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ON NECESSARY CONDITIONS OF OPTIMALITY IN LINEAR SPACES

Milan VLACH, Delft

1. Introduction. The approach of the present communication to necessary conditions of optimality is based on the fact that the optimality of an element can be expressed by stating that certain suitable sets have an empty intersection. Therefore, the following scheme is adopted.

Let \( G \) be a set, let \( \omega \) be a subset of \( G \) and let \( R \) be a reflexive and transitive binary relation on \( G \). An element \( x \) of \( G \) will be called optimal with respect to \( \omega \) and \( R \) - or more briefly \((\omega, R)\)-optimal - if

(a) \( x \in \omega \),

(b) \( y \in \omega \) and \( yRx \Rightarrow xRy \).

This scheme is clearly general enough to include both the problems of constrained optimization under scalar-valued criteria and the problems of constrained optimization under vector-valued criteria.

For \( x \in G \) let \( G^x_\omega \) mean \( \{ y \in G : yRx \} \) and \( \neg xRy \). We notice that

(a) if \( G^x_\omega \cap \omega \neq \emptyset \) then \( x \) is not \((\omega, R)\)-opti-
(b) if \( G^R_x \cap \omega = \emptyset \) and if \( x \in \omega \) then \( x \) is \((\omega, R)\)-optimal.

If we are interested only in necessary conditions of the optimality of a point \( x \) of \( \omega \) then it is sufficient to have at our disposal an "approximation" \( A(x, \omega) \) of the set \( \omega \) at the point \( x \) and an "approximation" \( B(x, G^R_x) \) of the set \( G^R_x \) at the point \( x \) possessing the property

\[
\omega \cap G^R_x = \emptyset \implies A(x, \omega) \cap B(x, G^R_x) = \emptyset.
\]

If we are interested also in sufficient conditions of the optimality, then, of course, also the converse implication is important. Without additional mathematical structures we can hardly construct suitable "approximations" different from the trivial ones, as for example \( A(x, \omega) = \omega \) or perhaps \( A(x, \omega) \subset \omega \).

In what follows we are going to present a relatively simple realization of the idea in real linear spaces. Nevertheless, a number of the known necessary conditions of the optimality are consequences of theorems, obtained by this approach. A rederivation of both classical and more recent necessary conditions will be the subject of another paper. Here, only simple illustrations are included for the reader's convenience. Moreover, the reader will find in [1] a suitable background for the present communication.

Finally, for the reader's convenience and to avoid misunderstandings, we recall some definitions and re-
suits from the theory of linear spaces. Let \( L \) be a
real linear space. A surrounded point of the set
\( A \subset L \) is a point \( x \in A \) possessing the property
that whatever the element \( u \in L \), some segment \([x, x + \alpha u] \)
where \( \alpha > 0 \), is contained in \( A \). Here
the segment \([x, x + \alpha u] \) is the set of all points
\( y \in L \) which are representable in the form
\( y = (1 - \lambda)x + \lambda(x + \alpha u) \), where \( 0 \leq \lambda \leq 1 \). A cone
in \( L \) is any set \( A \subset L \) satisfying the condition
\[ x \in A, \: \alpha > 0 \implies \alpha x \in A. \]
A linear functional is an additive and homogeneous
functional. The set of all linear functionals on \( L \) will
be denoted by \( L^* \). If \( L \) is a real linear topological
space then the set of all continuous linear functionals
on \( L \) will be denoted by \( L^* \).

