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ERROR - ESTIMATES FOR THE METHOD OF LEAST SQUARES OF FINDING EIGENVALUES AND EIGENFUNCTIONS

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In [1], [2], we considered the approximation of eigenvalues and eigenfunctions of a DS-operator. In this paper, we shall present a priori and a posteriori error estimates for the method of least squares of finding eigenvalues and eigenfunctions. Upper and lower error bounds are found.

We assume throughout that $A$ be a DS-operator with its domain in a real separable Hilbert space $H$, i.e., $A$ is a symmetric operator in $H$ such that the set of its eigenvalues is of the first category on the real axis and the spectrum $\sigma(A)$ is the closure of this set. Let $\{Y_i\}_{i=1}^{\infty}$ be a totally complete system. Suppose $A$ is such that the eigenvalues $\{\lambda_i\}$ of $A$ satisfy the relations

\[(I) \quad 0 < |\lambda_1| < |\lambda_2| \leq |\lambda_3| \leq ... \]

and $\lambda_i$ is simple.

Let $R_n$ and $R_m$ be subspaces of $H$ determined by functions $\{Y_i\}_{i=1}^{n}$ and $\{AY_i\}_{i=1}^{m}$, respectively. Let $g_i$ be a normalized eigenfunction of $A$ correspon-
ding to the eigenvalue $\lambda_i$. We shall denote the orthogonal projection of $\varphi_i$ on $R_m$ and $R_{m-1}$ by $\varphi_i^{(m)}$ and $\varphi_i^{(m)}$, respectively. By $T$ we shall mean the restriction of $A$ to $R_m$. Since $0 \in \sigma(A)$, it follows that $T$ and $T^{-1}$ are continuous linear operators on $R_m$ and $R_{m-1}$ respectively.

It has been shown in [1] that $q_\infty$ is an approximation to $|\lambda_i|$, where

$$q_\infty = \min_{\mu \in R_m} |A\mu|.$$  

From Theorem 3 of [2] it follows that there exist $\{\mu_n\}_{n=1}^\infty$ such that the following conditions are satisfied:

1) $\mu_n \in R_m$, $|\mu_n| = 1$,

(II)

2) $|A\mu_n| = q_\infty$,

3) $\lim_{n \to \infty} \mu_n = \varphi_i$,

4) $(\mu_n, \varphi_i) > 0$ for $n = 1, 2, 3, \ldots$.

1. In this section, we shall derive upper and lower bounds for $q_\infty - |\lambda_i|$. Before going further we note this useful fact:

Since $|\varphi_i| = 1$, it follows from the definition of orthogonal projection that

$$|\varphi_i - \varphi_i^{(m)}|^2 = 1 - |\varphi_i^{(m)}|^2,$$

$$|\varphi_i - x\varphi_i|^2 = 1 - |x\varphi_i|^2.$$  

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Now, we present a group of two results, which is useful to have on record for later use.

**Lemma 1.** With the assumption of (I), the following inequalities are valid for each positive integer $n$:

a) $\lambda_1^2 \cdot \| T^{-1}(m) q_1 \|^2 \geq 2 \cdot \| T^{-1}(m) q_1 \|^2 - \| q_1^{(m)} \|^2$,

b) $| \lambda_1 \| T^{-1}(m) q_1 \| \geq 1 - \| q_1 - q_1^{(m)} \|$

**Proof.** a) It follows from the definition of $q_1^{(m)}$ that

$$\| q_1 - q_1^{(m)} \|^2 = \| q_1 - \lambda_1 T^{-1}(m) q_1 \|^2 .$$

We have therefore

(2) $1 - \| q_1^{(m)} \|^2 \leq 1 + \lambda_1^2 \cdot \| T^{-1}(m) q_1 \|^2 - 2 \lambda_1 \cdot (q_1, T^{-1}(m) q_1) .$

The proof of a) follows at once from (2), because

$$\lambda_1 (q_1, T^{-1}(m) q_1) = (A q_1, T^{-1}(m) q_1) = (q_1^{(m)} q_1) = \| q_1^{(m)} \|^2 .$$

b) By Theorem 2 of [1] we have

$$| A \mu | > | \lambda_1 | \cdot | \mu |$$

for any $\mu \in \mathcal{E}(A)$.

