Stanislav Tomášek
Projectively generated topologies on tensor products

Commentationes Mathematicae Universitatis Carolinae, Vol. 11 (1970), No. 4, 745--768
Persistent URL: http://dml.cz/dmlcz/105311

Terms of use:
© Charles University in Prague, Faculty of Mathematics and Physics, 1970
Institute of Mathematics of the Academy of Sciences of the Czech Republic provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these Terms of use.

This paper has been digitized, optimized for electronic delivery and stamped with digital signature within the project DML-CZ: The Czech Digital Mathematics Library http://project.dml.cz
The purpose of this article is to introduce the inductive tensor topology in the class of all topological vector spaces (abbreviated TVS's) and to investigate its properties, especially, in connection with the projective tensor topology (cf.[11]). As to the methodology of this research, the present paper is a continuation of [10] and [11].

Let us note in advance and quite shortly, some general remarks concerning the subject of the following treatise. The taking of the tensor product under the inductive topology represents a bifunctor in the category \( \mathcal{C} \) of all TVS's (or in the category of all topological abelian groups). Further, if \( E \) and \( F \) are, e.g., in \( \mathcal{C} \), then \( E \oplus F \) topologized in such a manner is an object in \( \mathcal{C} \) - which, together with the bimorphism \( (x, y) \mapsto x \oplus y \), realizes the tensor product defined axiomatically in [61,p.134], in the category \( \mathcal{C} \).

The theory of the usual projective tensor topology possesses, in the class of all locally convex spaces, two different aspects - namely the metric and the projective ones. In what follows, we shall extend, as it was partially done also in [11], to the class of all TVS's, a part of the theory of topological tensor products which is based on the projective properties of the topology in question.
The metric properties of topological tensor products will be explored later in a separate paper.

1. Tensor products of abelian groups

In this section, we shall topologize the tensor product $E \otimes F$ of two abelian groups in a prescribed way as to make continuous certain families of $\mathbb{Z}$-linear mappings (with respect to the module $\mathbb{Z}$ of all integers) on $E \otimes F$.

Suppose that $E$ and $F$ are two fixed topological abelian groups. Denote by $\mathcal{Q}$ the category of all topological abelian groups and by $\mathcal{J}(G)$, $G \in \mathcal{Q}$, the collection of all $\mathbb{Z}$-bilinear separately continuous mappings from $E \times F$ into $G$. Let $\mathcal{J}$ be the union of all systems $\mathcal{J}(G)$, $G \in \mathcal{Q}$. If $\mu$ is a $\mathbb{Z}$-bilinear mapping of $E \times F$, then we often write $\tilde{\mu}$ for its associated $\mathbb{Z}$-linear extension to $E \otimes F$.

Any such transformation $\tilde{\mu}$, $\mu \in \mathcal{J}(G)$ determines a topology on $E \otimes F$, the so-called inverse image with respect to $\tilde{\mu}$ of the topology on $G$, compatible with its group structure. The least upper bound $T(\mathcal{J})$ of all group topologies determined by the family $\{\tilde{\mu}; \mu \in \mathcal{J}\}$ is the projectively generated topology by the family $\mathcal{J}$ and it will be termed the inductive tensor group topology on $E \otimes F$.

Theorem 1. If $E$ and $F$ are two abelian topological groups, then $T(\mathcal{J})$ is the unique group topology on $E \otimes F$ with the following properties:

1. The mapping $\phi : (x, y) \mapsto x \otimes y$ from $E \times F$ into $(E \otimes F, T(\mathcal{J}))$ is separately continuous.

2. For any $G \in \mathcal{Q}$ and for each $\mu \in \mathcal{J}(G)$ the $\mathbb{Z}$-linear extension $\tilde{\mu}$ is continuous on $E \otimes F$ under the
Proof. First, we prove the separate continuity of $g$. Let $W$ be an arbitrary neighborhood of the zero element in $E \otimes F$ under the topology $T(Y)$. There are some $G_{\alpha_i} \in G$, $\omega_{\alpha_i} \in Y(G_{\alpha_i})$, $1 \leq i \leq m$, such that for suitable neighborhoods $W_{\alpha_i}$ in $G_{\alpha_i}$ ($1 \leq i \leq m$), it holds

$$\bigcap_{i=1}^{m} \omega_{\alpha_i}^{-1}(W_{\alpha_i}) \subseteq W.$$ 

For a fixed $\psi$ in $F$ we take the neighborhoods $V_{\alpha_i}$ in $E$ so that $\omega_{\alpha_i}(V_{\alpha_i}, \psi) \subseteq W_{\alpha_i}$ ($1 \leq i \leq m$). Hence, for any $x$ in $V = \bigcap_{i=1}^{m} V_{\alpha_i}$ we obtain $x \otimes \psi \in W$. This proves the continuity of $x \mapsto x \otimes \psi$. Similarly, one proves the continuity of $\psi \mapsto x \otimes \psi$. The property $2^0$ follows immediately from the definition of $T(Y)$. The uniqueness of such a topology is clear.

