 5.** Let us consider \( \phi(\mu_o + \mu) \) over the space \( W^{k,p}_{\text{TL}} \). We admit the value \( +\infty \) for

\( \phi(\mu_o + \mu) \). From (3.5) and (3.8) we conclude

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On the ground of the property of $u$, we have

$$\phi(u_o + u) \geq c \sum \frac{d}{dx} G_i (D^i u_{o} + D^i u) - c_2 \| u_o + u \|_{w_{K-1}}^2 - c.$$  

On the other hand, $\| u_o + u \|_{w_{K-1}}^2 \geq c \| u_o + u \|_{w_{K-1}} - c$ holds and $\| u \|_G \leq 2 \int G(u(x)) \, dx$ for $\| u \|_G \geq 2$.

(See [4], Theorem 9.5.) Thus, we conclude that from any sequence $\{ u_{m} \} \to \infty$ it is possible to choose a subsequence $u_{m_k}$ for which $\phi(u_o + u_{m_k}) \to \infty$ and hence $\phi(u_o + u) \to \infty$ if $\| u \|_{w_{K-1}} \to \infty$.

The last statement is true in the case $\mu = 1$, too, by reason of the inequality $G_i \left( \frac{H}{K} \right) - c_1 | \mu | \geq c'_i G_i \left( \frac{H}{K} \right) - c$, for each $|i| = K$; for suitable constants $c'_i$ and $c$.

Let $\{ u_{m} \}$ be a minimizing sequence for the functional $\phi(u_o + u)$. By reason of the previous fact there exists $c$ such that $\| u_{m} \|_{w_{K+1}^K} \leq c$. Using Lemma 1 we find $u \in W_{0, K}^K$ and a suitable subsequence still called $u_{m}$ such that $u_{m} \rightharpoonup u$. With regard to Lemma 2 we have

$$\text{inf}_{u \in W_{0, K}^K} \phi(u_o + u) = \phi(u_o + u) \leq \liminf_{m \to \infty} \phi(u_o + u_{m}).$$

If $v \in W_{0, K}^K$ and $\frac{\partial^2 v}{\partial x^2} = \frac{\partial^2 u}{\partial x^2}$ on $\partial \Omega$

for $l = 0, 1, \ldots, K - 1$, then $v = v - u_o + u_o$ and $v - u_o \in W_{0, K}^K$ and hence $\phi(u_o + u) \leq \phi(v)$.

If (3.9) holds and $u_1, u_2$ are two points of minimum, then we have for $u = t u_1 + (1-t) u_2, t \in (0,1)$

$$\int_{\Omega} \left[ t f(x, D^i u_1) + (1-t) f(x, D^i u_2) \right] \, dx + \int_{\Omega} \left[ t g(x, D^i u_1) + (1-t) g(x, D^i u_2) \right] \, dx = 0.$$
and thus \( D^i u_1 = D^i u_2 \) for \( |i| = k \) almost everywhere in \( \Omega \). Considering the Dirichlet problem \( u_1(x) \equiv u_2(x) \) almost everywhere in \( \Omega \).

Remark 1. Theorem 5 remains true if we substitute (3.7) and (3.6) by

\[
(3.15) \quad \varphi(x, f_i^+) \geq 0 \quad \text{almost everywhere in } \Omega \quad \text{for all } |f_i^+| < \infty, \ i \in M_2 \quad \text{and } \frac{G_i(w)}{w} \in M_1 \quad \text{for all } i \in M.
\]

Indeed, if \( u_m \xrightarrow{L} u \), \( u_m, u \in \mathcal{V}_x \), then a suitable subsequence still called \( u_m \rightarrow D^i u(x) \) holds for all \( i \in M - L \), almost everywhere in \( \Omega \). Using Fatou's Lemma, we obtain

\[
\int_{\Omega} \varphi(x, D^i(u_0 + u)) \, dx \leq \liminf_{n \to \infty} \int_{\Omega} \varphi(x, D^i(u_0 + u_n)) \, dx
\]

and hence Lemma 2.