Letting $\mu = q_1 - \lambda_1 T^{-1}(m) q_1$, we see that

$$| \lambda_1 | \cdot \| q_1 - q_1^{(m)} \| = | A \mu | ,$$

whence follows

(3) $\| q_1 - q_1^{(m)} \| \geq \| q_1 - \lambda_1 T^{-1}(m) q_1 \| .$

It follows from $\| q_1 \| \leq | A \mu | + | \lambda_1 | \cdot \| T^{-1}(m) q_1 \|$ that

(4) $| \lambda_1 | \cdot \| T^{-1}(m) q_1 \| \geq 1 - \| q_1 - \lambda_1 T^{-1}(m) q_1 \| .$

Now, if we insert (3) in (4), we obtain the sta-
Corollary 1. For any $n$, we have $\|q_n\|^2 \leq \|q_n^{(m)}\|^2$.

Hence $\|q_n - q_n^{(m)}\|^2 \geq \|q_n - q_n^{(m)}\|^2$.

Proof: By the definition $q_n$, we have $q_n \geq |\lambda_n| > 0$ and

\begin{equation}
\|T^{-1}(q_n)\| \leq \frac{1}{q_n} \cdot \|q_n\|.
\end{equation}

The corollary follows easily from (5) and Lemma 1.

Remark 1. From the totally completeness of $\{q_i\}_{i=1}^\infty$ and the assumption $0 \in \sigma(A)$ it follows that

$$\lim_{n \to \infty} \|q_n^{(m)}\| = \lim_{n \to \infty} \|q_n\| = 1$$

and therefore

$$\lim_{n \to \infty} q_n^{(m)} = \lim_{n \to \infty} q_n = q_1.$$ 

Consequently, from Lemma 1 it follows $\lim_{n \to \infty} |\lambda_n|$.

$$\|T^{-1}(q_n)\| = 1.$$ 

Remark 2. There exists some $m_0 \geq 0$ such that

$$2 \cdot \|q_n\|^2 - \|q_n^{(m)}\|^2 \geq (1 - \|q_n - q_n^{(m)}\|)^2$$

for $m \geq m_0$.

Proof. From Remark 1 it follows that there exists $m_0$ such that $\|q_n - q_n^{(m)}\|^2 \leq \frac{2}{3} \cdot \|q_n - q_n^{(m)}\|^2$ for $m \geq m_0$. It follows that

$$\|q_n\|^2 \geq 1 - \frac{2}{3} \cdot \|q_n - q_n^{(m)}\|^2$$

for $m \geq m_0$.

When this is substituted in

$$2 \cdot \|q_n^{(m)}\|^2 - \|q_n^{(m)}\|^2 \geq 2 \cdot \|q_n\|^2 - 1 = 3 \cdot \|q_n\|^2 + \|q_n - q_n^{(m)}\|^2 - 2,$$

we obtain the statement.

An important tool in the proof of the next theorem is furnished by the following lemma.
Lemma 2. If we denote the product $(u_n, q_j)$ by $\alpha^{(m)}_i$, then under the assumption (I) we have
\[
(\alpha^{(m)}_i)^2 \geq 1 - \frac{q^2 - \lambda^2}{\lambda^2 - \lambda^2} \quad \text{for any } n.
\]

Proof. By Lemma 1 of [1], we have
\[
(6) \quad q^2 - \lambda^2 \geq \varepsilon \left( \lambda^2 - \lambda^2 \right) \| u^{(m)}_i \|^2,
\]
where $u^{(m)}_i$ is the orthogonal projection of $u_m$ on $H_i$ and $H_i$ is the closure of linear manifold generated by the eigenfunctions of $A$ associated with the eigenvalue $\lambda_i$. Since $|\lambda_2| > |\lambda_1|$ and $\| u_m \| = 1$, it follows from (6) that
\[
q^2 - \lambda^2 \geq \varepsilon \left( \lambda^2 - \lambda^2 \right) \| u^{(m)}_i \|^2,
\]
so that
\[
\| u^{(m)}_i \|^2 \geq 1 - \frac{q^2 - \lambda^2}{\lambda^2 - \lambda^2}.
\]
Now $u^{(m)}_i = (u_m, q_j) \cdot q_j$ and thus the proof is complete.

The following theorem is of fundamental importance.