Remark 1. (a) Let $E_i$, $F_i$ ($i = 1, 2$) be topological groups, $\omega_i$ ($i = 1, 2$) $\mathbb{Z}$-linear continuous mappings from $E_i$ into $F_i$ ($i = 1, 2$). Then the tensor mapping $\omega_1 \otimes \omega_2$ from $E_1 \otimes F_1$ into $E_2 \otimes F_2$ is continuous under the inductive tensor topologies.

(b) If $E_i$ ($1 \leq i \leq m$) and $F$ are topological groups, then the spaces $(\prod_{i=1}^{m} E_i) \otimes F$ and $\prod_{i=1}^{m} (E_i \otimes F)$ are topologically isomorphic under the inductive tensor topologies.

The next task is to describe the structure of neighborhoods of the inductive tensor topology. First, let us consider a somewhat modified situation of Theorem 1.

Let $\nu = (x_m, m \in N)$ and $\mu = (\nu_m, m \in N)$ be two, otherwise arbitrary, sequences of elements of $E$ and
\( F ( N ) \) is the set of all positive integers. For any two sequences \((U_n; n \in N)\) and \((V_n; n \in N)\) of neighborhoods in \( E \) and \( F \), respectively, we set

\[
\Omega ((U_n), (V_n)) = \bigcup_{n \in N} (U_n \otimes V_n + x_n \otimes V_n + \ldots + U_m \otimes V_m).
\]

If \( U_n \) and \( V_n \) are variables, then the family of all such subsets determines a base of a filter in \( E \otimes F \). If \( U_n' \) and \( V_n' \), \( n \in N \) are neighborhoods in \( E \) and \( F \), respectively, with \( U_n' + U_n' \subseteq U_n \), \( V_n' + V_n' \subseteq V_n \), then

\[
\Omega ((U_n'), (V_n')) + \Omega ((U_n'), (V_n')) \subseteq \Omega ((U_n), (V_n)).
\]

Hence, the family of all \( \Omega ((U_n), (V_n)) \) defines on \( E \otimes F \) a group topology (depending on \( \nu \) and \( \mu \)); it will be denoted by \( T(\nu, \mu) \).

**Theorem 2.** The topology \( T(\nu, \mu) \) is the unique group topology on \( E \otimes F \) with the properties:

1. The mappings \( x \to x \otimes \nu_n \), \( \gamma \to \gamma_n \otimes \gamma \) are continuous for each \( n \in N \).

2. If \( \mu \) is a \( \mathbb{Z} \)-bilinear separately continuous mapping with respect to \( \nu = \nu_m \) and \( x = \gamma_n \), \( n \in N \), of \( E \times F \) into a topological group \( G \), then \( \mu \) is continuous on \( E \otimes F \) under \( T(\nu, \mu) \).

**Proof.** The property 1 and the uniqueness is clear. We prove the property 2. Let \( \mu \) be a \( \mathbb{Z} \)-bilinear separately continuous mapping with respect to \( x = \gamma_n \), \( \psi = \psi_m \), \( n \in N \), of \( E \times F \) into \( G \). If \( W \) is a neighborhood in \( G \), then we take a sequence \((W_i; i \in N)\) of neighborhoods in \( G \) with \( W_i + W_i \subseteq W, \ldots, W_{n+1} + W_{n+1} \subseteq W_n, \ldots \). If \( \mu(x_i, \psi_i) \in W_{i+1}, \mu(x_i, \psi_i) \in W_{i+1} \), then evidently
\( \mathcal{N} \left( \bigcap \left( U_i, (V_i) \right) \right) \subseteq W \),

hence \( \mathcal{N} \) is continuous on \( E \otimes F \) under the topology \( T(\nu, \mu) \).

