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Josef Kolomý

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SOME REMARKS ON NONLINEAR FUNCTIONALS

Josef KOLOMÝ, Praha

§ 1. Introduction. In this note, we give sufficient conditions under which a convex functional possesses the Gâteaux or Fréchet derivatives at some point u_0 of a normed linear space X, strongly continuous Fréchet derivative on an open subset M of X. Theorem 2 deals with smoothness of nonlinear functionals. Using some known results from the geometry of Banach spaces, we obtain, as a simple consequence of Theorem 3, a representation theorem for linear continuous functionals on certain Banach spaces.

§ 2. Preliminaries. Let X be a real normed linear space, X^* its dual space, $S_1 = \{x \in X : \|x\| = 1\}$, $S_1^* = \{x^* \in X^* : \|x^*\| = 1\}$. A space X or its norm $\|\cdot\|$ is said to be strictly convex if for each $x, y \in X$, $\|x\| = \|y\| = 1$ there is $\|\frac{x+y}{2}\| < 1$. We shall use the symbols "$\longrightarrow$", "$\xrightarrow{w}$" to denote the strong and weak convergence in X. A functional f is said to be

(a) convex on a convex set $M \subset X$ if $f(\lambda x + (1-\lambda)y) \leqq \lambda f(x) + (1-\lambda) f(y)$ for each $x, y \in M$ and $\lambda \in [0,1]$;

(b) weakly continuous at $u_0 \in X$ if $u_n \xrightarrow{w} u_0$

implies $f(u_n) \longrightarrow f(u_0)$;

(c) smooth [1, 2] at some $u \in X$ if

$$\lim_{\|h\| \to 0} \frac{\Delta_f(u, h)}{\|h\|} = 0 ,$$

where $\Delta_f(u, h) = f(u+h) + f(u-h) - 2f(u)$, $h \in X$;

(d) uniformly smooth [2] on some set $E \subset X$, if for each $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ so that $|\Delta_f(u,h)| \leqq \varepsilon \|h\|$ holds for each $u \in E$ and $0 < \|h\| < \delta$.

For the Gâteaux, Fréchet differentials and derivatives we shall use the notions and notations given in Vajnberg's book [3,Chapt.I]. By $V_+ f(u_0, \cdot)$ we denote the one-sided Gâteaux differential of a convex functional f at $u_0 \in X$. We shall say that the Fréchet derivative $f'(u)$ of a functional f is strongly continuous at some $u_0 \in X$ if $u_n \xrightarrow{w} u_0$ implies $\|f'(u_n) - f'(u_0)\| \longrightarrow 0$ as $n \longrightarrow \infty$. According to Suchomlinov [4] f is said to have a bounded differential $dVf(u_0, \cdot)$ at $u_0 \in X$, if

$$f(u_0 + h) - f(u_0) = dVf(u_0, h) + \omega_f(u_0, h) , \quad h \in X ,$$

where $\lim_{\|h\| \to 0} (\omega_f(u_0, h)/\|h\|) = 0$ and $dVf(u_0, \cdot)$ is homogeneous in X and bounded in some neighborhood $U(0)$ of 0.

§ 3. We start with the following

Theorem 1. Let X be a reflexive Banach space, f a weakly continuous convex functional on X. Suppose f is uniformly smooth on an open subset $M \subset X$. Then f possesses a strongly continuous Fréchet derivative $f'(u)$ on M. Conversely, if f has a strongly continuous Fréchet derivative $f'(u)$ on M, then f is weakly continuous, uniformly smooth and Lipschitzian on each bounded weakly sequentially closed (= weakly closed by [5, § 24,1.(7)]) set E contained in M.

Proof. As f is uniformly smooth and continuous on M, according to Theorem 6 [2], f is uniformly Fréchet differentiable on M. Using the arguments of [6, Theorem 6], resp. [7, Lemma 2], we see that $f'(u)$ is strongly continuous on M. Conversely, if $f'(u)$ is strongly continuous on M, then by Vajnberg's theorem [3, Theorem 1.4] $f'(u)$ is compact and uniformly continuous on each bounded weakly closed set E contained in M. But the compactness of $f'(u)$ on E implies that f is weakly continuous on E (cf.[3]). $f'(u)$ being uniformly continuous on E, $f'(u)$ is bounded on E and hence Lipschitzian on E by the mean-value theorem. By Theorem 8 [2] f is uniformly smooth on E. This concludes the proof.

Theorem 2. Let X be a normed linear space, f a functional on X. Assume that there exist two functionals φ, g defined on a neighborhood $V(u_0)$ of some $u_0 \in X$ so that $u \in V(u_0) \Rightarrow \varphi(u) \leq f(u) \leq g(u)$ and $f(u_0) = \varphi(u_0) = g(u_0)$. Suppose that φ, g possess bounded differentials $dV\varphi(u_0, \cdot)$, $dVg(u_0, \cdot)$ at u_0. Then f is smooth at u_0.