Theorem 1. Let $A$ be a DS-operator and $\{ \mathcal{Y}_i \}_{i=1}^\infty$ a totally complete system. Suppose the eigenvalues $\{ \lambda_i \}_{i=1}^\infty$ of $A$ satisfy the relations $0 < |\lambda_1| < |\lambda_2| \leq |\lambda_3| \leq \ldots$ and $\lambda_i$ is simple. Construct the sequence of numbers $\{ q_n \}_{n=1}^\infty$ such that
\[
q_n = \min_{\| u \| = 1} \| Au \|
\]
where $R_n = \mathcal{X} \mathcal{I} \{ \mathcal{Y}_j \}_{j=1}^n$.

Let $u^{(m)}_i$ be the orthogonal projection of a normalized eigenfunction $q_j$ corresponding to $\lambda_i$ on $R_n$. - 467 -
Let $\{q_i\}_{i=1}^\infty$ and $m_0$ be a positive integer such that $\langle q_i, q_1 \rangle = 0$ and $\langle q_i, \phi_1 \rangle = 0$. Then there exist constants $C_1$ and $C_2$ such that $C_1 \langle q_i, \phi_1 \rangle \leq C_2 \langle q_i, \phi_1 \rangle$ for $m \geq m_0$.

Proof. Suppose $m \geq m_0$. Then $\|q_i\| = 0$. By the definition of $q_m$, we have

$$q_m - |\lambda_1| \leq C \langle q_i, \phi_1 \rangle^2 \leq \langle q_i, \phi_1 \rangle^2 \leq \langle q_i, \phi_1 \rangle^2,$$

where $C = \|T^{-1}q_i \|^{-1} \langle q_i, \phi_1 \rangle^2$. From Lemma 1 and (8) it follows that

$$q_m - |\lambda_1| \leq C \langle q_i, \phi_1 \rangle^2 + \langle q_i, \phi_1 \rangle^2 \| q_i \| \leq \langle q_i, \phi_1 \rangle^2 \leq \langle q_i, \phi_1 \rangle^2,$$

for $m \geq m_0$. Since

$$\frac{\langle q_i, \phi_1 \rangle^2}{\|q_i, \phi_1 \|^2} \geq |\lambda_1|,$$

we have

$$C \leq \frac{1}{2 |\lambda_1| \|q_i, \phi_1 \|^2}.$$

From this and Lemma 1 we obtain

$$C \leq \frac{|\lambda_1|}{2 \langle q_i, \phi_1 \rangle^2 \|q_i, \phi_1 \|^2} \leq \langle q_i, \phi_1 \rangle^2 \|q_i, \phi_1 \|^2 \|q_i, \phi_1 \|^2.$$

Letting $C_1 = \frac{1}{2} |\lambda_1| \langle q_i, \phi_1 \rangle^2 \|q_i, \phi_1 \|^2$, from (9) and (1) it follows
To prove the second part of (7) we construct \( u_m \)
such that the conditions (1) are satisfied. Then

\[
q_m - \lambda_1 = \| A u_m - \lambda_1 \varphi_i \|^2 + 2 \lambda_1 (u_m - \varphi_i, \varphi_i) \geq \\
\geq \lambda_1^2 \| \varphi_i - (m) \|^2 + 2 \lambda_1^2 (\alpha_i^{(m)})^2 - 1,
\]

where \( \alpha_i^{(m)} = (u_m, \varphi_i) \).

Using Lemma 2, we have

\[
q_m - \lambda_1^2 \geq \lambda_1^2 \| \varphi_i - (m) \|^2 - 2 \lambda_1^2 (q_m - \lambda_1^2) \cdot (\lambda_2 - \lambda_1^{-1})
\]

whence with the notation

\[
x = q_m - \lambda_1, a = (\lambda_2 - \lambda_1^2) (\lambda_2 + \lambda_1^2)^{-1}, \lambda_1^2 \| \varphi_i - (m) \|^2, b = 2! \lambda_1
\]

one finds

\[
(x + b) \geq a.
\]

After some computation we find that the solution

of (10) satisfies the inequality

\[
x = q_m - \lambda_1 \geq \\
\geq C_2 \cdot \| \varphi_i - (m) \|^2,
\]

where

\[
C_2 = 2! \lambda_1 \cdot (\lambda_2 - \lambda_1^2) \cdot (5 \lambda_2^2 + 3 \lambda_1^2)^{-1}.
\]

Thus the proof is complete.

Remark 3. Theorem 1 is valid in the case when \( \lambda_1 \)
is a multiple eigenvalue of \( A \).