The topology \( T(\nu, \mu) \) is projectively generated, according to Theorem 2, by a more extensive family of \( \mathbb{Z} \)-bilinear mappings than \( \mathcal{Y} \), whence \( T(\nu, \mu) \geq T(\mathcal{Y}) \). One might, therefore, expect that it will yield a certain approximation for the inductive topology. Indeed, it holds:

**Theorem 3.** The inductive tensor topology \( T(\mathcal{Y}) \) is the greatest lower bound of all group topologies \( T(\nu, \mu) \), where \( \nu \) and \( \mu \) run over the collection of all sequences in \( E \) and \( F \).

**Proof.** Denote by \( T' \) the greatest lower bound of the system \( (T(\nu, \mu); \nu, \mu) \). It suffices to prove that \( T' \leq T(\mathcal{Y}) \). Let \( x_0 \) and \( y_0 \) be arbitrary points of \( E \) and \( F \). Taking \( x_0 \) and \( y_0 \) such that \( x_0 \in \nu_0, y_0 \in \mu_0 \), we see that the canonical mappings \( x \rightarrow x \otimes y_0, y \rightarrow x_0 \otimes y \) are continuous if we consider \( E \otimes F \) under the topology \( T(\nu_0, \mu_0) \), hence the stated mappings are continuous also from \( E \) and \( F \), respectively, into \( (E \otimes F, T') \). From Theorem 1 we may now derive that \( T(\mathcal{Y}) \geq T' \).

Thus, the topology \( T(\mathcal{Y}) \) is the least upper bound of the topologies generated by single mappings \( \mu \in \mathcal{Y} \) and, at the same time, the greatest lower bound of the topologies \( T(\nu, \mu), \nu \) and \( \mu \) variables. If we intend now to describe the structure of neighborhoods of the inductive tensor topology, then the first stated property may afford scarcely a convenient approach for principal reasons to the imme-
mediate solution of the presented problem. This is why we hav-
e introduced the topologies $T(\varphi, \mu)$ by means of which we shall now construct the base of neighborhoods of $T(\mathcal{Y})$.

For any sequence $\mu = (\psi_n; n \in \mathbb{N})$ in $F$ and for any sequence $\nu = (x_n; n \in \mathbb{N})$ in $E$ we take neighborhoods $U_{i0}^\mu$ and $V_{i0}^\nu$, $i \in \mathbb{N}$, the choice is otherwise arbitrary, of the zero element in $E$ and $F$. Denote

$$
(2) \quad \Phi_0 = \bigcup_{\nu, \mu} \bigcup_{i_0} \left( U_{i_0}^\mu \otimes \psi_i + x_i \otimes V_{i_0}^\nu + \ldots \right.
\quad \left. + U_{m_0}^\mu \otimes \psi_m + x_m \otimes V_{m_0}^\nu \right),
$$

where $\nu$ and $\mu$ run over the collection of all sequences in $E$ and $F$. What we are going to do is, roughly speaking, to enlarge the subsets of the form (2) in such a way in order to obtain subsets in $E \otimes F$ satisfying the additive axiom of some group topology.

First, take neighborhoods $U_{i1}^\mu$, $V_{i1}^\nu$ in the spaces $E$, $F$ for any $i = 1, 2, \ldots$ and for each $\mu, \nu$ and such that

$$
U_{i1}^\mu + U_{i1}^\mu \subseteq U_{i0}^\mu, \quad V_{i1}^\nu + V_{i1}^\nu \subseteq V_{i0}^\nu.
$$

We now form the subsets $(n = 1, 2, \ldots )$

$$
(3) \quad (U_{i1}^\mu \otimes \psi_i + U_{i1}^\mu \otimes \psi_i + U_{21}^\mu \otimes \psi_2 + U_{21}^\mu \otimes \psi_2 + \ldots 
\quad \ldots + U_{m1}^\mu \otimes \psi_m + x_m \otimes V_{m1}^\nu + x_m \otimes V_{m1}^\nu + x_m \otimes V_{m1}^\nu + \ldots 
\quad \ldots + x_m \otimes V_{m1}^\nu + x_m \otimes V_{m1}^\nu )
$$

for any two sequences $\mu = (\psi_i; i \in \mathbb{N})$, $\mu' = (\psi_i'; i \in \mathbb{N})$ in $F$ and $\nu = (x_i; i \in \mathbb{N})$, $\nu' = (x_i'; i \in \mathbb{N})$ in $E$. The union of all subsets given by formula (3) which are con-
structed to all couples of sequences $(\mu, \mu')$ and $(\nu, \nu')$,
will be denoted by $\Phi_0$. Define $\Phi_1 = \Phi_0 \cup \Phi_1$. Rearranging the systems of sequences $\{\mu^1, \nu^1, (\mu', \mu'')\}$ and $\{\nu, \nu'\}$, we repeat the same procedure with $\Phi_1$ (with respect to the new indices) and obtain the subset $\Phi_2$.

