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Commentationes Mathematicae Universitatis Carolinae, Vol. 11 (1970), No. 3, 559--582

Persistent URL: http://dml.cz/dmlcz/105298

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ON ONE-PARAMETER FAMILIES OF DIFFEOMORPHISMS

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This paper is concerned with diffeomorphisms of manifolds, depending on a parameter. This means that we shall consider mappings \( f : P \times M \to M \), where \( P \) is a 1-dimensional \( C^\infty \) manifold, \( M \) is an \( m \)-dimensional \( C^\infty \) manifold, \( f \) is \( C^\infty \) and such that for every \( \mu \in P \), the mapping \( f_\mu : M \to M \) given by \( f_\mu(m) = f(\mu, m) \) is a diffeomorphism. Given \( P \), \( M \), we denote by \( \mathcal{F} \) the set of all mappings \( f \) with the above properties, endowed with the \( C^\infty \) Whitney topology. We shall be interested in the generic behavior of the periodic points of \( f_\mu \) (i.e. fixed points of \( f_\mu \) and its iterates) if \( \mu \) is varied.

We say that a property is generic in \( \mathcal{F} \) if it is valid for every \( f \) from a residual subset of \( \mathcal{F} \).

The first part of our results (§ 1) concerns the case of arbitrary \( n \), the second (§ 2) takes place for \( n = 2 \).

The problems studied in this paper are to a great extent motivated by differential equations, where problems of dependence of critical points and periodic trajecto-

This research was partly done under the support of NASA (NGR 24-005-063) during the author's stay at the University of Minnesota.

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ries on a parameter are frequent.

The present research has been stimulated by the work of K.R. Meyer [1] on two dimensional symplectic diffeomorphisms, to whom the author is indebted for valuable discussions. Similar problems have been studied by J. Sotomayor [2] whose work deals with two-dimensional flows. His setting of the problem and results are of a somewhat different character.

§ 1

Denote by \( Z_k = Z_k(f) \subset P \times M \) the set of all \( k \)-periodic points of \( f \), i.e. \( Z_k = \{(p, m)| f^k(m) = m, f^j(m) \neq m \text{ for } 0 < j < k \} \). In this section, we shall study the sets \( Z_k \). \( \kappa \) will be called the prime period of a point \((p, m) \in Z_k \).

A closed subset \( Q \) of \( P \times M \) will be called invariant, if \( \{(p, f(p, m))|(p, m) \in Q\} \subset Q \) and \( \{(p, f^{-1}(m))|(p, m) \in Q\} \subset Q \). By the orbit of a point \((p, m) \) we shall understand the set of all points \((p, f_k^j(m)), \kappa \) integer.

Lemma 1. For every \( f \) from a certain open and dense subset \( \mathcal{F}' \) of \( \mathcal{F}, Z_1 \) is a closed one-dimensional submanifold of \( P \times M \).

Proof. It is obvious that \( Z_1 \) is closed. Associate with every \( f \in \mathcal{F} \) a mapping \( F: P \times M \to P \times M \) given by \( F(p, m) = (m, f(p, m)) \). Then, \( Z_1 = F^{-1}(\Delta) \) where \( \Delta \) is the diagonal in \( M \times M \) and by the transversa-
lity theorems 18.2, 19.1 of [3], the set of \( f \)'s for which \( F \) meets \( \Delta \) transversally, is open and dense in \( \mathcal{F} \). The statement of the lemma follows by the implicit function theorem.

Denote by \( X_1 \) the set of those points \((p, m)\) in \( Z_1 \) for which \( df_{\nu}(m) - \text{id} \) (or, \( dF(p, m) \)) is singular (i.e. at least one eigenvalue of \( df_{\nu}(m) \) is equal 1). Further, denote by \( J = \delta_{\nu} \times \delta_{\mathcal{M}} \) the imbedding of \( Z_1 \) into \( P \times M \). From the implicit function theorem it follows that \( X_1 \) is exactly the set of those points \( x \in Z_1 \) for which \( T_{\delta_{\nu}}(x) \) meets the submanifold \((TP)_0\) of those points from \( TP \) satisfying \( df_{\nu} = 0 \).

**Lemma 2.** For every \( f \) from an open and dense subset \( \mathbb{F}_1'' \) of \( \mathbb{F}_1'' \), \( T_{\delta_{\nu}}(x) \) meets \((TP)_0\) transversally.

**Corollary 1.** For \( f \in \mathbb{F}_1'' \), if \((p, m) \in X_1 \), then there is a coordinate neighbourhood \((W, \xi \times \xi), \xi = U \times V, (p, m) \) such that \( \mu \times \xi (p, m) = (0, 0), Z_1 \cap W \) can be parameterized by \( x_1 \), i.e. \( \xi \times \xi (Z_1 \cap W) = \{(\xi, \xi)|\xi = \phi_\iota(x_1), x_1 = \phi_\iota(x_1), 2 \leq i \leq m, x_1 \in J \} \) where \( \phi \) is \( C^\iota \), \( 0 \in J \), \( J \) is an interval, and \( \frac{d^2 \phi_\iota}{dx_1^2}(0) > 0 \). (The last inequality is the coordinate representation of the transversality condition of Lemma 2.)

Based upon this corollary, we shall call the points of \( X_1 \) collapse (fixed) points. Namely, there are exactly two points in \( Z_1 \cap W \) with fixed \( \mu > 0 \) small enough; these points collapse at \( \mu = 0 \) and disappear for \( \mu < 0 \).
Corollary 2. For every $f \in \mathcal{F}'$, the fixed points of $f$ are isolated for every $\mu \in P$.

Corollary 3. For $f \in \mathcal{F}'$, $\chi_q$ is discrete.