Remark 4. From the proof of Theorem 1 it follows

that the right hand side of the inequality (7) is va-

lid for any DS-operator such that \( 0 \not\in \sigma(A) \).
2. Bearing in mind the considerations of the previous section, we now find a priori bounds for the approximations $u_m$ to an eigenfunction $\varphi_i$. To establish these bounds we require the following Lemma 3.

**Lemma 3.** Under the hypotheses as in Theorem 1, we have for $m \geq m_*$

(a) $\| A u_m - A \varphi_i \| ^2 \leq (q_m^2 - \lambda_i^2) \cdot (\lambda_2^2 + \lambda_i^2) \cdot (\lambda_2^2 - \lambda_i^2)^{-1}$,

(b) $\| u_m - \varphi_i \| ^2 \leq (q_m^2 - \lambda_i^2) (\lambda_2^2 - \lambda_i^2)^{-1}$,

(c) $\lambda_2^2 - \lambda_i^2 \leq D \cdot \| \varphi_i - c_2 \varphi_i \| ^2$,

where $D = \lambda_2^2 \cdot (1 - \| \varphi_i - c_2 \varphi_i \| ^2)^{-1}$.

**Proof.** In a similar way, by methods analogous to those employed in the proof of Theorem 1, we can obtain

(11) $q_m^2 - \lambda_i^2 \leq \| T^{-1} c_{i1} \varphi_i \| ^2 \cdot (\| c_{i2} \varphi_i \| ^2 - \| c_{i1} \varphi_i \| ^2)$.

From Lemma 1 and (1) it follows the inequality (c).

To prove (a) we write

$\| A u_m - A \varphi_i \| ^2 = \| A u_m \| ^2 + \lambda_i^2 - 2 (A u_m, A \varphi_i) = q_m^2 - \lambda_i^2 + 2 \lambda_i^2 (1 - \alpha_{i1})$.

where $\alpha_{i1} = (u_m, \varphi_i)$.

Since $\alpha_{i1} \in (0, 1)$, we see that

$\| A u_m - A \varphi_i \| ^2 \leq q_m^2 - \lambda_i^2 + 2 \lambda_i^2 (1 - (\alpha_{i1})^2)$,

and the inequality (a) follows from Lemma 2.

The proof of (b) follows at once from Lemma 2, because $(\alpha_{i1})^2 \leq \alpha_{i1}$, $\| u_m \| = 1$ and $\| \varphi_i \| = 1$.

The following theorem is a consequence of Lemma 3.
Theorem 2. Under the hypotheses as in Theorem 1 there exist the constants $C_2$ and $C_3$ which do not depend on $m$ such that for $m \geq n_0$

\[ |\lambda_1| \cdot \| \varphi - (m) \varphi \| \leq \| A \mu_m - A \varphi \| \leq C_2 \cdot \| \varphi - (m) \varphi \| \]

\[ \| \varphi - (m) \varphi \| \leq \| \mu_m - \varphi \| \leq C_3 \cdot \| \varphi - (m) \varphi \| , \]

where $\varphi^{(m)}$ is the orthogonal projection of $\varphi$ on $R_m = \mathcal{L} \{ \varphi_i \}_{i=1}^m$.

Proof. The right sides of these inequalities follow at once from Lemma 3. Since $\lambda_1 \neq 0$ from the definition of orthogonal projection it follows

\[ \| A \mu_m - A \varphi \|^2 = \lambda_1^2 \frac{A \mu_m}{\lambda_1} - \varphi \|^2 \geq \lambda_1^2 \cdot \| \varphi - (m) \varphi \|^2 \]

Thus all is proved.

Remark 5. Theorem 2 is valid in the case when $\lambda_1$ is a multiple eigenvalue of $A$.

3. In this section, we find a posteriori bounds for the errors in the approximations $\varphi$ and $\mu_m$ to the eigenvalue $\lambda_1$ and the eigenfunction $\varphi$, respectively.

Under the hypotheses as in Theorem 1, we construct the sequence $\{ \mu_m \}_{m=1}^\infty$ such that the condition (I) is satisfied. To simplify our notation in this section let $\epsilon_m = \| A \mu_m - \epsilon \varphi \|$, where $\epsilon = \text{sign} \lambda_1$.