Suppose that $\Phi_0 \subseteq \Phi_1 \subseteq \ldots \subseteq \Phi_{n-1}$ have been determined; let the defining subsets for $\Phi_{n-1}$ be of the form

$$U_{i, \alpha}^\mu \otimes V_{i, \alpha}^\nu, ~ x_{i, \alpha}^\mu \otimes x_{i, \alpha}^\nu,$$

where $\mu$ and $\nu$ stand for indices corresponding to the family $\Phi_{n-1}$. We now select neighborhoods

$$U_{i, \alpha+1}^\mu, ~ V_{i, \alpha+1}^\nu$$

with

$$U_{i, \alpha}^\mu + U_{i, \alpha+1}^\mu \subseteq U_{i, \alpha}^\mu, ~ V_{i, \alpha}^\nu + V_{i, \alpha+1}^\nu \subseteq V_{i, \alpha}^\nu.$$

Define, for any couple $(\mu', \mu''), (\nu', \nu'')$ of indices, the subsets of the form (3), where $U_{i, \alpha}^\mu$, $V_{i, \alpha}^\nu$, and $U_{i, \alpha}^{\mu'}$, $V_{i, \alpha}^{\nu'}$ are replaced by $U_{i, \alpha+1}^\mu$, $V_{i, \alpha+1}^\nu$, and by $U_{i, \alpha+1}^{\mu'}$, $V_{i, \alpha+1}^{\nu'}$. The union of all such subsets with respect to the couples $(\mu, \mu')$, $(\nu, \nu')$ determines now $\Phi_{n+1}$.

Putting $\Phi_{n+1} = \Phi_0 \cup \Phi_{n+1}$, we obtain an increasing sequence $(\Phi_n; n \geq 1)$ and, finally, we define

$$(4) \quad \Phi_\omega = \bigcup_{n \geq 1} \Phi_n.$$

Hence, we construct to any subset $\Phi_\alpha$ of the form (2) the set $\Phi_\omega$ (the last one may be called the additive envelope of $\Phi_\alpha$ in spite of a certain indeterminacy consisting in the liberty of choices of $U_{i, \alpha}^\mu$ and $V_{i, \alpha}^\nu$). The system of all such $\Phi_\omega$ forms a base of neighborhoods of a group topology in $E \otimes F$.

To prove this statement, it suffices to determine a subset $Y_\omega$ of the form (4) satisfying, moreover, the inclusion $Y_\omega + + Y_\omega \subseteq Y_\omega$ for any $\Phi_\omega$. Namely, we may define the subset $Y_\omega$ by

$$Y_\omega = \bigcup_{\mu, \nu, \alpha \geq 1} (U_{\mu, \alpha}^\mu \otimes V_{\nu, \alpha}^\nu + x_{\mu, \alpha}^\mu \otimes x_{\nu, \alpha}^\nu + \ldots + U_{\mu, 1}^\mu \otimes V_{\nu, 1}^\nu + x_{\mu, 1}^\mu \otimes x_{\nu, 1}^\nu).$$
and proceed in the same way as before (i.e., \( y_{m} \) depends on the neighborhoods \( U_{i, a+1}^{a}, V_{i, a+1}^{a} \)). Evidently, 
\( y_{w} = \bigcup y_{m} \) satisfies the requested relation.

**Theorem 4.** The inductive tensor topology \( T(y) \) on \( E \otimes F \) coincides with the group topology which has the subsets of the form (4) as a base of neighborhoods.

**Proof.** It suffices to prove that the group topology \( T \) defined above satisfies the properties \( 1^o \) and \( 2^o \) of Theorem 1. But \( 1^o \) is clear and the proof of \( 2^o \) may be carried out as in Theorem 2.

**Remark 2.** (a) Suppose that \( \mu_{i} (i = 1, 2) \) in Remark 1 (a), are both open. Then \( \mu_{1} \otimes \mu_{2} \) is an open mapping from \( E_{1} \otimes F_{1} \) onto \( E_{2} \otimes F_{2} \) under the inductive tensor topologies. The proof follows from the construction of the neighborhoods \( \phi_{w} \).

