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MODIFICATIONS OF CLOSURE COLLECTIONS

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Let $\mathcal{S} = \{(S_U, \tau_U); \rho_{UV}; X\}$ be a presheaf of closure spaces over X (i.e. $\rho_{UV}: (S_U, \tau_U) \to (S_V, \tau_V)$ are continuous maps), $\mu = \{\tau_U; U\}$ its closure collection. If for every U and every open covering $\mathcal{V}$ of U there is $\tau_U = \varprojlim_{V \in \mathcal{V}} \tau_V$, we call μ projective collection.

To every μ there exists a finest projective collection μ' coarser than μ (see also [1]). The main result is Theorem (1.20) which shows how we can get the projective modification μ' of μ in case of locally compact X and finitely projective collection μ. From this follows a method of construction of the modification μ' to an arbitrary μ and, moreover, the characterization of projective collections (see (1.22, 23).

<u>Notations</u>.

0.1. We denote by $\mathcal{B}(X)$ the set of all open subsets of a topological space X

0.2. Let (X, t) be a closure space, M its subset.

A) Every filter base of t-neighborhoods of M is denoted by $\Delta(M; t)$.

B) If $\mathcal{F}$ is such a filter in X that for every $F \in \mathcal{F}$ there is $M \subset F$, we say that $\mathcal{F}$ is a filter round M.

C) The relation "a closure u is finer than v" will be denoted by $u \leq v$.

0.3. In the set X, let us have a nonempty family Ω of closures. The coarsest (finest) closure in X, finer (coarser) than every closure from Ω will be denoted by $\varprojlim \Omega$ ($\varinjlim \Omega$).

0.4. Let $U \in \mathcal{B}(X)$. By the symbol Π_U (Π_U°) we denote the set of all open coverings (of all finite open coverings) of the set U.

0.5. Agreement. When speaking about a compact set in a topological space X, we suppose that X is Hausdorff.

§ 1. Projective modifications

(1.1) Notations. For a presheaf $\mathcal{S} = \{(S_U, \tau_U); \rho_{UV}; X\}$ of closure space let

(1.2) $$\mu = \{\tau_U;\ U \in \mathcal{B}(X)\},$$

or briefly $\mu\{\tau_U\}$. A collection μ will be called closure collection of $\mathcal{S}$ or briefly collection. Further, we say that $\mu = \{\tau_U\}$ is finer than $\mu' = \{\tau'_U\}$ (briefly $\mu \leq \mu'$), if every τ_U is finer than τ'_U. If $\Omega \neq \emptyset$ is a family of collections of the presheaf $\mathcal{S}$, then by $\varprojlim \Omega$ resp. $\varinjlim \Omega$ we denote the closure collection

(1.3) $\mu^1 = \{\varprojlim_{\mu \in \Omega} \tau_U^{\mu}\}$, resp. $\mu^2 = \{\varinjlim_{\mu \in \Omega} \tau_U^{\mu}\}$.

From the properties of projective (inductive) limits it follows easily that $\mu^1 = \{\tau_U^1\}$ and $\mu^2 = \{\tau_U^2\}$ from (1.3) are again closure collections, i.e. that the maps $\varrho_{UV} : (S_U, \tau_U^i) \longrightarrow (S_V, \tau_V^i)$ $i = 1, 2$ are all continuous.

(1.4) Definition, notation. If $\mathcal{U} \in \mathcal{B}(X)$ and $\mathcal{V} \in \Pi_U$, we have a collection of maps

$$\Delta_{\mathcal{V}} = \{\varrho_{UV} ; V \in \mathcal{V}, \varrho_{UV} : (S_U, \tau_U) \longrightarrow (S_V, \tau_V)\} .$$

Then we call $\mu = \{\tau_U\}$ projective, if for every $\mathcal{U} \in \mathcal{B}(X)$ and $\mathcal{V} \in \Pi_U$

(1.5) $$\tau_U = \varprojlim_{V \in \mathcal{V}} \tau_V$$

with respect to the set of maps $\Delta_{\mathcal{V}}$.

For a collection μ let

(1.6) $\Omega(\mu) = \{\nu ; \nu$ is a projective collection, $\mu \leq \nu\}$.

(1.7) Proposition. $\mu' = \varprojlim \Omega(\mu) \in \Omega(\mu)$.

(1.8) Definition. The collection μ' will be called projective modification of μ .

We can see that to every μ there exists its projective modification (see also [1]).

(1.9) Notation. Let $\mu = \{\tau_U\}$ be a collection. For any $\mathcal{U} \in \mathcal{B}(X)$ let us set

(1.10) $$\tau_{U,\mathcal{V}} = \varprojlim_{V \in \mathcal{V}} \tau_V \qquad \text{for } \mathcal{V} \in \Pi_U ,$$

(1.11) $\tau_U^* = \varinjlim_{\mathcal{V} \in \Pi_U} \tau_{U,\mathcal{V}}$; $\mu^* = \{ \tau_U^* ; U \in \mathcal{B}(X) \}$.