Proof of Lemma 2. Openness. Assume $f \in \mathcal{F}'$. We cover $\mathcal{Z}_q$ by a countable number of coordinate neighbourhoods $(U_{\alpha} \times V_{\alpha}, \mu_{\alpha} \times \chi_{\alpha})$. Using the implicit function formula for second derivatives, we can express the transversality condition of Lemma 2 by inequalities

$$\pi_{\alpha} \neq 0,$$

where $\pi_{\alpha}$ are polynomials in $(\mu_{\alpha} \times \chi_{\alpha}) \circ f \circ (\mu_{\alpha} \times \chi_{\alpha})^{-1}$ and its first and second derivatives. Restricting suitably the coordinate neighbourhoods, we can assume that $|\pi_{\alpha}|$ are bounded away from zero by positive constants $\varepsilon_{\alpha}$. If $\mathcal{F}$ is close enough to $f$ (in the $C^\infty$ Whitney topology), $\mathcal{Z}_q(\mathcal{F})$ will be contained in $U_{\alpha} \times V_{\alpha}$ and $\pi_{\alpha}(\mathcal{F})$ will be non zero on $U_{\alpha} \times V_{\alpha}$. Consequently, $\mathcal{Z}_q(\mathcal{F})$ will satisfy the transversality condition.

For the proof of density, we first prove the following lemma:

Lemma 3. Denote $B_q(\varepsilon) = \{x \in \mathbb{R}^2 \mid |x| < \varepsilon\}$, $|\cdot|$ being the Euclidean norm. Let $f \in \mathcal{F}'$ and let $(W_i, \mu \times \chi)$, $W = U \times V$ be a coordinate neighbourhood in $P \times M$ such that $\mu(U) = B_q(1)$, $\chi(V) = B_n(1)$ and $W \cap \mathcal{Z}_q$ is connected. Denote $W_i = U_i \times V_i = (\mu \times \chi^{-1})(B_q(i/3) \times B_n(i/3))$, $i = 1, 2$. Then, in any neighbourhood $\mathcal{Q}$ of $f$ in $\mathcal{F}'$, there is an $\mathcal{F}$ which coincides with $f$ outside $W$ and such that $T_p(\mathcal{Z}_q(\mathcal{F}) \cap W_i)$ meets $(TP)_p$ transversally, $T_p(\cdot)$ being the projection of $T(\cdot)$ into $TP$. 

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Proof. Denote by $\mathcal{G}$ the set of all $C^\infty$ maps of $Z_1 \cap W$ into $U$, $\hat{\mathcal{G}} = \{T_\varphi | \varphi \in \mathcal{G}\}$. We consider $\hat{\mathcal{G}}$ as a submanifold of the Banach manifold $\mathcal{G}$ of all $C^\infty$ maps $T(Z_1 \cap W) \to TP$. By Theorem 19.1 of [3], there is a $\gamma \in \hat{\mathcal{G}}$, arbitrary $C^\infty$-close to $\hat{\gamma}_p$ such that $T\gamma$ meets $(TP)_p$ transversally. In particular, $\gamma$ can be chosen so that $|\mu \circ \gamma - \mu \circ \hat{\gamma}_p| \leq 1/4$. Let $\varphi$ be a $C^\infty$ bump function such that $\varphi = 1$ on $W_1$, $\varphi = 0$ on $W \setminus W_2$. Define $\varphi(x) = \mu^{-1}(\mu \circ \hat{\gamma}_p + \varphi(\gamma \circ \hat{\gamma}_p))$. Then, $\varphi$ meets $(TP)_p$ transversally in $W_1$ and coincides with $\hat{\gamma}_p$ outside $W_2$.

Since $W$ is isomorphic with a subset of $\mathbb{R}^{m+1}$ and $(\mu \times x)(Z_1 \cap W)$ is a $C^\infty$ curve in $\mathbb{R}^{m+1}$, there is a $C^\infty$ tubular neighbourhood of $Z_1 \cap W$, $\mathcal{H}: Z_1 \cap W \times B_m(1) \to W$ such that $\mathcal{H}(x, 0) = \hat{\gamma}(x)$ (for the concept of tubular neighbourhood cf. [4]). This tubular neighbourhood can be constructed e.g. so that $(\mu \times x) \circ \mathcal{H}(x, B_m(1))$ lies in the $n$-hyperplane passing through $(\mu \times x)(x)$ and orthogonal to the tangent to $(\mu \times x)(Z \cap W)$ at $(0, 0)$.

Denote $\mathcal{H}_1$, $\mathcal{H}_2$ the natural projections of $Z_1 \cap W \times B_m(1)$ into $Z_1 \cap W$ and $B_m(1)$ respectively, $\psi: \mathbb{R}^{m+1} \to \mathbb{R}$ a $C^\infty$ bump function such that $\psi = 1$ on $B_m(1/2)$ and $\psi = 0$ outside $B_m(1)$. We define

$$\tilde{f}(\mu, m) = f(\mu^{-1}(\mu \circ \mathcal{H}_1 \mathcal{H}_2^{-1}(\mu, m)) + \psi \mathcal{H}_2^{-1}(\mu, m) \cdot \mu \circ \mathcal{H}_1^{-1}(\mu, m); m)$$

for $(\mu, m) \in \mathcal{H}[(Z_1 \cap W) \times B_m(1)]$. 

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\( \tilde{f}(\mu, m) = f(\mu, m) \) elsewhere.

Then, \( Z^C(\tilde{f}) \cap W = (\varphi \times \tilde{\varphi}M)Z^C(\tilde{f}) \cap W \), \( \tilde{f} \) coincides with \( f \) outside \( U \) and \( \tilde{f} \) can be made arbitrary close to \( f \) by choosing \( \varphi \) sufficiently close to \( \tilde{\varphi} \). This proves the lemma.