**Lemma 1** ([2], Chapter 3, Section 3.2, Theorem 1).

Let \( M, N \) be convex subsets of a real linear space
\( L \) and assume that \( N \) has a surrounded point but no
point of \( M \) is a surrounded point of \( N \). Then \( M, N \)
can be separated, that is, there is a nontrivial linear
functional \( f \) and a real number \( \lambda \) such that
\[ x \in M \implies f(x) \geq \lambda, \]
\[ x \in N \implies f(x) \leq \lambda. \]

**Lemma 2** ([3], Chapter I, Paragraph 4). Let \( L \) be
a real linear topological space. If \( f \in L^* \) and if
there is a non-empty open subset \( U \) of \( L \) and a real
number $t$ such that $f(x) = t$ for all $x \in U$, then $f \in L^*$

2. **Conical approximations.** Let $L$ be a real linear space, let $Q$ be a subset of $L$ and let $x$ be a point of $L$. We define approximations $C(x, Q)$ and $K(x, Q)$ of the set $Q$ at the point $x$ as follows:

$m \in C(x, Q) \iff m \in L$ and $\forall \exists \varepsilon > 0 \in (0, \varepsilon): x + \alpha m \in Q$,

$m \in K(x, Q) \iff m \in L$ and $\exists \forall \varepsilon > 0 \in (0, \varepsilon): x + \alpha m \in Q$.

We notice the following properties of $C(x, Q)$ and $K(x, Q)$:

(a) $K(x, Q) \subseteq C(x, Q)$,

(b) $C(x, Q)$ and $K(x, Q)$ are cones in $L$,

(c) $K(x, Q_1 \cap Q_2) = K(x, Q_1) \cap K(x, Q_2)$,

(d) $C(x, Q_1 \cap Q_2) \supseteq K(x, Q_1) \cap C(x, Q_2)$.

**Theorem 1.** Let $G$ be a real linear space. Then $\omega \cap G_x^R = \emptyset \implies [C(x, \omega) \cap K(x, G_x^R)] \cup [K(x, \omega) \cap C(x, G_x^R)] = \emptyset$.

**Proof.** If $m \in C(x, \omega) \cap K(x, G_x^R)$ then

$\exists \forall \exists \varepsilon > 0 \in (0, \varepsilon): x + \alpha m \in G_x^R$. At the same time there is $\alpha' \in (0, \varepsilon)$ such that $x + \alpha' m \in \omega$. Thus $\omega = x + \alpha' m$ belongs to $\omega \cap G_x^R$. If $m \in K(x, \omega) \cap$
\[ \text{Illustration. Let } xR^2 \text{ mean } f(x) \neq f(y), \text{ where } f \text{ is a functional defined on a real linear space } G \text{ and let } \omega \in G. \]

(a) If there is an element \( u \in K(x, \omega) \) such that the limit

\[ f'(x, u) = \lim_{\alpha \to 0^+} \frac{f(x + \alpha u) - f(x)}{\alpha} \]

exists and is negative, then \( x \) is not \((\omega, R)\)-optimal (since there is \( \varepsilon > 0 \), such that \( f(x + \alpha u) < f(x) \) for all \( \alpha \in (0, \varepsilon) \), so that \( u \) belongs also to

\[ K(x, G^R_x) \subset C(x, G^R_x). \]

(b) If \( f'(x, u) \) is defined \( f'(x, \cdot) \) need not be linear) for all \( u \in K(x, \omega) \) and if

\[ \inf_{u \in K(x, \omega)} f'(x, u) < 0 \]

then \( x \) is not \((\omega, R)\)-optimal

\( (x \in \omega \Rightarrow \theta \in K(x, \omega); f'(x, \theta) = 0) \).

(c) (See [4], p. 77) Let the functional \( f \) be defined on a region \( \omega \) of a real Banach space \( G \) and let \( x_0 \) be an interior point of \( \omega \), at which \( f \) has a linear Gâteaux differential. Then in order that the point \( x_0 \) be \((\omega, R)\)-optimal, it is necessary that \( \text{grad} f(x_0) = \theta \)

\( (\text{grad} f(x_0) = \theta \Rightarrow \text{grad} f(x_0)(u) < 0 \text{ for some } u \in G; K(x_0, \omega) = G) \).