Proof. To prove our theorem we use the following remark: a functional f is smooth at $u_0 \Longleftrightarrow$ the limit

$$\lim_{t \to 0_+} \frac{1}{t} \, |\Delta_f(u_0, th)| = 0 \tag{1}$$

is uniform with respect to $h \in X$, $\|h\| = 1$. Indeed, assume (1) is satisfied and f is not smooth at u_0. Then there exists $\varepsilon_0 > 0$ and the sequence $(\tilde{h}_n) \in X$ so that $\|\tilde{h}_n\| < n^{-1}$ and

(2) $$\frac{1}{\|\tilde{h}_n\|} \, |\Delta_f(u_0, \tilde{h}_n)| > \varepsilon_0 \, .$$

Set $h_n = \tilde{h}_n \|\tilde{h}_n\|^{-1}$, $t_n = \|\tilde{h}_n\|$. Then $\tilde{h}_n = t_n h_n$, $t_n \to 0_+$ as $n \to \infty$ and $\|h_n\| = 1$. Hence by (2) we have that $\frac{1}{t_n} |\Delta_f(u_0, t_n h_n)| > \varepsilon_0$, which contradicts our hypothesis. The second part is obvious.

Let $h \in X$ be arbitrary (but fixed), $\|h\| = 1$. For sufficiently small $t > 0$ we have that $u_0 + th \in V(u_0)$, $u_0 - th \in V(u_0)$. Hence our hypotheses imply that $(t > 0, \|h\| = 1)$

$$\frac{1}{t} \Delta_\varphi(u_0 th) \leqq \frac{1}{t} \Delta_f(u_0, th) \leqq \frac{1}{t} \Delta_g(u_0, th) \, .$$

We have

$$g(u_0 + th) - g(u_0) = t\, dVg(u_0, h) + \omega_g(u_0, th),$$

$$g(u_0 - th) - g(u_0) = -t\, dVg(u_0, h) + \omega_g(u_0, t(-h)),$$

and two similar relations for the functional φ. Hence

$$\omega_\varphi(u_0, th) + \omega_\varphi(u_0, t(-h)) \leqq \Delta_f(u_0, th) \leqq$$
$$\leqq \omega_g(u_0, th) + \omega_g(u_0, t(-h)) \, .$$

From these inequalities it follows that uniformly on S_1

$\lim\limits_{t \to 0_+} \frac{1}{t} \Delta_f(u_0, th) = 0$. According to the above mentioned remark f is smooth at u_0.

Theorem 3. Let X be a normed linear space, f, g functionals on X, f convex so that $f(u_0) = g(u_0)$ and $f(u) \leqq g(u)$ for each $u \in V(u_0)$, where $V(u_0)$ is some neighborhood of $u_0 \in X$. Assume g has the Gâteaux (Fréchet) derivative $g'(u_0)$ at u_0. Then f possesses the Gâteaux (Fréchet) derivative $f'(u_0)$ at u_0 and $f'(u_0) = g'(u_0)$. In particular, if f is linear and continuous, then $f(h) = g'(u_0)h$ for each $h \in X$.

Proof. Let h be an arbitrary (but fixed) element of X. For sufficiently small t, $0 < t < t_0(h)$ our assumptions imply that

$$\frac{1}{t}(g(u_0 + th) - g(u_0)) \geqq \frac{1}{t}(f(u_0 + th) - f(u_0)) \geqq$$

$$\geqq \frac{1}{t}(f(u_0) - f(u_0 - th)) \geqq \frac{1}{t}(g(u_0) - g(u_0 - th)).$$

By our hypothesis $\lim_{t \to 0_+} \frac{1}{t}(g(u_0 + th) - g(u_0)) =$

$$= g'(u_0)h, \ \lim_{t \to 0_+} \frac{1}{t}(g(u_0) - g(u_0 - th)) = g'(u_0)h$$

for each $h \in X$. Hence $V_+ f(u_0, h) = g'(u_0)h$ for each $h \in X$. As $V_+ f(u_0, h) = -V_+ f(u_0, -h) = g'(u_0)h$, $h \in X$, f possesses the Gâteaux derivative $f'(u_0)$ at u_0 and $f'(u_0) = g'(u_0)$. Assume g has the Fréchet derivative $g'(u_0)$ at u_0. By the assertion just proved f has the Gâteaux derivative $f'(u_0)$ at u_0 and $f'(u_0) = g'(u_0)$. Consider $h \in X$, $\| h \| = 1$ and $t > 0$.