To prove the density part of Lemma 2, we find a countable family of coordinate neighbourhoods

\( (W_\alpha, \omega_\alpha \times \tilde{\omega}_\alpha) \) in such a way that every \( (W_\alpha, \omega_\alpha \times \tilde{\omega}_\alpha) \) satisfies the assumptions of Lemma 3 and \( Z^C(\tilde{f}) \subset \bigcup \alpha W^C_\alpha \) (the subscript 1 used as in Lemma 3). Then, we apply Lemma 3 stepwise for every \( \alpha \) and choose the approximation of \( \tilde{f} \) at every step so close that the transversality condition is not destroyed in \( \bigcup \alpha W^C_{\alpha,1} \). This is possible due to the first part of the proof.

The next lemma examines the behaviour of \( f \) in the neighbourhood of a collapse point.

**Lemma 4.** For every \( f \) from an open and dense subset \( \mathcal{F}'' \) of \( \mathcal{F}'' \), the following is true:

(a) for every \( (\mu_0, m_0) \in \mathcal{X} \), one eigenvalue of \( df_{\mu_0}(m_0) \) is 1, the moduli of the others being different from 1,

(b) locally, \( (\mu_0, m_0) \) divides \( \mathcal{E} \setminus \{(\mu_0, m_0)\} \) into two components and the number of eigenvalues of \( df_{\mu} \) with modulus 1 at points from different components of \( \mathcal{E} \setminus \{(\mu_0, m_0)\} \) differs by 1.

(c) There is a neighbourhood \( W \) of \( (\mu_0, m_0) \) such that
contains no invariant set.

Proof. Since \((p_0, m_0) \in X_4\) and \(f \in F''\), \(df^{p_0}(m_0)\) has 1 as an eigenvalue. This eigenvalue is simple because of Lemma 1.

If \((p_0, m_0) \in X_4\) and \(f \in F''\), then there is a coordinate neighbourhood \((W, \mu \times x)\) of \((p_0, m_0)\), \(W = U \times V\) such that \((\mu \times x)(p_0, m_0) = (0, 0)\) and \(f\) can be in these coordinates represented by

1. \(x_1' = x_1 + \alpha \mu + \beta x_1^2 + \omega (\mu, x_1, y)\),
2. \(y' = A y + \chi (\mu, x_1, y)\)

where \(y = (x_2, \ldots, x_n)\), the primed coordinates are those of the images, \(\alpha < 0\),

3. \(\chi(0, 0, 0) = 0, \omega (\mu, x, 0) = \sigma (|\mu| + x^2)\).

Note that from the form of (2) it follows that every fixed point in \(W\) satisfies \(y = 0\) (\(W\) possibly restricted).

We denote by \(F''''\) the set of all \(f \in F''\), in the representation (1), (2) of which (i) \(\beta \neq 0\) and (ii) the eigenvalues of \(A\) have moduli \(\neq 1\). It is obvious that the meaning of these conditions is independent of the choice of coordinates. Also, (ii) is equivalent with (a). We show that \(F''''\) is open dense.

Openness follows easily from the continuous dependence of the eigenvalues on \(f\). To prove density, we note that there is a real \(\sigma\) arbitrarily small in abso-
lute value such that $\beta + \sigma \neq 0$ and for any eigenvalue $\lambda$ of $d\varphi_{m_0}^n (m_0)$, $|\lambda + \sigma| \neq 1$. We change $f$ into $\mathcal{F}$ by changing the terms $A y$ and $\beta x_1^2$ in the representation (1), (2) of $f$ into $(A + \psi (\mu, x) \sigma E) y$ and $(\beta + \psi (\mu, x) \sigma) x_1^2$ ($E$ being the unity matrix) respectively, where $\psi (\mu, x)$ is a $C^\infty$ bump function vanishing outside $\mathcal{W}$, and equal 1 at $(0,0)$. By the choice of a sufficiently small $\sigma$, $\mathcal{F}$ can be made sufficiently close to $f$. $d\varphi_{m} (m)$ will then satisfy (a) and we do not introduce any new fixed points. Since $X_1$ is discrete for $f \in \mathcal{F}''$, this proves the density of $\mathcal{F}''$.

To prove (b) we note that if $f$ satisfies (a), only one eigenvalue can cross the unit circle at $(\mu, m_0)$ and this eigenvalue is the eigenvalue of the restriction of $d\varphi_{m}$ to the manifold $y = 0$, $d\varphi_{m} | y = 0$. This mapping is represented by (1) with $y = 0$.

Assume $\beta > 0$ (in the other case we change the sign of $x_1$). To prove (c), we note first that $A$ is similar to a matrix $(B \, 0 \, 0 \, C)$, i.e. there is a nonsingular matrix $Q$ such that $Q^{-1} A Q = (B \, 0 \, 0 \, C)$, where the moduli of eigenvalues of $B$ and $C$ are $<1$ and $>1$ respectively. Applying first the linear coordinate transformation $y = Q (\frac{A}{B})$ and then $x = w^{*+} (x_1, \mu) + \xi$

$\mu = w^{*-} (x_1, \xi) + \eta$ where $x = w^{*+} (x_1, \mu)$ and $\mu = w^{*-} (x_1, \xi)$ ($w^{*+}$, $w^{*-}$ being $C^\infty$) are the equations of the center-stable and center-unstable mani-
folds respectively (cf. [3], Appendix C (1), (1) and (2) is transformed into

(4) $\xi' = \xi + \alpha \mu + \beta \xi^2 + \Xi(\mu, \xi, \eta, \zeta)$,

(5) $\eta' = B \eta + \Theta(\mu, \xi, \eta, \zeta)$,

(6) $\zeta' = C \zeta + \Omega(\mu, \xi, \eta, \zeta)$

where $\alpha < 0$, $\Xi$, $\Theta$, $\Omega$ are $C^\infty$ and

(7) $\Theta(\mu, \xi, 0, \zeta) = 0$, $\Omega(\mu, \xi, 0, \eta) = 0$,

$\Xi(\mu, \xi, \eta, \zeta) = \sigma(|\mu| + \xi^2)$, $d \Xi(0, 0, 0, 0) = 0$, $d \Theta(0, 0, 0, 0) = 0$, $d \Omega(0, 0, 0, 0) = 0$.