The approximations \( K(x, G) \) and \( C(x, G) \) are
insufficient in some important cases. However, more accurate approximations require beside the linearity also a convenient topology, as a rule. Thus, it is natural to consider now real linear topological spaces. For the sake of simplicity we consider first the case of real normed spaces.

Let $L$ be a real normed space, let $Q$ be a subset of $L$ and let $x$ be a point of $L$. We define approximations $M(x, Q)$ and $N(x, Q)$ of the set $Q$ at the point $x$ as follows:

\[ u \in M(x, Q) \iff u \in L \quad \text{and} \quad \forall \varepsilon > 0 \exists v \in \mathcal{O}(u, \varepsilon) \quad : \quad x + \alpha v \in Q, \]

\[ u \in N(x, Q) \iff u \in L \quad \text{and} \quad \exists \forall \varepsilon > 0 \exists v \in \mathcal{O}(u, \varepsilon) \quad : \quad x + \alpha v \in Q. \]

Here $\mathcal{O}(u, \varepsilon)$ denotes the set $\{v \in L : |v - u| < \varepsilon\}$.

Let us again notice a few useful properties of $M(x, Q)$ and $N(x, Q)$:

(a) $N(x, Q) \subseteq \mathcal{K}(x, Q)$, $\overline{C(x, Q)} \subseteq M(x, Q)$;

(b) $M(x, Q)$ is a closed cone in $L$;

(c) $N(x, Q)$ is an open cone in $L$;

(c') $N(x, Q_1 \cap Q_2) = N(x, Q_1) \cap N(x, Q_2)$;

(d) $M(x, Q_1 \cap Q_2) \supseteq N(x, Q_1) \cap M(x, Q_2)$.

Theorem 2. Let $G$ be a real normed space. Then

\[ \omega \cap C^R_x = \emptyset \Rightarrow [M(x, \omega) \cap N(x, G^R_x)] U [N(x, \omega) \cap M(x, G^R_x)] = \emptyset. \]
Proof. If \( \mu \in M(x, \omega) \cap N(x, G_x^R) \) then there is \( \varepsilon > 0 \) such that

\[ \alpha \in (0, \varepsilon), \quad |\nu - \mu| < \varepsilon \quad \Rightarrow \quad x + \alpha \nu \in G_x^R. \]

Since \( \mu \in M(x, \omega) \) for every \( \varepsilon > 0 \) there are \( \beta_\varepsilon \in \varepsilon (0, \varepsilon) \) and \( \omega_\varepsilon \in 0(\mu, \varepsilon) \) such that \( x + \beta_\varepsilon \omega_\varepsilon \in \omega \). Consequently, \( x + \beta_\varepsilon \omega_\varepsilon \in \omega \cap G_x^R \). If \( \mu \in N(x, \omega) \cap M(x, G_x^R) \) then the similar argumentation completes the proof.

Remark. It is not difficult to verify that

\[ \omega \cap G_x^R = \emptyset \quad \Rightarrow \quad K(x, \omega) \cap K(x, G_x^R) = \emptyset. \]

It is obvious that

\[ \omega \cap G_x^R = \emptyset \quad \Rightarrow \quad M(x, \omega) \cap N(x, G_x^R) = \emptyset. \]

However, there are \( G, \omega, R \) and \( x \) such that

\[ \omega \cap G_x^R = \emptyset \quad \text{and} \quad M(x, \omega) \cap K(x, G_x^R) = \emptyset. \]

Illustration. Let \( xR_f \) again mean \( f(x) \neq f(y) \), where \( f \) is a functional on \( G \). If at a point \( x \in \omega \), \( f'(x, \mu) \) exists for all \( \mu \) and if \( f'(x, \cdot) \) is not only continuous but the functions \( o_{x, \mu} (\alpha) \), where

\[ f(x + \alpha \mu) = f(x) + \alpha f'(x, \mu) + o_{x, \mu} (\alpha), \]

\[ \lim_{\alpha \to 0^+} \frac{o_{x, \mu} (\alpha)}{\alpha} = 0, \]

satisfy also the condition

\[ \forall \varepsilon > 0 \exists \varepsilon (0, \varepsilon) : \left| \frac{o_{x, \mu} (\alpha)}{\alpha} \right| < \varepsilon \]
then

$$\inf_{\mu \in M(x, \omega)} f'(x, \mu) \neq 0 \Rightarrow \omega \text{ is not } (\omega, R)-\text{optimal.}$$

For other examples in real normed spaces we refer to [11], [5] and [6].