For sufficiently small $t > 0$, $u_0 + th \in V(0)$. According to Lemma 2 [8] (here $V. f(u_0, h) = f'(u_0)h$)

$$0 \leq \frac{1}{t} \omega_f(u_o, th) \leq \frac{1}{t}(g(u_o + th) - g(u_o)) - g'(u_o)h,$$

where $\omega_f(u_o, h) = f(u_o + h) - f(u_o) - f'(u_o)h$. Therefore $\lim_{t \to 0_+} \frac{1}{t} \omega_f(u_o, th) = 0$ uniformly with respect to $h \in X$, $\|h\| = 1$. Similarly, one gets that $\lim_{t \to 0_-} \frac{1}{t} \omega_f(u_o, th) = \lim_{t \to 0_+} (-\frac{1}{t} \omega_f(u_o, (-t)h)) = 0$ uniformly with respect to $h \in X$, $\|h\| = 1$. Hence f possesses the Fréchet derivative $f'(u_o)$ at u_o and $f'(u_o) = g'(u_o)$. In the case f is linear and continuous, $f(h) = f'(u_o)h = g'(u_o)h$, $h \in X$. This concludes the proof.

The second assertion of Theorem 3 generalizes the result of Theorem 4 [9]. At the same time, the proof of this assertion, presented here, is more simple than that of Theorem 4 [9].

The result just proved gives a representation theorem for linear continuous functionals on reflexive Banach spaces. First of all, we introduce some known results concerning the geometry of Banach spaces.

(a) Let X be a reflexive strictly convex Banach space, f a linear continuous functional on X. It is well-known that reflexivity of X guarantees the existence of $u_f \in X$ so that $\|u_f\| = 1$ and $f(u_f) = \|f\| \cdot \|u_f\|$ (a simple consequence of reflexivity of X and the Hahn-Banach theorem). Moreover, the strict convexity of X implies the uniqueness of such point $u_f \in X$.

(b) At first suppose that $(X, \|\cdot\|)$ is a separable Banach space. According to the Day's results [5,§ 26] a separable Banach space $(X, \|\cdot\|)$ admits a norm equivalent to $\|\cdot\|$ which is simultaneously Gâteaux differentiable and strictly convex. Following Köthe [5, § 26, 9.5] we introduce here the explicit form of such a norm.

Let (x_i) be a sequence dense in $S_1 = \{x \in X : \|x\| = 1\}$. Set

$$|||x^*|||_{X^*} = \|x^*\|_{X^*} + \|\left(\frac{x^*(x_i)}{2^i}\right)_{i=1}^{i=\infty}\|_{\ell_2} .$$

This norm is strictly convex and equivalent to $\|\cdot\|$. Moreover, the set $\{x^* \in X^* : |||x^*||| \leqq 1\}$ is w^*-closed. Let $(x_i^*)_{i=1}^{i=\infty}$, $x_i^* \in S_1^*$ be a total set (i.e. if $x_i^*(x) = 0$ for each $i = 1, 2, \dots,$ then $x = 0$). Introduce a new norm by

$$|||x|||_X = \sup_{|||x^*|||_{X^*} \leqq 1} |\langle x^*, x\rangle|$$

(this norm is Gâteaux-differentiable and equivalent to $\|\cdot\|$) and set (cf. [5, § 26, 3.2])

$$|x|_X = |||x|||_X + \|Tx\|_{\ell_2}, \quad Tx = \left(\frac{x_i^*(x)}{2^i}\right)_{i=1}^{i=\infty} .$$

The norm $|\cdot|_X$ is equivalent to the original norm $\|\cdot\|$ and it is simultaneously strictly convex and Gâteaux-differentiable.

Now suppose that $(X, \|\cdot\|)$ is reflexive and separable Banach space. Then $(X, |\cdot|_X)$ is reflexive strictly convex Banach space with the Gâteaux-differentiable norm $|\cdot|_X$. Hence, according to (a) for each linear continuous functional f on $(X, \|\cdot\|)$ there exists a unique point $u_f \in X$ so that $|u_f|_X = 1$ and $f(u_f) = |f|_X \cdot |u_f|_X$. The differentiability property of $|\cdot|_X$

will be used later.

(c) Suppose that $(X, \|\cdot\|)$ is a reflexive Banach space. According to [10, Theorem 5.2] there exists a new norm $\|\cdot\|_1$ which is equivalent to $\|\cdot\|$ and which is simultaneously strictly convex and Gâteaux differentiable. However, this new norm $\|\cdot\|_1$ cannot be explicitly expressed.

Theorem 3 and the above discussions yield the following (compare [10, Lemma 3]).