From (5) and (7) it follows that the orbit of every point $(\mu, m)$ which is contained entirely in some sufficiently small neighbourhood of $(\mu_0, m_0)$ satisfies $\eta(f^k_\mu (m)) \to 0$ for $k \to \infty$ and $\zeta(f^k_\mu (m)) \to 0$ for $k \to -\infty$. Thus, if there is an invariant set contained in this neighbourhood, it must be a part of the manifold $\eta = 0$, $\zeta = 0$. In particular, this implies

(8) $\eta(Z_1 \cap W) = 0$, $\zeta(Z_1 \cap W) = 0$

($W$ possibly restricted).

(1) Actually, Appendix C in [3] deals with flows rather than mappings. Therefore, in order to use its results directly, we have to construct a flow from $f$ as in [5] and then return to $f$ by considering the cross-section mapping.
We therefore consider the restriction of $f$ to the center manifold $\eta = 0$, $\xi = 0$, the representation of which is given by

$$
\xi' = \xi + \alpha \mu + \beta \xi^2 + \epsilon (\mu, \xi, 0, 0).
$$

It follows from Corollary 1 and (8) that for $\mu > 0$ fixed, $Z_1 \cap W$ consists of two points

$$(\mu, \xi_1(\mu), 0, 0), (\mu, \xi_2(\mu), 0, 0)$$

satisfying

$$\xi_1(\mu) < 0, \xi_2(\mu) > 0$$

and

$$
\alpha_1 (\mu)^{1/2} \leq |\xi_i(\mu)| \leq \alpha_2 (\mu)^{1/2}, i = 1, 2
$$

for some positive constants $\alpha_1, \alpha_2$. From (9) and (10) it follows

$$
\xi' - \xi > 0 \quad \text{for} \mu \leq 0,
$$

$$
\xi_1(\mu) < \xi < 0 \quad \text{for} \mu > 0, \xi = 0,
$$

$$
\xi' - \xi > 0 \quad \text{for} \mu > 0, (-4\alpha \beta^{-4}(\mu)^{1/2} < \xi < \xi_2(\mu), \xi' - \xi > 0 \quad \text{for} \xi > \xi_2(\mu).
$$

Since $\xi' - \xi$ can change its sign only at fixed points, for $\mu > 0$ from (12), (13) we conclude $\xi_1(\mu) < \xi < \xi_2(\mu)$, $\xi' - \xi > 0$ for $\xi = \xi_2(\mu)$.

This, together with (11), proves (c).

To prove (b) we note that if $f \in F_1''$, then only one eigenvalue of $df_{\mu}$ can cross the unit circle at $(\eta_0, m_0)$ and this eigenvalue is the eigenvalue of the restriction of $df_{\mu}$ to the manifold $\eta = 0$, $\xi = 0$.
which is represented by (9). From (13) it follows
\[
\frac{d F'}{d F} (\mu, F_i(\mu)) = 1 + 2 \beta F_i + \sigma (F_i)
\] which implies
\[
\frac{d F'}{d F} (\mu, F_i(\mu)) < 1, \frac{d F'}{d F} (\mu, F_i(\mu)) > 1 \text{ for small } \mu > 0.
\]
This completes the proof.

We summarize the results of Lemmas 1 - 4 together with their generalization for periodic points with higher prime period in the following theorem.

Denote \( X_{f^k} = Z_{f^k} \cap X_f (f^k) \).

**Theorem 1.** For every \( f \) from a residual subset \( \mathcal{F} \in \mathcal{F} \):

(i) \( Z_{f^k} \) are 1-dimensional submanifolds of \( P \times M \); \( Z_f \) is closed;

(ii) for fixed \( \mu \), the \( k \)-periodic points of \( f_{\mu} \) are isolated;

(iii) \( X_{f^k} \) is discrete;

(iv) for every \( (\mu, m) \in Z_{f^k} \setminus X_{f^k} \), there is a neighborhood \( W = U \times V \) of \( (\mu, m) \) and a \( C^\infty \) function \( \varphi : U \to V \) such that \( Z_{f^k} \cap W \) is the graph of \( \varphi \);

(v) for every \( (\mu_0, m_0) \in X_{f^k} \), there is a coordinate neighbourhood \( (W, \mu \times x) \) of \( (\mu_0, m_0), (\mu \times x)(\mu_0, m_0) = (0, 0) \) such that

(a) there is a \( C^\infty \) function \( \psi : U \cap W \to W \), \( U \subset \mathbb{R} \) open, such that \( Z_{f^k} \cap W = \{ \psi (x_1) | x_1 \in U, \varphi = id, \frac{d^2 \psi \circ x_1}{d x_1^2} (0) > 0 \).
(b) \( df_p(m) \) has one eigenvalue 1, the others having moduli different from 1; the number of eigenvalues with moduli \( > 1 \) in the components \( x_i > 0 \), and \( x_i < 0 \) of \( Z_m \cap W \) is constant and differ by one;

(c) \( W \setminus Z_m \) contains no invariant set.

Proof. The statement for \( \lambda = 1 \) is proven in Lemmas 1 - 4. To prove the rest, we denote by \( \mathcal{F}_1(\mathcal{U}) \) the set of all \( f \in \mathcal{F} \) such that \( f|_{\mathcal{U}} \) satisfies (i) - (v) for \( 1 \leq \lambda \leq \ell \).

Let \( \varphi \) be a \( C^\infty \) Riemannian metric on \( P \times M, \{ K_\varphi \} \) an increasing sequence of compact sets, \( \bigcup K_\varphi = P \times M \). Denote \( B(N, \sigma) = \{ (\eta, m) \mid d(\eta, (\eta, m)) < \sigma \} \) for \( N \subset P \times M \).

