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Remarks on doubly substochastic rectangular matrices


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REMARKS ON DOUBLY SUBSTOCHASTIC RECTANGULAR MATRICES

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In the present paper we show applications of doubly stochastic unit matrix $E$ of the type $(m, n)$ introduced in [1], to doubly substochastic matrices. We obtain a generalization of some results due to L. Mirsky for the case $m = n$ (see [2]).

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2. Doubly stochastic and substochastic maps

1. Orderings and doubly substochastic matrices
Suppose $m, n$ are two positive integers.

Let $\mathbb{R}_m$ be the euclidean space of the dimension $m$.

Define $\beta, \gamma = \sum_{k=1}^n \beta_k \gamma_k$ for $\beta, \gamma \in \mathbb{R}_m$, $a \leq c$ iff $a_1 \leq c_1$, $a_2 \leq c_2$, ..., $a_m \leq c_m$ for $a, c \in \mathbb{R}_m$, $x^+ = (x_1^+, x_2^+, ..., x_m^+)$ and $x^- = (x_1^-, x_2^-, ..., x_m^-)$ for $x \in \mathbb{R}_m$.

If $Z$ is a subset of $\mathbb{R}_m$ then we put:
$Z^+ = \{x^+; x \in Z\}$, $\hat{Z} = \{\hat{x} \in \mathbb{R}_m; \hat{x} \geq 0\}$ for all $x \in Z$. 

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and \( \hat{Z}^+ = \{ \hat{z} \in \mathbb{R}_m \mid \hat{z} \cdot z \geq 0 \} \) for all \( z \in \mathbb{Z}^+ \).

(1.1) **Proposition.** Let \( \mathcal{V}_m = \{ \partial \in \mathbb{R}_m \mid \partial_1 \geq \partial_2 \geq \ldots \geq \partial_m \} \).

Suppose \( c^1 = (1,1,0,\ldots;0) \), \( c^2 = (0,1,1,0,\ldots,0) \), \ldots, \( c^{m-1} = (0,0,0,1,\ldots,1) \) and \( c^m = (0,\ldots,0,1) \) are elements of \( \mathbb{R}_m \). Then the convex cone \( \mathcal{V}_m \) is generated by the elements \( c^1, c^2, \ldots, c^{m-1}, c^m \) and the convex cone \( \mathcal{V}_m^+ \) is generated by the elements \( c^1, c^2, \ldots, c^{m-1}, c^m \).

**Proof.** According to [1] \( d \in \mathcal{V}_m^+ \) (\( d \in \mathcal{V}_m \) resp.) if and only if \( d_1 + d_2 + \ldots + d_m \geq 0 \) for \( \kappa = 1, 2, \ldots, m \) (for \( \kappa = 1, 2, \ldots, m - 1 \) and \( d_1 + d_2 + \ldots + d_m = 0 \) resp.). Hence \( d \in \mathcal{V}_m^+ \) (\( d \in \mathcal{V}_m \) resp.) if and only if there are nonnegative numbers \( \gamma_1, \gamma_2, \ldots, \gamma_m \) (nonnegative numbers \( \gamma_1, \gamma_2, \ldots, \gamma_m \), where \( \gamma_m = 0 \) resp.) such that
\[
d = \gamma_1 c^1 + \gamma_2 c^2 + \ldots + \gamma_{m-1} c^{m-1} + \gamma_m c^m.
\]

(1.2) **Definition.** A matrix \( S = (S_{j,k}) \in \mathbb{R}^{m \times n} \) of the type \((m, n)\) will be called doubly substochastic iff
\[
S_{j,k} \geq 0, \sum_{k=1}^{n} S_{j,k} = 1 \quad \text{and} \quad \frac{1}{m} \sum_{j=1}^{m} S_{j,k} \leq \frac{1}{m} \quad \text{for} \quad j = 1, 2, \ldots, m \quad \text{and} \quad k = 1, 2, \ldots, n.
\]

The set of all doubly substochastic matrices of the type \((m, n)\) will be denoted by \( \mathcal{D}_{m,n} \).

(1.3) **Definition.** Suppose \( S = (S_{j,k}) \in \mathcal{D}_{m,n} \) and \( T = (t_{j,k}) \in \mathcal{D}_{m,n} \). Then we define the following orderings on the set \( \mathcal{D}_{m,n} \):
\[
\preceq, \leq \quad \text{and} \quad \succeq, \geq, \quad \text{where}
\]
\[
1^o \quad S \preceq T \iff S_{j,k} \leq t_{j,k} \quad \text{for all} \quad j \quad \text{and} \quad k.
\]
2° $S \preceq T$ iff $S \vartheta \cdot x \preceq T \vartheta \cdot x$ for all $x \in \mathcal{V}_m$ and $\vartheta \in \mathcal{V}_m$.

3° $S \succeq T$ iff $S \vartheta \cdot x \succeq T \vartheta \cdot x$ for all $x \in \mathcal{V}_m^+$ and $\vartheta \in \mathcal{V}_m^+$.