Remark 2. In the case \( k = 1 \) Condition (3.2) can be weakened to (3.2') with respect to the results of J. Serrin [9] (Theorem 12).

(3.2') 1) \( \varphi(x, u, f_i^+) \geq 0 \) is continuous in all variables, \( |i| = 1 \).

2) \( \varphi(x, u, f_i^+) \) is convex in \( f \) for each \( x \in \Omega, |u| < \infty \).

Without loss of generality it is possible to suppose in (3.2) or (3.2') \( \varphi(x, f_i^+) \geq -c \) only.

Examples. Theorem 5 is applicable in the following types of examples:

a) \( \phi(u) = \int \sqrt{\frac{\partial u}{\partial x} \ln^2(1 + |\partial u/\partial x|) + (\partial u/\partial y) \ln^2(1 + |\partial u/\partial y|)} + 1 \, dx \, dy \)
- \int_\Omega u \, dx \, dy.

b) \( \phi(u) = \int_\Omega \sqrt{\left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial u}{\partial y} \right)^2} \, dx \, dy - \int_\Omega u \, dx \, dy. \)

c) \( \phi(u) = \int_\Omega \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial u}{\partial y} \right)^2 \, dx \, dy - \int_\Omega u \, dx \, dy. \)

d) \( \phi(u) = \int_\Omega \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial u}{\partial y} \right)^2 + \delta_0 (u(x,y)) \right) - u \right) \, dx \, dy, \)

where \( 0 \leq \nu(t) \in M_1. \)

§ 4.

In this section we establish a weak solution of those equations when the growth (2.1) or (2.2) is given by the class \( M_2. \) We shall consider \( \partial \Omega \in C^1. \) Let \( u_0 \in W_0^{1,2} \) give us a boundary value.

**Theorem 6.** Suppose (2.1), (2.4) and let the functional from (2.5) have the form (3.1). Suppose (3.2) to (3.5), (3.7), (3.8) and \( \rho_i(u) \in M_3 \) for all \( i \in M_2. \) Then there exists the weak solution of the Dirichlet's problem (2.3). In the case (2.8a) the solution is unique.

By reason of Lemma 3, §1 we must prove at first that (2.5) defines a functional over \( W_0^{1,2}. \)

\( a_i(x, D^2(u_0 + tv)) \) are measurable functions on \( \Omega \times (0, 1). \) Using (2.1), we have

\( (4.1) \sum_{i \in M} |D_i \nu ||a_i(x, D^2(u_0 + tv))| \leq c \left( \sum_{i \in M} |D_i \nu | + \right. \)
+ \sum_{i,j \in M} |D^i v||q_{i,j}(D^j(u_0 + tv))|.

Let us consider some member from the right side of (4.1) by t fixed. Using Lemma 3, §1, we obtain successively

\[ (4.2) \int_\Omega |D^i v||q_{i,j}(D^j(u_0 + tv))| \, dx = \]

\[ \leq \|D^i v\|_{L_2(\Omega)} \cdot \|q_{i,j}(D^j(u_0 + tv))\|_{L_2(\Omega)} \leq \|D^i v\|_{L_2(\Omega)} (4 + \]

\[ + \int_\Omega P_{ij}(D^j(u_0 + tv)) \, dx \leq \|D^i v\|_{L_2(\Omega)} (c + \]

\[ + \int_\Omega G_j(D^j(u_0 + tv)) \, dx \leq \|D^i v\|_{L_2(\Omega)} (c + \int_\Omega G_j(2D^j u_0) \, dx + \]

\[ + \int_\Omega G_j(2D^j u) \, dx). \]

Thus, on account of (4.1), (4.2) and (2.1), the functional (2.5) is well defined over $W^k_\Omega$. In addition, it is bounded on the bounded sets, because of Lemma 3, § 1.

By Theorem 5, the functional (2.5) attains its minimum at a point $v \in W^k_\Omega$.