We show that the sets \( \mathcal{F}_{i, j} = \mathcal{F}_j(K_\varphi \setminus B(\bigcup \{ K_\varphi \setminus Z_m, \ell^{-i} \}) \) are open and dense. Since \( \mathcal{F}_i = \bigcap \mathcal{F}_{i, j} \), this will complete the proof.

To prove density, we cover \( Z_1 \cap K_\varphi \setminus B\bigcup \{ K_\varphi \setminus Z_m, \ell^{-i} \} \) by a countable family \( \{ W_i \} \) of open sets such that \( W_i \cap f(W_i) \cap \ldots \cap f^{i-i}(W_i) = \emptyset \) and \( W_i \cap Z_m = \emptyset \), \( \lambda < \ell \).

Using Lemmas 1 - 4 we find that \( f^\varphi \) can be arbitrarily closely approximated by a map \( \lambda \) such that \( \lambda \in \mathcal{F}_j(W_i) \) and \( \lambda \) coincides with \( f^\varphi \) outside \( W \). We denote
\[
\mathcal{F} = \{ \begin{array}{ll}
\varphi^{-1} \lambda & \text{on } W_i, \\
\varphi & \text{outside } W_i.
\end{array}
\]

Then, if \( \lambda \) is close enough to \( f^\varphi \), \( W_i \cap \mathcal{F}(W_i) \cap \ldots \cap f^{i-i}(W_i) = \emptyset \), \( \mathcal{F} \) and, therefore, \( \mathcal{F} \in \mathcal{F}_{i, j}(W_i) \).
Repeating this for every \( i \) and taking into account the openness of \( \mathcal{F}_i(W_i) \), one concludes the proof of density of \( \mathcal{F}_x \).

For the proof of openness we note that since
\[
K \setminus \overline{B(\bigcup_{n \leq i} Z_n, \ell^{-1})}
\]
is compact, from \( f \in \mathcal{F}_x \) it follows \( f \in \mathcal{F}_x(K \setminus \overline{B(\bigcup_{n \leq i} Z_n, \ell^{-1} - \sigma')}) \) for some small \( \sigma > 0 \).
If \( \tilde{f} \) is close enough to \( f \), \( \bigcup_{n \leq i} Z_n(\tilde{f}) \in B(\bigcup_{n \leq i} Z_n(f), \sigma') \).
Thus,
\[
(14) \quad B(\bigcup_{n \leq i} Z_n(\tilde{f}), \ell^{-1}) \supset B(\bigcup_{n \leq i} Z_n(f), \ell^{-1} - \sigma') .
\]
The openness of \( \mathcal{F}_x \) follows now from (14), Lemmas 1–4 and the fact that \( \tilde{f} \) is arbitrarily close to \( f \) if \( \tilde{f} \) is close enough to \( f \).

Remarks. 1. In case \( m = 2 \), the points of one component of \( Z \setminus W \setminus \{(n_x, m_x)\} \) are saddles, the points of the other are either sources or sinks.

2. The set \( \mathcal{F}_x \) of those \( f \in \mathcal{F} \) satisfying (i)–(v) of Theorem 1 for \( k = 1 \) is open dense in \( \mathcal{F} \).

\[\text{§ 2.}\]

The sets \( Z_n \) for \( n > 1 \) are not closed in general. A point from \( \overline{Z_n} \setminus Z_n \) is also a periodic point, its prime period being a divisor of \( n \). We shall call the points of \( \overline{Z_n} \setminus Z_n \) branching (\( \ell \)-periodic, according to their prime period) points. In this section, we shall study the behaviour of \( f \) in the neighbourhood of bran-
ching points in the case $n = 2$ which allows us to obtain some information about the sets $Z_{\Phi \lambda}$.

If $f \in \mathcal{F}_1$, a $\lambda$-periodic point $(\mu, m)$ can be a branching point only if $d\phi_{\lambda}^n(m)$ has some root of unity different from 1 as an eigenvalue. For, if $d\phi_{\lambda}^n(m)$ has no root of unity as an eigenvalue, $d\phi_{\lambda}^n(m) - id$ is regular for every $\nu > 0$ and by the implicit function theorem there is a unique $C^\nu$ 1-dimensional submanifold of periodic points with (not necessarily prime) period $\nu \Phi \lambda$, $\nu > 0$; thus, this manifold coincides with $Z_{\Phi \lambda}$ for every $\nu > 0$. The case of 1 being an eigenvalue is covered by Theorem 1.

Therefore, we need first to know how the eigenvalues cross the unit circle if $\mu$ is changed, in the generic case.

Henceforth we shall assume $n = 2$ without repeating it. Let $f \in \mathcal{F}_1$ and denote $D_{\Phi \lambda} = \{(\mu, m) \in Z_{\Phi \lambda} | d\phi_{\lambda}^n(m)$ has double eigenvalues$\}$.

From the implicit function theorem it follows that the eigenvalues $\lambda_{1,2}^{(\lambda)}$ of $d\phi_{\lambda}^n(m)$ are $C^\nu$ functions on $Z_{\Phi \lambda} \setminus D_{\Phi \lambda}$.

Denote by $S$ the unit circle in the complex plane.

**Theorem 2**. For a residual subset $\mathcal{F}_2$ of $\mathcal{F}$, $\mathcal{F}_2 \subset \mathcal{F}_1$:

(i) $\lambda_i^{(\lambda)}(D_{\Phi \lambda}) \cap S = \emptyset$, $i = 1, 2$,

(ii) $\lambda_i^{(\lambda)}$, $i = 1, 2$ meet $S$ transversally.
(iii) If, for some $(\mu, m) \in \mathbb{Z}_M$, $\lambda_1^{(\mu)}(\mu, m) \in S$, then either $\lambda_2^{(\mu)} \notin S$ or $\lambda_1^{(\mu)}(\mu, m)$ is not a root of 1.

**Corollary.** Generically, $(\mu, m)$ can be a branching point only if one of the eigenvalues of $df^{(\mu)}_\mu(m)$ is $-1$, the other being real $\neq 1$. We denote by $\chi_{\mu}$ the set of such points.