3. Convexity. Since the necessary conditions of the optimality assert that an intersection of some sets is empty, it is possible, in the case that the sets $\omega$ and $G^\omega$ or their suitable approximations are convex, to use various separation theorems. With regard to the properties (c) and (d) of the approximations $C(x, Q)$, $K(x, Q)$, $M(x, Q)$ and $N(x, Q)$ and with regard to the fact that the set $\omega$ is often given as an intersection of some sets (for example, as the set of all solutions to a given system of equations and/or inequalities), it is desirable to develop separation theorems for the case of finite families of convex sets. Some theorems of this sort have been stated in [1] and [7]. These results are straightforward consequences of the following theorems.

**Theorem 3.** Let $A_o$ be a non-empty convex set of a real linear space $L$. Let $A_1, A_2, \ldots, A_m$ be convex sets of $L$ such that every $A_i$ ($i = 1, 2, \ldots, m$) has a surrounded point. If $\bigcap_{i=1}^{m} A_i = \emptyset$ then there are linear functionals $f_1, f_2, \ldots, f_m$ and real numbers $\lambda_1, \lambda_2, \ldots, \lambda_m$ such that

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(1) \( \sum_{i=0}^{n} f_i = \theta \) and not all \( f_i \) are trivial,

(2) \( \sum_{i=0}^{n} \lambda_i \leq 0 \),

(3) \( x \in A_i \implies f_i(x) \leq \lambda_i \) for \( i = 0, 1, \ldots, m \).

Proof. Let us define

\[ N = A_1 \times A_2 \times \cdots \times A_n, \]
\[ M = \{(x, x, \ldots, x) : x \in A_0\} . \]

Since \( M, N \) are disjoint non-empty convex subsets of \( L \times L \times \cdots \times L \) and since \( N \) has a surrounded point, it follows from Lemma 1 that there is \( f \in (L \times L \times \cdots \times L)^* \) and a real number \( \lambda \) such that \( f \neq \theta \) and

(a) \( x \in N \implies f(x) \leq \lambda \),

(b) \( x \in M \implies f(x) \geq \lambda \).

Since \( (L \times L \times \cdots \times L)^* \) is isomorphic to \( L^* \times L^* \times \cdots \times L^* \), there are \( f_1, f_2, \ldots, f_m \in L^* \) such that

\( f = f_1 + f_2 + \cdots + f_m \). Denoting \( \lambda_i = \sup_{x \in A_i} f_i(x) \) \( (i = 1, 2, \ldots, m) \) we obtain \( \sum_{i=1}^{m} \lambda_i \leq \lambda \) and \( x \in A_i \implies f_i(x) \leq \lambda_i \). We complete the proof by defining \( f_0 = -f, \lambda_0 = -\lambda \).

**Theorem 4.** Let \( A_0 \) be a non-empty convex set of a real linear space \( L \). Let \( A_1, A_2, \ldots, A_m \) be non-empty algebraically open convex sets of \( L \). If there are \( f_0, f_1, \ldots, f_m \in L^* \) and real numbers
\[ \lambda_0, \lambda_1, \ldots, \lambda_m \] such that (1), (2) and (3) from Theorem 3 are valid, then \( \bigcap_{i=0}^{m} A_i = \emptyset \).