(1.4) Lemma. If $S \preceq T$ ($S \succeq T$ resp.) then $S \preceq T$.

(1.5) Lemma. Let $S \in \mathcal{D}_{m,n}$. Then there is a doubly stochastic matrix $Q \in \mathcal{D}_{m,n}$ (see [1]) such that $S \preceq Q$.

Proof. This lemma follows from Lemma 9.1 in [3].

(1.6) Theorem. Let $E$ be the doubly stochastic unit matrix of the type $(m,n)$ (see [1], 5). Then $S \preceq E$ for all $S \in \mathcal{D}_{m,n}$.

Proof. Let $S \in \mathcal{D}_{m,n}$. Then there is a doubly stochastic matrix $Q \in \mathcal{D}_{m,n}$ such that $S \preceq Q$ by (1.5). According to Theorem (5.3) in [1] $Q \preceq E$. Hence $S \preceq Q \preceq E$ by (1.4) and $S \preceq E$.

(1.7) Corollary. Let $U = \{ x \otimes \vartheta; x \in \mathcal{R}_m, x_1 > x_2 > \ldots > x_m > 0, \vartheta \in \mathcal{R}_n, \vartheta_1 > \vartheta_2 > \ldots > \vartheta_n > 0 \}$ (see [1], 5.5). Then $E$ is a $U$-exposed element of the set $\mathcal{D}_{m,n}$, i.e. $E \vartheta \cdot x > S \vartheta \cdot x$ for all $x \in \mathcal{R}_m, \vartheta \in \mathcal{R}_n$ and $S \in \mathcal{D}_{m,n}$ such that $x_1 > x_2 > \ldots > x_m > 0, \vartheta_1 > \vartheta_2 > \ldots > \vartheta_n > 0$ and $S \preceq E$.

2. Doubly stochastic and substochastic maps

Suppose $a$ is an element of the set $\mathcal{V}_m$ and $\vartheta$ is an element of the set $\mathcal{V}_n$. 

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Let $E$ be the doubly stochastic unit matrix of the type $(m, m)$.

(2.1) Lemma. 1° The following inclusion holds:

$$R^+_m \cup \hat{V}^-_m \subseteq \hat{V}^+_m.$$ 

$$2° \quad E \cdot e^+ - a \in \hat{V}^+_m \iff E \cdot e^+ - a \in \hat{V}^+_m.$$ 

(2.2) Theorem. The following conditions are equivalent:

1° There is a doubly stochastic matrix $Q \in D_{m,m}$ such that $a \in Q \cdot e$.

$$2° \quad E \cdot e^+ - a \in \hat{V}^+_m.$$ 

$$3° \quad a \cdot v^\kappa \leq e \cdot w^\kappa \text{ for } \kappa = 1, 2, ..., m,$$ 

where the vectors $v^\kappa$ and $w^\kappa$ are defined in [1].

Proof. 1° $\Rightarrow$ 2°: If $a \in Q \cdot e$, where $Q \in D_{m,m}$, put $d = Q \cdot e$. Then $d - a \in R^+_m$ and $E \cdot e - d \in \hat{V}^-_m$ by (5.3) in [1]. Using Lemma (2.1) we obtain Property 2°.

2° $\Rightarrow$ 1°: Let $E \cdot e - a \in \hat{V}^+_m$. Put $c = E \cdot e$ and $\xi = (c_1, c_2, ..., c_{m-1}, c_m)$, where $\xi_m = a_1 + a_2 + ... + a_m - c_1 - c_2 - ... - c_{m-1}$. Then $\xi \in V_m$; further $\xi \leq c$ and $\xi - a \in \hat{V}_m$. Therefore there is a doubly stochastic matrix $R \in D_{m,m}$ such that $a = R \xi$ by (6.3) in [1].

Hence $a = R \xi \leq Rc = R \cdot e^+ \cdot e$ and $a = R \cdot e \in D_{m,m}$ by (2.2) in [1].

2° $\Rightarrow$ 3°: $E \cdot e - a \in \hat{V}^+_m \iff a \cdot x \leq E \cdot e \cdot x$ for all $x \in V^+_m$. Since $E \cdot e \cdot v^\kappa = e \cdot \hat{V}^+_m$ for $\kappa = 1, 2, ..., m$ and since the conus $V^+_m$ is generated by the vectors $v^1, v^2, ..., v^m$, we obtain
the equivalence of the conditions 2° and 3°.

(2.3) Theorem. There is a matrix $S \in \mathbb{D}_{m,n}$ such that
\[ a \leq S \tau \text{ if and only if } E \tau^+ - a \in \check{V}^+_m. \]

Proof. 1° Let $a \leq S \tau$, where $S \in \mathbb{D}_{m,n}$. Then
there is a doubly stochastic matrix $Q \in \mathbb{D}_{m,n}$ such
that $S \leq Q$ by (1.5). We obtain the following inequa-
lities: $a \leq S \tau \leq S \tau^+ \leq Q \tau^+$. Hence $E \tau^+ - a \in \check{V}^+_m$ by (2.2).