**Proof of Theorem 2.** We prove the statement of the theorem for $\mu = 1$ (fixed points), the generalization to the case $\mu > 1$ being similar as in the proof of Theorem 1.

From Theorem 1, (iv) and its proof it follows that for every $f \in \mathcal{F}_1$, if some eigenvalue meets $S$ at $1$, it is single and meets $S$ transversally. Therefore, we can restrict our attention to $S \setminus \{1\}$.

Let $f \in \mathcal{F}_1$, where $\mathcal{F}_1$ is defined at the end of §4, $(\mu, m) \in \mathbb{Z}_M \setminus X_1$. Then, according to Theorem 1, (iv), there is a coordinate neighbourhood $(W, \mu \times \mathbb{R})$, $W = U \times V$ such that $\mu(\mu) = 0, \mu(m) = 0$ and the representation of $f$ in these coordinates is given by

$$x' = A(\mu)x + \omega(\mu, x),$$

where $\omega(\mu, 0) = 0$, $d\omega(0, 0) = 0$.

The subset of matrices with both eigenvalues on the unit circle is a submanifold $\mathcal{C}_\mu$ of co-dimension 1 in $\text{GL}(2)$ (it is the set of matrices $A$ such that $\det A = 1$). Further, the set of all $2 \times 2$ matrices with
eigenvalues being \( \ell \)-th roots of unity (the unity matrix \( E \) excluded), \( \mathcal{U}_2 \) is a 2-dimensional submanifold of \( \text{GL}(2) \), given by \( \det A = 1, \text{tr} A = \alpha_j + \alpha_j^{-1} \) for \( \ell \) odd, and a union of the 2-dimensional manifold given as for \( \ell \) odd and the isolated matrix \(-E\) for \( \ell \) even, where \( \alpha_j \) are the \( \ell \)-th roots of unity, lying in the open upper complex halfplane.

Using the elementary transversality theorem, we can approximate the function \( A : \mu(U) \to \text{GL}(2) \) arbitrarily closely by \( \tilde{A} : \mu(U) \to \text{GL}(2) \) so that

\[ \tilde{A} \text{ coincides with } A \text{ outside } U_1, \quad \tilde{U}_1 \subset \mu(U), \]

\( \tilde{A} \) meets \( \mathcal{U} \) transversally and does not meet \( \mathcal{U}_2 \) at all for \( \mu \in U_2 \), \( U_2 \) open, \( \tilde{U}_2 \subset U_1 \). As a consequence we obtain that \( \tilde{A}(\mu) \) does not have \(-1\) as double eigenvalue for any \( \mu \in U_2 \). This implies that the eigenvalues \( \lambda_1, \lambda_2 \) are \( \mathcal{C}^\infty \) functions of matrices in the neighbourhood of any \( A(\mu) \), some eigenvalue of which is \(-1\).

Therefore, in the neighbourhood of the values of \( \tilde{A}(\mu) \), \( \mu \in U_2 \), the subsets of \( \text{GL}(2) \), given by \( \lambda_1 = -1 \) and \( \lambda_2 = -1 \) are submanifolds of co-dimension 1. Thus, we can use the transversality theorem again (for \( \tilde{A} \) and \( \mathcal{U}_2 \)) to obtain that arbitrarily near \( \tilde{A} \) (and, thus, \( A \)) there is a function \( \tilde{A} : \mu(U) \to \text{GL}(2) \) such that for \( \mu \in U_2 \), \(-1\) is not double eigenvalue of \( \tilde{A}(\mu) \), and the eigenvalues \( \lambda_1 \) and \( \lambda_2 \) cross the unit circle transversally at the points which are not \( \ell \)-th roots.
of unity.

Let $V_2, V_1$ be open, $\overline{V}_2 \subset V_1$, $\overline{V}_1 \subset V$, let $\varphi(\mu, \lambda)$ be a bump function such that $\varphi(\mu, \lambda) = 1$ for $(\mu, \lambda) \in U_2 \times V_2$, $\varphi(\mu, \lambda) = 0$ outside $U_1 \times V_1$. We denote by $f$ the map that coincides with $f$ outside $U \times V$ and is given in $W$ by its coordinate representation

$$x' = [A(\mu) + \varphi(\mu, \lambda)(\tilde{A}(\mu) - A(\mu))] x + \Omega(\mu, \lambda).$$

Then, if $A$ is chosen close enough to $A$, $f$ is arbitrarily close to $f$, satisfies (i),(ii) and

(iii) if $\lambda_2 \notin S$, or $\lambda_4$ is not an $\ell$-th root of unity, in $U_2 \times V_2$.

As usual, we can prove that $f$ can be approximated by a function $\tilde{f}$ having Properties (i),(ii),(iii) over all $Z_1 \setminus X_1$ by covering $Z_1 \setminus X_1$ by a countable family of coordinate neighbourhoods. It is obvious that the set of $f$'s, having Properties (i),(ii),(iii) is open.

Since the subset $\mathcal{F}_{21} \subset \mathcal{F}$ of maps, having Properties (i),(ii),(iii) for $A = 1$ is the intersection of the sets $\mathcal{F}_{212} \subset \mathcal{F}$, satisfying (i),(ii),(iii), the proof of Theorem 2 for $A = 1$ is completed.

Remark. Note that the subset $\mathcal{F}_{212} \subset \mathcal{F}$ of maps, all iterates up to order $A$, of which satisfy (iii), is open dense in $\mathcal{F}$.

We shall now study the behaviour of $f$ in the neighbourhood of a branching point.
Theorem 3. Assume $\kappa \geq 3$. Then, for a residual subset $\mathcal{F}_2$ of $\mathcal{F}$, $\mathcal{F}_2 \subset \mathcal{F}_2$, the following is valid:

(i) $Y_{\mathcal{K}}$ coincides with the set of $\mathcal{K}$-periodic branching points.