Our next principal result is Theorem 3. An important tool in the proof of this theorem is furnished by the
Lemma 4. Suppose \( n \) is such that \( \alpha_i^{(m)} > 0 \) and \( |\lambda_2| > q_n \). Then

(a) \[ q_n - |\lambda_2| \leq D_1 \sigma_n^2, \]

where \[ D_1 = \frac{4q_n^2 + \lambda_2^2 - \lambda_1^2}{q_n \cdot (\lambda_2^2 - \lambda_n^2)} \]

(b) \[ q_n - |\lambda_2| \geq D_2 \sigma_n^2, \]

where \[ D_2 = \frac{1}{2} (\lambda_2^2 - \lambda_1^2) \cdot q_n \cdot (\sqrt{2} + \sqrt{\frac{\lambda_2^2}{\lambda_1^2}} + 1)^2 \]

(c) \[ \|A u_m - A \varphi_i\| \leq D_3 \sigma_n', \]

where \[ D_3 = 5 \lambda_2^2 \cdot [\left(\lambda_2^2 - \lambda_1^2\right) \left(\lambda_2^2 - q_n^2\right)]^{-\frac{1}{2}} \]

(d) \[ \|A u_m - A \varphi_i\| \geq D_4 \sigma_n', \]

where \[ D_4 = (\lambda_2^2 - \lambda_1^2) \cdot [\sqrt{2} \sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_2^2}]^{-1} \cdot \left(\frac{|\lambda_2|}{q_n}\right)^{\frac{1}{2}} \]

(e) \[ \|u_m - \varphi_i\| \leq D_5 \sigma_n', \]

where \[ D_5 = 5 \cdot |\lambda_2| \cdot \left[\left(\lambda_2^2 - \lambda_1^2\right) \left(\lambda_2^2 - q_n^2\right)\right]^{-\frac{1}{2}} \]

**Proof.** Since \( \lambda_{i} = |\lambda_i| \) and \( \|A u_m\| = q_n \), we have

(12) \[ \|A u_m - e q_m \varphi_i\|^2 = 2 q_n (q_n - |\lambda_i| \cdot \alpha_i^{(m)}) \]

where \( \alpha_i^{(m)} = (u_m, \varphi_i) \).

If we subtract the following identity
from (12), we obtain

\[ 2 \sigma_1^{(m)} q_m (Q_n - \lambda) = y (\gamma + 2 q_n \cdot q_m - \mu_n) \],

where \( y = \| A u_m - e q_n \| - q_n \cdot q_m - \mu_n \| \).

Since \( q_n \geq \lambda \), it follows that \( y \geq 0 \) and

\[ c_n' = \| A u_m - e q_n \| + \gamma q_n \cdot q_m - \mu_n \| \geq y \).

Hence we have from (13)

\[ 2 \sigma_1^{(m)} (Q_n - \lambda) \leq c_n' (c_n' + 2 q_n \cdot \mu_n) \).

It follows immediately from Lemma 3

\[ \| u_m - q \| \leq 2 \cdot \sqrt{\frac{q}{\lambda^2 - \lambda^2}} \cdot \sqrt{q - 1 \lambda_n} \cdot \sqrt{q - 1 \lambda_n}.\]

Using this in (14), we obtain

\[ a x^2 \leq c + b x, \]

where

\[ x = \sqrt{q_n - 1 \lambda_n}, \quad a = 2 q_n \sigma_1^{(m)}, \quad b = 4 q_n \sigma_1' \cdot \sqrt{\frac{q_n}{\lambda^2 - \lambda^2}}. \]

After some computation we may find that

\[ x \geq \sigma_n^2. \] This proves (a).

To prove (b) observe that

\[ c_n' = \| A u_m - e q_n \| + q_n \cdot \mu_n \| \).

By the definition of \( m_n \) in Theorem 2, we have that

\[ m_n \geq m_n. \] Since \( q_n \geq \lambda_n \), it now follows from Lemma 3 that

\[ \| q_n - \mu_n \| \leq 2 \cdot \sqrt{\frac{q_n}{\lambda^2 - \lambda^2}} \cdot \sqrt{q_n - 1 \lambda_n}. \]
Assume that \( q_n > |\lambda_1| \). Then, by (15) and (12)

\[
(16) \quad c_n' \leq C \cdot \sqrt{q_n - |\lambda_1|},
\]

where

\[
C = 2 \cdot \sqrt{\frac{q_n^2}{\lambda_2^2 - \lambda_1^2}} + \sqrt{2q_n} \cdot \sqrt{1 + \frac{|\lambda_1|(1 - \alpha_{m}^{(m)})}{q_n - |\lambda_1|}}.
\]