2° Let $E \tau^+ - a \in \check{V}^+_m$. Then according to Theo-
rem (2.2) there is a doubly stochastic matrix $Q \in \mathbb{D}_{m,n}$
such that $a \leq Q \tau^+$. Define $S = (\lambda^*_{j,k})_{j,k} \in \mathbb{D}_{m,n}$:
\[ \lambda^*_{j,k} = Q_{j,k} (Q_{j,k} = 0 \text{ resp.}) \text{ if } j \in \{1, 2, \ldots, m\}, \]
\[ k \in \{1, 2, \ldots, n\} \text{ and } Q_{j,k} > 0 \text{ ( } Q_{j,k} = 0 \text{ resp.}). \]
Then $Q \tau^+ = S \tau$. Hence $a \leq S \tau$.

(2.4) Note. If $a \in V^+_m$, $\tau \in V_m$ and $E \tau^+ - a \in \check{V}^+_m$
then there is a doubly substochastic matrix
\[ S = (\lambda^*_{j,k})_{j,k} \in \mathbb{D}_{m,n} \text{ such that } a = S \tau, \text{ where } \lambda^*_{j,k} = \]
\[ = 0 \text{ if } j \in \{1, 2, \ldots, m\}, k \in \{1, 2, \ldots, n\}, \text{ and } Q_{j,k} \leq 0 \text{ or } a_j = 0. \]

Proof. Suppose that $a \in V^+_m$, $\tau \in V_m$ and
$E \tau^+ - a \in \check{V}^+_m$. Then there is a matrix $R = (\kappa^*_{j,k})_{j,k} \in \mathbb{D}_{m,n}$
such that $a \leq R \tau$ by (2.3). We can suppose
that $\kappa^*_{j,k} = 0$ if $Q_{j,k} \leq 0$ for all $j$ and $k$.

Put $\lambda_j = \frac{a_j}{\sum_{k=1}^{n} \kappa^*_{j,k} Q_{j,k}}$ ( $\lambda_j = 0$ if $j \in \{1, 2, \ldots, m\}$
and $a_j = 0$ ), $\lambda^*_{j,k} = \lambda_j \kappa^*_{j,k}$ and $S = (\lambda^*_{j,k})_{j,k}$ for
all $j$ and $k$. Clearly, $0 \leq \lambda_j \leq 1$, $S \in \mathbb{D}_{m,n}$ and

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a = S\ell r.

(2.5) Corollary. Suppose \( a \in V_m \) and \( \ell r \in V_m \). Then there is a doubly substochastic matrix \( S \in \mathcal{D}_{m,n} \) such that \( a = S\ell r \) if and only if \( E\ell r^+ - a \in \hat{V}_m^+ \) and

\[ E_m'(E\ell r^+ + a) \in \hat{V}_m^+ \text{, where } E_m' \text{ is the converse-permutation matrix of the set } \mathcal{D}_{m,n} \text{ (see [11], 4).} \]

Proof. 1° If \( a = S\ell r \), where \( S \in \mathcal{D}_{m,n} \) then

\[ E_m'(-a) = E_m'(-a) = E_m'\hat{E}_m^+(E_m'(-\ell r)), E_m'(-a) \in V_m \text{ and} \]

\[ E_m'(-\ell r) \in V_m \text{. According to our theorem (2.3) and to Theorem (5.8), in [11] } E\ell r^+ - a \in \hat{V}_m^+ \text{ and} \]

\[ E_m'(E\ell r^+ + a) = E.E_m'(-\ell r)^+ - E_m'(-a) \in \hat{V}_m^+ \text{.} \]

2° Suppose that \( E\ell r^+ - a \in \hat{V}_m^+ \) and \( E_m'(E\ell r^+ + a) \in \hat{V}_m^+ \). Then \( E\ell r^+ - a \in \hat{V}_m^+ \) and \( E.E_m'(-\ell r)^+ - E_m' a^- \in \hat{V}_m^+ \) by (2.1). Using the note (2.4) we obtain substochastic matrices \( R = (\kappa_{j,k})_{j,k} \) and \( T = (t_{j,k})_{j,k} \) in \( \mathcal{D}_{m,n} \) such that \( a^+ = R\ell r \), \( a^- = T(-\ell r) \),

where \( \kappa_{j,k} = 0 \) (\( t_{j,k} = 0 \) resp.) if \( j \in \{1, 2, \ldots, m\}^3 \), \( k \in \{1, 2, \ldots, m\} \) and \( \ell r_k \leq 0 \) or \( a^- \leq 0 \) (\( \ell r_k \geq 0 \) or \( a^- \geq 0 \) resp.).

Put \( S = R + T \). Then \( S \in \mathcal{D}_{m,n} \) and

\[ a = a^+ - a^- = (R + T)\ell r = S\ell r \text{.} \]

The proof is complete.

References


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