(b) Let \( E \) and \( F \) be two topological groups, \( M \) a subgroup in \( E \), \( N \) a subgroup in \( F \). Then the quotient group \( E \otimes F / \Gamma(M, N) \) is topologically isomorphic with \( (E/M) \otimes (F/N) \), where \( \Gamma(M, N) \) stands for the subgroup in \( E \otimes F \) generated by all \( x \otimes y, x \in M \) or \( y \in N \).

**Remark 3.** Suppose that \( \{ C_{a}(X), \partial_{a} \} \) is a chain complex, \( C_{a}(X) \) a topological abelian group, \( \partial_{a}: C_{a}(X) \rightarrow C_{a-1}(X) \) a continuous boundary homomorphism. The category of all such chains complexes \( X \) will be denoted by \( \partial_{y} \). A pair of mappings \( (\varphi, \psi) \) is termed to form a direct couple if the sequence

\[ 0 \rightarrow X \xrightarrow{\varphi} X' \xrightarrow{\psi} X'' \rightarrow 0 \]

is exact, \( \varphi \) and \( \psi \) are open homomorphisms and \( g_{m}(C_{m}(X)) \)
possess topological complements in $C_n(X')$, $m, m \in N$, where $X$, $X'$ and $X''$ are chain complexes.

Any such couple defines, in the same way as in [2] (see pp.126-127), a continuous boundary homomorphism $\partial_x : H_n(X'') \rightarrow H_{n-1}(X)$. Similarly as in Theorem 4.9 of [2] (cf. p.130), we obtain that the system $(H_n(X), f_m, \partial_x)$ is a homology theory on the $\mathcal{H}$-category $\mathcal{C}_\mathcal{H}$ of chain complexes with couples (with direct couples) into the category $\mathcal{C}$. Let $\mathcal{G}$ be the functor on $\mathcal{C}_\mathcal{H}$ with direct couples defined by

$\mathcal{G} : K \rightarrow K \otimes G$ (with $\mathcal{W}$ or with $T(\mathcal{Y})$-topology; cf. [11]),

$\mathcal{G} : f \rightarrow f \otimes 1_G$, where $1_G$ is the identical mapping of $G$. Then $\mathcal{G}$ is (compare with Theorem 11.2 of [2], p. 150) a covariant $\mathcal{H}$-functor on $\mathcal{C}_\mathcal{H}$. Hence, $\mathcal{G}$ defines a homology theory with the coefficient group $G$ on $\mathcal{C}_\mathcal{H}$ (with the homology theory on $\mathcal{C}_\mathcal{H}$).

2. Inductive tensor topologies and vector spaces

In further discussion, we consider vector spaces over the same and usual field of scalars.

Let $E$ and $F$ be two fixed TVS's. The class of all TVS's will be denoted by $\mathcal{E}$. For any $G \in \mathcal{E}$ we mean by $\mathcal{F}(G)$ the family of all separately continuous bilinear mappings of $E \times F$ into $G$; $\mathcal{Y}$ stands for the union of all systems $\mathcal{F}(G)$, $G \in \mathcal{E}$. The vector topology $T(\mathcal{Y})$ on $E \otimes F$ generated by the family $\mathcal{Y}$ will be termed the inductive tensor topology on $E \otimes F$.
Theorem 3. The topology $T(Y)$ is the unique vector topology on $E \otimes F$ with the properties:

1° The canonical bilinear mapping $\gamma : E \times F \to E \otimes F$ is separately continuous.

2° For any separately continuous bilinear mapping $\mu$ of $E \times F$ into a TVS $G$, its linear extension $\widetilde{\mu}$ is continuous on $(E \otimes F, T(Y))$.

The proof is analogous to that of Theorem 1.

If $\mathcal{E}'$ is the class of all locally convex spaces, then $Y'$ means the union of all systems $\mathcal{Y}(G), G \in \mathcal{E}'$. It is immediate that the projectively generated topology $T(Y')$ by the system $\mathcal{Y}'$ is locally convex and that $T(Y') \leq T(Y)$ on $E \otimes F$.

The topology $T(Y')$ will be called the inductive tensor $\ell$-topology on $E \otimes F$.

Theorem 6. (a) The inductive tensor $\ell$-topology $T(Y')$ is the uniquely determined locally convex topology on $E \otimes F$ satisfying the property 1° of Theorem 5 and:

2° For any $G \in \mathcal{E}'$ and for each $\mu \in \mathcal{Y}'(G)$ the linear extension $\widetilde{\mu}$ is continuous on $(E \otimes F, T(Y'))$.