Since \( \alpha_{m}^{(m)} > 0 \), we see from Lemma 2 that

\[
(17) \quad 1 + \frac{|\lambda_1|(1 - \alpha_{m}^{(m)})}{q_n - |\lambda_1|} \leq 1 + \frac{|\lambda_1(q_m + |\lambda_1|)}{\lambda_2^2 - \lambda_1^2} = \frac{\lambda_2^2 + |\lambda_1^2| \cdot q_m}{\lambda_2^2 - \lambda_1^2}.
\]

The inequality (b) now follows from (16) and (17) in the case of \( q_n > |\lambda_1| \). It is readily verified that (b) is also valid in the case of \( q_n = |\lambda_1| \).

The proof of (c) and (e) follows at once from (a) and Lemma 3 because \( q_n + |\lambda_1| \leq 2q_n \). It is readily verified that

\[
|A u_n - Aq_1|^2 \geq q_n^2 - \lambda_1^2 \geq 2 \cdot |\lambda_1| \cdot (q_n - |\lambda_1|)
\]

and from (b) it follows the inequality (d). This completes the proof.

From Lemma 4 (c) and from \( c_n \to 0 \) it follows

\[
\lim_{n \to \infty} A u_n = Aq_1. \quad \text{Consequently, there exists } m_1 \quad \text{such that for } n \geq m_1
\]

\[
(18) \quad \text{sign} (A u_m, u_m) = \text{sign} \lambda_1 = e.
\]

Therefore

\[
(19) \quad \|A u_m - e q_n u_m\|^2 = 2q_n \cdot (q_n - |(A u_m, u_m)|) \quad \text{for } n \geq m_1.
\]

From Lemma 4 and (19) we deduce the following
Theorem 3. Under the hypotheses as in Theorem 1 there exist the constants $K_1, K_2, K_3, K_4, K_5$ which do not depend on $m$ and an integer $n_1$ such that for $m \geq n_1$

\[ K_2 \cdot \varepsilon_m^2 \leq q_m - |\lambda_i| \leq K_4 \cdot \varepsilon_m^2 , \]

\[ K_5 \cdot \varepsilon_m \leq ||A_{\mu_n} - A_{\mu_i}|| \leq K_3 \cdot \varepsilon_m , \]

\[ ||\mu_n - \mu_i|| \leq K_6 \cdot \varepsilon_m , \]

where $\varepsilon_m = q_m - |(A_{\mu_n}, \mu_n)|$.

Remark 5. From (18) it follows that

\[ \lim_{n \to \infty} \left[ q_n \cdot \text{sign} (A_{\mu_n}, \mu_n) \right] = \lambda_i . \]

4. In all previous sections we have been concerned with setting up error bounds of approximations for $\lambda_i$ and $\mu_i$. In order to obtain error bounds for $\lambda_i, i > 1$, we shall assume that

(III) $\lambda_i$ is not an accumulation point of the spectrum $\sigma(A)$.

For the sake of simplicity, we shall suppose that

(IV) $\lambda_i$ is simple and $0 \not\in \sigma(A)$.

Select $\mu$ in such a way that

1) $\mu \in \sigma(A)$,

(V)

2) $|\mu - \lambda_i| < |\mu - t|$ for any $t \in \sigma(A), t \neq \lambda_i$.

From Theorem 3 of [11] it follows that $\lim_{n \to \infty} q_n =$
Then \( \mu + q_m \) or \( \mu - q_m \) is the approximation to \( \lambda_i \).
Denote this approximation by \( \lambda^{(m)}_i \). Let \( \varphi_i \) be a normalized eigenfunction corresponding to \( \lambda_i \), and \( \varphi^{(m)}_i \) and \( \varphi^{(m)}_i \) orthogonal projections of \( \varphi_i \) on \( R_m = \mathcal{L}\{\varphi_i\}_{i=1}^m \) and \( R_m = \mathcal{L}\{A\varphi_i\}_{i=1}^m \), respectively.

If we apply the above results with \( (A - \mu I) \) in place of \( A \), then we obtain error bounds of approximations for \( \lambda_i \) and \( \varphi_i \). As an immediate consequence of Theorems 1, 2, 3 and the following Lemma 5, we have

**Theorem 4.** Under the assumptions (III) - (V) we construct \( \{\mu_m\}_{m=1}^M \) such that the following conditions are satisfied:

1) \( \mu_m \in R_m, \|\mu_m\| = 1 \),
2) \( q_m = \|A\mu_m - \mu\mu_m\| \),
3) \( \langle \mu_m, \mu_{m+1} \rangle \geq 0 \).