We shall construct a Gâteaux differential $v$ only in some directions; precisely, we shall prove

\[ (4.3) \lim_{t \to 0} \frac{\phi(v + tv) - \phi(v)}{t} = 0 \text{ for each } \phi \in \mathcal{D}(\Omega). \]

We use the idea of [8], (Theorem 5.1) and [2], (Theorem 2.1). Let us denote $a_{\alpha}(x, \xi_j)$ the mollified function of $a_{\alpha}(x, \xi_j)$ by $x \in \Omega$ fixed (see (*) of Lemma 2, § 1). Let $k \leq 10$ be fixed. There holds

\[ (4.4) |a_{\alpha}(x, \xi_j)| \leq c(1 + \sum_{j \in M} q_{i,j}(G_j(2\xi_j)))). \]
for all $i \in M$.

(4.5) $|a_{i,j}(x, \xi_i^j)| \leq c(h)(1 + \sum_{j \in M} G_j^i(G_j (2 \eta_i^j)))$,

where $a_{i,j}(x, \xi_i^j) = \frac{\partial a_{i,j}(x, \xi_i^j)}{\partial \xi_i^j}$ for all $i, j, l \in M$.

By means of $a_{i,j}(x, \xi_i^j)$ let us define the functional $\phi_j(\nu)$ from (2.5). Similarly as in [5], we obtain, with respect to (4.2), (4.4) and (4.5),

(4.6) $\phi_j(\nu + \nu') - \phi_j(\nu) = \int_0^1 \int_{a \in M} D^i a_{i,j}(x, D^j(u_0 + + \nu + \nu')) dx - F(\nu')$.

The inner integral is a continuous function in $\nu$,

because of

$a_{i,j}(x, D^j(u_0 + \nu + \nu')) \to a_{i,j}(x, D^j(u_0 + \nu + \nu'))$

for almost all $x \in \Omega$ and

(4.7) $\|a_{i,j}(x, D^j(u_0 + \nu + \nu'))\|_{L_2} \leq c$

for all $\nu \in (0, 1)$.

Using the Valeé–Poussin's theorem analogically as in the proof of Theorem 3, § 2, $D^i a_{i,j}(x, D^j(u_0 + \nu + \nu'))$

have the uniformly absolutely continuous integrals.

($\mathfrak{H}, \mathcal{A}(x)$ being fixed.)

Thus, for suitable $\nu_0 \in (0, 1)$

$\phi_j(\nu + \nu') - \phi(\nu) = \sum_{i \in M} \int_{a \in M} D^i a_{i,j}(x, D^j(u_0 + \nu + + \nu')) dx - F(\nu')$

holds and hence there exists a derivative in the direction $\nu'$, from which

(4.8) $\frac{d\phi_j(\nu + \nu')}{d\nu} = \phi_j(\nu')$
\[ \frac{1}{c} \int_0^T \sum_{i \in M} D^i g a_{i,\nu}(x, D^i (u_0 + \nu + t F)) \, dx - F(\mathbf{\nu}) \]

holds almost everywhere in \( \Omega \).

By the reason of (4.1), (4.2) and (4.4) the Lebesgue's theorem gives \( \phi_n(\nu) \xrightarrow{n \to 0} \phi(\nu) \). Now, we are allowed to let \( \mathbf{\nu} \) become to infinity in (4.8). The inner integral in (4.8) is again continuous at \( t = 0 \). Thus, in the point \( \nu \) of the minimum we obtain

\[ \int_\Omega \sum_{i \in M} D^i g a_i(x, D^i (u_0 + \nu)) \, dx - F(\mathbf{\nu}) = 0 \]

for each \( \mathbf{\nu} \in D(\Omega) \). But, \( \overline{D(\Omega)} = W^{1,\infty}_0(\Omega) \) and the theorem is proved.

**Examples.**

a) Let us construct the Euler's equation to \( \phi(\nu) \) from Example a), § 3. This equation possesses a weak solution.

b) Let us consider the example from § 2, where \( \mathbf{\nu}_2(\mathbf{\omega}) \in M_2 \) and \( \partial \Omega \in C^1 \). There exists a weak solution of (2.23) in Case b.

**References**


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