(ii) For every $(\nu, m) \in Y_{\mathcal{K}}$ there is a coordinate neighbourhood $(W, \mu \times \chi)$, $W = U \times V$ of $(\nu, m)$ such that $\mu (\nu) = 0$, $\chi (m) = 0$, $Z_{\mathcal{K}} \cap W = U \times \{0\}$

and

(a) $Z_{\mathcal{K}} \cap W$ consists of two components, separated by $(\nu, m)$; all points of $Z_{\mathcal{K}} \cap W$ satisfy $\mu > 0$ and $Z_{\mathcal{K}} \cap W \cup f(\nu, m)$ is a $C^1$ (but not $C^2$) submanifold of $W$.

(b) Either the points of $Z_{\mathcal{K}} \cap W$ are sinks for $\mu > 0$ saddles for $\mu = 0$ (degenerated for $\mu = 0$), and the points of $Z_{\mathcal{K}} \cap W$ are saddles, or the same is true with sink replaced by saddle and conversely, or one of the above cases is true for the inverse of $f$.

(c) $W \setminus (Z_{\mathcal{K}} \cup Z_{\mathcal{K}}^\mu)$ contains no invariant set of $f_{\mathcal{K}}$.

Proof. We again prove the theorem for $\mathcal{K} = 1$, the generalization for $\mathcal{K} > 1$ being similar as in the proof of Theorem 1.

Assume $f \in \mathcal{F}_2$. Then, one eigenvalue of $df_{\mathcal{K}}(m)$ is $-1$, the other, $\lambda$, is not on $\mathcal{S}$. We can assume $|\lambda| < 1$, in the other case we consider the inverse of $f$.

As in the proof of Theorem 1, using [3], Appendix C, we find that there is a coordinate neighbourhood
\((W, (\mu \times x)), W = U \times V, (\mu \times x)(b_0, m_0) = (0, 0)\) such that the local representation of \(f\) in the coordinates \((\mu, x)\) is given by

\[
\begin{align*}
\dot{x}_1 &= -x_1 + \alpha(\mu x_1^2 + \beta x_1^3 + \gamma x_1^4 + \omega(x_1, x_2)), \\
\dot{x}_2 &= \lambda x_2 + \vartheta(\mu, x_1, x_2),
\end{align*}
\]

where \(\omega, \vartheta\) are \(C^\infty\) and

\[
\begin{align*}
\vartheta(\mu, x_1, 0) &= 0, \quad \partial \vartheta(0, 0) = 0, \\
\omega(\mu, x_1, x_2) &= (|x_1^3| + |\mu x_1^2| + |x_2|).
\end{align*}
\]

Similarly, as in the proof of Lemma 4, it can be shown that every \(f\) can be arbitrarily closely approximated in \(\mathcal{F}_{31}\) by a map the local representation of which satisfies \(\beta^2 + \gamma = 0\) at every point from \(Y\). We denote \(\mathcal{F}_{31}\) the set of such maps. The openness of \(\mathcal{F}_{31}\) is obvious.

We prove that if \(f \notin \mathcal{F}_{31}\) then \(f\) satisfies (i), (ii), of this theorem for \(k = 1\). We shall analyze the case \(\alpha > 0, \beta^2 + \gamma < 0\). The other cases can be transformed to the above case by a suitable change of coordinates or lead to other cases of (ii b), which can be analyzed similarly.

From (15), (16) we obtain the representation of the second iterate of \(f|_{x_2 = 0}\)

\[
\begin{align*}
\dot{x}_1'' &= x_1 - 2 \alpha(\mu x_1^2 - 2(\beta^2 + \gamma)x_1^3 + \omega_2(x_1, x_2),
\end{align*}
\]

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where \( \omega_2(\mu, x) = \left( |\mu x_1| + |x_1^3| \right) \). By a change of variables \( x_1 = \nu^2 \xi, \mu = \nu^2 \) for \( \mu > 0 \), (18) is transformed into

\[
(19) \quad \xi'' = \xi - 2\nu^2 \left[ \alpha \xi + (\beta^2 + \gamma) \xi^3 \right] + \chi(\nu, \xi),
\]

where \( \chi(\nu, \xi) = \nu^{-1} \omega_2(\nu^2, \nu \xi) \) is \( C^{\infty} \) for \( \nu > 0 \) and satisfies

\[
(20) \quad \chi(\nu, \xi) = \sigma(\nu^2).
\]

\( \xi \) is a 2-periodic point of \( f_\mu \mid x_2 = 0 \) for \( \nu > 0 \) if \( \xi \) satisfies

\[
(21) \quad \alpha \xi + (\beta^2 + \gamma) \xi^3 - \chi_1(\nu, \xi) = 0,
\]

where \( \chi_1(\nu, \xi) = \nu^2 \chi(\nu, \xi) \). From (20) it follows that if we define \( \chi_1(0, \xi) = 0 \), then \( \chi_1 \) is \( C^{\infty} \) for \( \nu \geq 0 \) and, in the case \( \nu = 3 \), that

\[
\frac{\partial \chi_1}{\partial \xi}
\]

is continuous.