Then there exist an integer \( m_1 \) and the constants \( C_1, \ldots, C_4, C_3, C_1, K_1, K_2, K_3, K_4 \) which do not depend on \( m \) such that for \( m \geq m_1 \)

(a) \( C_2 \sigma_n^2 \leq |\lambda_i - \lambda_i^{(m)}| \leq C_4 \sigma_n^2 \),
(b) \( \kappa_m \leq \|\mu_m - \varphi_i\| \leq C_3 \sigma_n \),
(c) \( C_4 \kappa_m \leq \|A\mu_m - A\varphi_i\| \leq C_4 \sigma_n \),

where \( \sigma_n = \|\varphi_i - \varphi^{(m)}_i\| \) and \( \kappa_m = \|\varphi_i - \varphi^{(m)}_i\| \).
Lemma 5. Let \((\cdot)_{\lambda_i} (\mu)\) be the orthogonal projection of \(g_i\) on \(R_{\lambda_i} = \mathbb{L}(A - \mu I) Y_{i_j} j=1\). Under the assumptions (III) - (V) we have

\[ \| g_i - \omega_i^{(\lambda)} \| \leq \| g_i - \omega_i^{(\mu)} \|^2 \leq D_1 \cdot \| g_i - \omega_i^{(\lambda)} \| , \]

where

\[ D_1 = \frac{1}{\lambda_i - \mu} \cdot \inf_{t \in \sigma(A)} \left| 1 - \frac{\mu}{t} \right| , \]

\[ D_2 = \frac{1}{\lambda_i - \mu} \cdot \sup_{t \in \sigma(A)} \left| 1 - \frac{\mu}{t} \right| . \]

Proof. It follows by the definition of \((\cdot)_{\lambda_i} (\mu)\) that

\[ (20) \| g_i - \omega_i^{(\lambda)} \| = \min_{\mu \in R_{\lambda_i}} \| g_i - (A - \mu I) \mu \| . \]

Since \(0 \notin \sigma(A)\) and \((\mu) \notin \sigma(A)\), there exist \(A^{-1}\) and \((A - \mu I)^{-1}\). Then

\[ (21) \| g_i - (A - \mu I) \mu \| = IB[(A - \mu I)^{-1} g_i - A \mu] \| , \mu \in R_{\lambda_i} , \]

where \(B = (A - \mu I) A^{-1}\) and \(I\) is the identity operator.

Letting \(\mu = A^{-1} \omega_i (\cdot)_{\lambda_i} (\mu) \cdot \frac{\lambda_i}{\lambda_i - \mu} \), it follows from

(20) and (21) that

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(22) \[ \| \varphi_i - \langle m \rangle \varphi_i \| \leq \frac{\lambda_i}{\lambda_i - \mu} \cdot \| B (\varphi_i - \langle m \rangle \varphi_i) \| . \]

Then, since \( A \) is a DS-operator, we have

(23) \[ \| B \| \leq \sup_{t \in \sigma(A)} \left| 1 - \frac{\mu}{t} \right| . \]

(24) \[ \| B v \| \geq \| v \| \cdot \inf_{t \in \sigma(A)} \left| 1 - \frac{\mu}{t} \right| \text{ for any } v \in \mathcal{R}(A) . \]

Thus, by (23) and (22)

\[ \| \varphi_i - \langle m \rangle \varphi_i \| \leq D_i \cdot \| \varphi_i - \langle m \rangle \varphi_i \|. \]

It is readily verified that

(25) \[ \| \varphi_i - \langle m \rangle \varphi_i \| = \min_{\mu \in \mathcal{R}_m} \| B [\varphi_i - A \mu] \| \cdot \left| \frac{\lambda_i}{\lambda_i - \mu} \right| . \]

It follows now from (24) and (25) that

\[ \| \varphi_i - \langle m \rangle \varphi_i \| \geq D_i \cdot \| \varphi_i - \langle m \rangle \varphi_i \|. \]

**Remark 6.** In the case of multiple eigenvalue Theorem 4 is valid, if \( \mu_m \) satisfies \( 3^\circ \) \( (\mu_m, \mu_{m+1}) \geq \varepsilon > 0 \) in place of 3).

**References**


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