**Proof.** If \( x \in \bigcap_{i=0}^{m} A_i \) then \( 0 = \sum_{i=0}^{m} \lambda_i \) and thus \( \lambda_i x = \lambda_i \) for \( i = 0, 1, \ldots, m \). However, this is impossible, since there is \( \lambda_{i_0} \in \{1, 2, \ldots, m\} \) such that \( f_{i_0} \neq \emptyset \) and since \( A_{i_0} \) is a non-empty algebraically open set.

**Theorem 3.** Let \( A_0 \) be a non-empty convex set of a real linear topological space \( L \). Let \( A_1, A_2, \ldots, A_m \) be convex sets of \( L \) such that every \( A_i \) \( (i = 1, 2, \ldots, m) \) has an inner point. If \( \bigcap_{i=0}^{m} A_i = \emptyset \) then there are \( f_0, f_1, \ldots, f_m \in L^* \) and real numbers \( \lambda_0, \lambda_1, \ldots, \lambda_m \) such that (1), (2) and (3) from Theorem 3 are valid.

**Proof.** Every inner point of \( A_i \) is a surrounded point of \( A_i \). Hence there are \( f_0, f_1, \ldots, f_m \in L^* \) and \( \lambda_0, \lambda_1, \ldots, \lambda_m \) satisfying (1), (2) and (3). Since

\[ f_i(x) = \lambda_i \quad \text{for} \quad x \in A_i \quad (i = 1, 2, \ldots, m) \]

there are non-empty open sets \( U_i \) and real numbers \( t_i \) such that

\[ \forall x \in U_i \Rightarrow f_i(x) = t_i, \quad i = 1, 2, \ldots, m. \]

Consequently, by Lemma 2, \( f_i \) is continuous for \( i = 1, 2, \ldots, m \). The functional \( f_0 \) is also continuous, since

\[ f_0 = -\sum_{i=1}^{m} f_i. \]

**Theorem 4.** Let \( A_0 \) be a non-empty convex set of a real linear topological space \( L \). Let \( A_1, A_2, \ldots, A_m \) be non-empty open convex sets of \( L \). If there are such
and such real numbers $\lambda_0, \lambda_1, \ldots, \lambda_m$
that (1), (2) and (3) from Theorem 3 are valid, then

$$\bigcap_{i=0}^{m} A_i = \emptyset.$$  

**Proof.** The sets $A_1, A_2, \ldots, A_m$ are algebraically open.

**Remark.** We have already mentioned that the corresponding assertions from [1] and [7] easily follow from the theorems of this section. During preparation of this text some other results of this sort were published ([81]). Also these assertions are straightforward consequences of the theorems presented here.

**Corollary 1** [8]. Let $A_0$ be a non-empty convex cone of a real linear space $L$. Let $A_1, A_2, \ldots, A_m$ be non-empty algebraically open convex cones in $L$. The intersection

$$\bigcap_{i=0}^{m} A_i$$  

is empty if and only if there are additive functionals $f_0, f_1, \ldots, f_m$ (not all trivial) such that

$$\sum_{i=0}^{m} f_i = \Theta \quad \text{and} \quad x \in A_i \Rightarrow f_i(x) \neq 0$$  

for $i = 0, 1, \ldots, m$.

**Proof.** We use Theorem 3 and notice that $\Theta$ is not a surrounded point of $L \setminus A_i$. Hence every $\lambda_i$ is non-negative and thus $\lambda_i = 0$ for all $i$.

**Corollary 2** [8]. Let $A_0$ be a non-empty convex cone of a real linear topological space $L$. Let $A_1, A_2, \ldots, A_m$ be non-empty open convex cones in $L$. The intersection

$$\bigcap_{i=0}^{m} A_i$$  

is empty if and only if there are continuous additive functionals $f_0, f_1, \ldots, f_m$ (not all...
trivial) such that \( \sum_{i=0}^{n} f_i = \Theta \) and \( x \in A_i \Rightarrow \)
\[ \Rightarrow f_i(x) \leq 0. \]

References


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