(1.12) Proposition. Let $\mu = \{ \tau_U \}$ be a collection.

A) The maps $\rho_{UV} : (S_U, \tau_U^*) \longrightarrow (S_V, \tau_V^*)$ are continuous and therefore μ^* is a collection.

B) There is $\mu \leq \mu^* \leq \mu'$.

C) The equality $\mu = \mu'$ holds iff $\mu = \mu^*$.

D) If $(\mu^*) = \mu^*$, there is $\mu^* = \mu'$.

(1.13) Definition. We say that a collection $\mu = \{ \tau_U \}$ is finitely projective, if for every $U \in \mathcal{B}(X)$ and every $\mathcal{V} \in \Pi_U^\circ$ there is

(1.14) $$\tau_U = \varprojlim_{V \in \mathcal{V}} \tau_V \ .$$

(1.15) Proposition. To every collection μ there exists a collection μ^+ such that

(a) $\mu^+ \geq \mu$,

(b) μ^+ is finitely projective,

(c) if ν is a collection satisfying (a,b), then $\mu^+ \leq \nu$,

(d) if we denote $\tilde{\Omega}(\mu) = \{ \nu ; \mu \leq \nu , \nu$ is a finitely projective collection $\}$,

then $\mu^+ = \varprojlim \tilde{\Omega}(\mu)$.

(e) if we denote $\mu^+ = \{ \tau_U^+ \}$, then for every $U \in \mathcal{B}(X)$

(1.16) $$\tau_U^+ = \varinjlim_{\mathcal{V} \in \Pi_U^\circ} \tau_{U,\mathcal{V}} \ .$$

(1.17) Definition. The collection μ^+ is called finite projective madification of μ .

(1.18) <u>Notation</u>. For $U \in \mathcal{B}(X)$, $a \in S_U$, let

(1.19) $\mathcal{B}(a) = \{ \rho_{UV}^{-1}(W^V); V \in \mathcal{B}(U)$, $\overline{V} \subset U$ is compact, $W^V \in \Delta(\rho_{UV}(a); \tau_V)\}$.

It is clear that $\mathcal{B}(a)$ is a filter base round a in S_U . These bases form there a closure which we denote by $\tilde{\tilde{\tau}}_U$. The set $\tilde{\mu} = \{\tilde{\tilde{\tau}}_U; U \in \mathcal{B}(X)\}$ is clearly a collection coarser than μ .

(1.20) <u>Theorem</u>. Let X be locally compact, $\mathcal{S} = \{(S_U, \tau_U); \rho_{UV}; X\}$ a presheaf over X , and $\mu = \{\tau_U\}$ its closure collection.

If $\mu = \mu^+$, then $\mu' = \mu^* = \tilde{\mu}$.

(1.21) <u>Corollary</u>. Let X be locally compact, μ a collection. Then $(\mu^+)^* = \mu'$.

(1.22) <u>Corollary</u>. If X is locally compact, then the collection μ can be projectively modified in two steps. First, we do the finite projective modification μ^+ following (1.16), and then the modification $(\mu^+)^*$ of μ^+ by the help of the bases $\mathcal{B}(a)$ from (1.19).

(1.23) <u>Corollary</u>. If X is locally compact and $\mu = \mu'$, then for $U \in \mathcal{B}(X)$, $a \in S_U$ the bases $\mathcal{B}(a)$ and $\Delta(a; \tau_U)$ are equivalent. Therefore we have the following description of the projective collections μ for a locally compact X : " μ is projective iff it is finitely projective and the bases $\mathcal{B}(a)$ from (1.19) and $\Delta(a; \tau_U)$ are equivalent for all $U \in \mathcal{B}(X)$, $a \in S_U$ ".

(1.24) <u>Example</u>. Let $\mathcal{S} = \{(S_U, \tau_U); \rho_{UV}; E_m\}$ be a presheaf of some sets of continuous functions on $\mathcal{U} \in \mathcal{B}(E_m)$, τ_U the closure of uniform convergence. Then for $\mu = \{\tau_U\}$ one can easily find that

(a) $\mu = \mu^+$,

(b) $\mu' = \mu^* = \{\tau'_U\}$,

where τ'_U for $\mathcal{U} \in \mathcal{B}(E_m)$ is the closure of locally uniform convergence.

It is clear that nothing will change in this example, if we take for X instead of E_m an arbitrary localy compact topological space.

R e f e r e n c e

[1] Z. FROLÍK: Structure projective and structure inductive presheaves. Celebrazioni archimedee del secolo XX:Simposio di topologia,1964.

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