For \( \nu = 0 \), (21) has two non-zero solutions

\[
\xi_1(0) = \left( -\alpha (\beta^2 + \gamma)^{-1} \right)^{1/2}, \quad \xi_2(0) = \left[ -\alpha (\beta^2 + \gamma)^{-1} \right]^{1/2}.
\]

Using the implicit function theorem of [6] and returning to the coordinates \( \mu, x_1 \) we obtain that for \( \mu > 0 \) sufficiently small there are two 2-periodic points (1 orbit) of \( f_\mu \mid x_2 = 0 \) with coordinates

\[
(22) \quad x_1(\mu) = -\left[ -\alpha (\beta^2 + \gamma)^{-1} \right]^{1/2} + \psi(\mu),
\]

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\[ x_{12}(\mu) = [-\alpha (\beta^2 + \gamma)^{-1} \mu]^{\frac{1}{2}} + \psi_2(\mu), \]

where \( \psi_1, \psi_2 \) are \( C^{\infty} \) and satisfy \( \psi_1'(\mu) = \sigma'(\mu) \); the eigenvalue of \( \frac{df_{\mu}^2}{x_2} \) at the points \( x_{11}, x_{12} \) is equal \( 1 + 4 \alpha \mu + \sigma'(\mu) \). Since from (16) it follows that the other eigenvalue of \( \frac{df_{\mu}^2}{x_2} \) at the points

\[(\mu, x_{11}(\mu), 0), (\mu, x_{12}(\mu), 0) \]

is of modulus less than one, this proves that the points \( (\mu, x_{11}(\mu), 0), (\mu, x_{12}(\mu), 0) \) are saddles for small \( \mu \). From (15), (16) it follows further that for small \( |\mu| \), the points of \( Z_4 \) are sinks for \( \mu > 0 \) and saddles for \( \mu < 0 \). This proves (ii b) if we show that \( Z_2 \cap W \) (\( W \) possibly restricted) does not contain other points except of the points \( (\mu, x_{1i}(\mu), 0), i = 1, 2 \).

From (16), (17) it follows that every orbit that remains in \( |x| < \sigma \) (\( \sigma \) sufficiently small independent of \( \mu \) for \( |\mu| \) small), approaches the submanifold \( x_2 = 0 \) (in the positive sense). Therefore, in order to prove (ii c) and thus also to complete the proof of (ii b) it suffices to prove that for sufficiently small \( \mu \) the only periodic points of \( f_{\mu} |_{x_2 = 0} \) for \( |x_2| < \sigma' < \sigma, \sigma' \) sufficiently small, are the points \( x_{1i}(\mu), i = 1, 2 \), and 0.

From (17) it follows that

\[(23) \quad x_1'' - x_1 < 0 \quad \text{for} \quad \mu < 0, x_1 < 0, \]
(24) \( x''_2 - x_1 > 0 \) for \( \mu \leq 0 \), \( x_1 > 0 \),
(25) \( x''_1 - x_1 > 0 \) for \( \mu > 0 \),
\[
x_1 > \left[-4 \alpha (\gamma + \beta^2)^{-1} (\mu) \right]^{1/2},
\]
(26) \( x''_1 - x_1 < 0 \) for \( \mu > 0 \),
\[
x_1 < \left[-4 \alpha (\gamma + \beta^2)^{-1} (\mu) \right]^{1/2},
\]
and \( |\mu| < \sigma_2^\ast, |x_1| < \sigma_2^\ast, \sigma_2^\ast \) being sufficiently small. From (23), (24), it follows that the orbit of every point with \( 0 > (\mu > -\sigma_2^\ast, |x_1| < \sigma_2^\ast \) leaves
\( |x_1| < \sigma_2^\ast \). From (22), (23), (24) and the implicit function argument used after (21) it follows that there are no periodic points with \( |x_1| < \left[-4 \alpha (\beta^2 + \gamma)^{-1} (\mu) \right]^{1/2} \)
except of the points \( x_{11}(\mu), x_{12}(\mu) \). From this, (25),
(26) and (19) it follows \( x''_1 - x_1 < 0 \) for \( \sigma_2^\ast < x_1 < x_{11}(\mu) \) or \( 0 < x_1 < x_{12}(\mu) \) and \( x''_1 - x_1 > 0 \)
for \( x_{11}(\mu) < x_1 < 0 \) or \( x_{12}(\mu) < x_1 < \sigma_2^\ast, (\mu > 0) \),
so that every orbit both in the positive and negative sense tends to one of the points \( 0, x_{11}(\mu), x_{12}(\mu) \).
This completes the proof of (iv c).

To complete the proof of (ii a), we denote by \( \varphi(x_1) \)
the real function, defined as the inverse of the functions
\( x_1 = x_{11}(\mu) \) for \( x_1 < 0 \) and \( x_1 = x_{12}(\mu) \) for
\( x_1 > 0 \). From (22) it follows
(27) \[
\lim_{x_1 \to 0^-} \varphi(x_1) = \lim_{x_1 \to 0^+} \varphi(x_1) = \frac{d \varphi^+(0)}{dx_1} = \frac{d \varphi^-(0)}{dx_1} = 0.
\]
Further, from the fact that the points \((\mu, x_{12}(\mu), 0)\)
\((\mu, x_{12}(\mu), 0)\) are nondegenerated for \(\mu > 0\) it follows that \(\varphi\) is \(C^\infty\). Using (22) and the implicit function theorem we obtain

\[
\frac{d\varphi}{dx_1} = -\left[\alpha^{-1}(\beta^2 + \gamma)\mu\right]^{1/2} + \sigma(\mu^{1/2}) \quad \text{for} \quad x_1 < 0,
\]

\[
\frac{d\varphi}{dx_1} = \left[\alpha^{-1}(\beta^2 + \gamma)\mu\right]^{1/2} + \sigma(\mu^{1/2}) \quad \text{for} \quad x_1 > 0.
\]

This, together with (27) shows that \(\varphi\) can be completed into a \(C^1\) function (which is not \(C^2\)) in some neighbourhood of 0 by defining \(\varphi(0) = 0\).

As a corollary of Theorem 1 and 3 we obtain

**Theorem 4.** Let \(\kappa > 2\). Then for every \(f \in \mathcal{F}_\kappa\):

(i) for \(\kappa\) odd, \(Z_{\kappa,\kappa}\) is a closed submanifold of \(P \times M\),

(ii) for \(\kappa\) even, \(Z_{\kappa,\kappa}\) is a closed \(C^1\) (but not \(C^2\)) submanifold of \(P \times M\); \(\overline{Z_{\kappa,\kappa}} \setminus Z_{\kappa,\kappa}\) is discrete and coincides with \(Y_{\kappa/2}\).

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(Oblatum 16.2.1970)