<u>Theorem 4</u>. Let $(X, \|\cdot\|)$ be a reflexive strictly convex Banach space whose norm $\|\cdot\|$ is Gâteaux differentiable at all nonzero points. Then for each linear continuous functional f on X there exists a unique point $u_f \in X$, $\|u_f\| = 1$ so that $f(h) = g'(u_f)h$, $h \in X$, where $g(h) = \|f\| \cdot \|h\|$. The same conclusion is valid for separable reflexive Banach spaces (reflexive Banach spaces) by taking $g(u) = |u|_X$ $(g(u) = \|u\|_1)$.

<u>Example</u>. The Riesz-Fréchet representation theorem. Let X be a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$, f a linear continuous functional at X. Set $g(u) = \|f\| \cdot \|u\|$, $u \in X$. There exists a unique point $u_f \in X$, $\|u_f\| = 1$ so that $f(u_f) = \|f\| \cdot \|u_f\|$. Then

$$g'(u_f)h = \frac{\langle u_f, h \rangle}{\|u_f\|} \|f\| = \langle u_f, h \rangle \|f\|, \; h \in X .$$

Hence $f(h) = \langle u_1, h \rangle$, $h \in X$, where $u_1 = u_f \|f\|$ and $\|u_1\| = \|f\|$.

Representation of linear continuous functionals on l_p, $L_p (p > 1)$ follows at once from the result of Mazur [12] and Theorem 4. See also [11, Theorem II]. Let us note that another simple proof of the Riesz theorem for L_p $(p > 1)$ based on variational argument was proposed by R. Bellman [13]. Using the basic results concerning the orthogonality in normed linear spaces, R.C. James [14, Theorem 8.2] has obtained the following result: Let X be a uniformly convex Banach space whose norm is Gâteaux differentiable at all nonzero points, then for a linear functional f with $\| f \| = 1$ there is one and only one element $u_0 \in X$ such that $\| u_0 \| = 1$ and $f(h) = Dg(u_0, h) \equiv -a$, where a is the number for which $u_0 \perp a u_0 + h$, $h \in X$ and $Dg(u_0, \cdot)$ is the Gâteaux differential of the norm at u_0.

To complete the results which concern the representation theorems of linear continuous functionals on Banach spaces (from the point of view of the differentiability of the norm), we also introduce here the result of Bonic and Reis [15, Lemma (d)]: Let X be a Banach space whose norm is Fréchet differentiable away from the origin. If $x^* \in X^*$ and for some $x \in X$, $x \neq 0$, $\|x^*\| = \|x\|$ and $x^*(x) = \|x\|^2$, then $x^*(h) = dg_1(x, h)$, $h \in X$, where $dg_1(x, \cdot)$ is the Fréchet differential of $g_1(x) = \|x\|^2/2$ at x.

<u>Remark 1</u>. The following assertion is a slight modification of Theorem 3 [16]. Let X be a normed linear space, f a convex functional on X. Then for any (but fixed)

$u_0 \in X$ we have that

$$(u_0 + \mathrm{Ker}\, V_+ f(u_0, \cdot)) \cap E = \emptyset ,$$

where $E = \{ u \in X : f(u) < f(u_0) \}$ and

$\mathrm{Ker}\, V_+ f(u_0, \cdot) = \{ h \in X : V_+ f(u_0, h) = 0 \}$.

Remark 2. In Math.Rev.39, No 4 # 4670, the reviewer pointed out that the proofs of Theorem 1 and Theorem 6 [8] contain a gap. I cannot agree with the second reviewer's objection that the complement of a dense set Z_n (the proof of Theorem 6 [8]) need not be of the first category in the Banach space X. The reviewer perhaps overlooked the fact that each Z_n is a G_δ -set ([8, p.745$_2$ from below]). As Z_n are G_δ-sets dense in X, Z_n are the sets of the second category in X and hence $X - Z_n$ must be of the first category in X. Indeed, $Z_n = \bigcap_{m=1}^{\infty} G_{n\,m}$, where $G_{n\,m}$ are open and dense in X (since Z_n is dense in X). Hence $X - G_{n\,m}$ are nowhere dense and $X - Z_n = X - \bigcap_{m=1}^{\infty} G_{nm} = \bigcup_{m=1}^{\infty} (X - G_{n\,m})$. Therefore $X - Z_n$ are the sets of the first category in X.

The gap of the proof of Theorem 1 [8] can be removed as follows: Replace the inequality (1) [8,p.737], containing e^*, by $\| V^2 F(x, h, k) \| \leq C \| h \| \, \| k \|$, h, $k \in X$, $x \in E$, where C does not depend on e^*. This relation follows immediately from a simple fact that a p -homogeneous operator F ($F(tu) = t^p F(u)$, $p > 0$, $u \in X$, t real) which is demicontinuous at 0 is continuous at 0 .

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Matematický ústav
Karlova universita
Sokolovská 83 Praha 8
Československo

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