(b) The topology $T(Y')$ is the finest locally convex topology $t$ on $E \otimes F$ with $t \leq T(Y)$.

(c) If $E$ and $F$ are locally convex, then $T(Y')$ coincides with the usual inductive tensor topology (in the sense of A. Grothendieck; cf.[3]) on $E \otimes F$.

(d) Let $(E, t_1)$ and $(F, t_2)$ be two TVS's. Denote by $t'_1$ and $t'_2$ the finest locally convex topologies on $E$ and $F$, respectively, satisfying $t'_1 \leq t_1$ and $t'_2 \leq t_2$.
If \( E' = (E, t_1') \), \( F' = (F, t_2') \), then the spaces \( E' \otimes F' \) and \( E \otimes F \) are topologically isomorphic under the inductive \( \ell \)-topologies.

The proof is clear.

**Remark 4.** The tensor products \((E \otimes F, T(\mathcal{F}))\) and \((E \otimes F, T(\mathcal{F}'))\) have the same topological dual, but, in general, they need not be topologically isomorphic. It is easy to verify that, e.g., on the metrizable space of all measurable functions on a measure space \((X, \mathcal{S}, \mu)\) (cf. [5], § 42), the topology \( T(\mathcal{F}')\) is actually strictly coarser than \( T(\mathcal{F})\).

**Theorem 7.** (a) For any TVS’s \( E \) and \( F \) the topology \( T(\mathcal{F}) \) is the greatest lower bound of the group topologies \( T(\mathcal{F}), T(\mathcal{F}'), \) determined by the neighborhoods of the form (1).

(b) The topology \( T(\mathcal{F}) \) is defined as the vector topology on \( E \otimes F \), \( E \) and \( F \) TVS’s, having the subsets of the form (4) as a base of neighborhoods.

(c) The convex envelopes of all \( \Phi_0 \) (see formula (2)) form the base of neighborhoods of the inductive tensor \( \ell \)-topology \( T(\mathcal{F}'). \)

**Proof.** The subset \( \Phi_0 \) (see formula (2)) is evidently absorbent and balanced whenever all \( U_{i0}^{\alpha}, V_{i0}^{\beta} \) are balanced neighborhoods. The additive axiom is clear. This implies (c). The statement (b) follows from Theorem 4, because \( \Phi_{\mathcal{F}} \) is absorbent and balanced whenever all \( U_{i,\alpha}^{\omega}, V_{i,\alpha}^{\omega} \) are of this kind. The statement (a) is clear.

**Remark 5.** If \( E \) and \( F \) are two TVS’s, then the sequences \( \mu \) and \( \nu \) in (2), Section 1, may be selected from algebraic bases of the vector spaces \( E \) and \( F \). In fact, any
subset of the form (4) is absorbent in $E \otimes F$ in this case.

Theorem 8. If $E$ and $F$ have countable algebraic bases

$(x_n; n \in N)$ and $(y_n; n \in N)$, then the subsets

(5) $\bigcap \left(\bigcup \left( U_n \in V_n \right) \right) = \bigcup_{i=1}^{\infty} \left( U_i \in V_i + x_i \in y_i + ... \right.

\left. ... + U_N \in V_N \right)

$U_n$ neighborhoods in $E$ and $V_n$ neighborhoods in $F$, determine the inductive tensor topology $T(\mathcal{Y})$.

The proof is elementary.

Corollary. Suppose that $E$ and $F$ are two locally convex spaces with countable algebraic bases. Then the inductive topology $T(\mathcal{Y})$ coincides with the inductive $\mathcal{L}$-topology $T(\mathcal{Y}')$.

Remark 6. We could also consider the tensor product $E \otimes F$ under one-sided topologies for any pair of TVS's.

For symmetrical reasons we shall illustrate this idea only for the one-sided topology generated by the space $E$. Suppose further, without loss of generality and for simplification, that $(y_n; n \in N)$ is a countable algebraic basis of $F$. Define a topology $T(E)$ by a base of neighborhoods

$\Omega(U_m) = \bigcup_{i=1}^{\infty} \left( U_i \in V_i + U_m \otimes y_m \right).

The following properties are now immediate:

(a) $T(E)$ is a vector topology on $E \otimes F$.

(b) $T(E)$ is separated whenever $E$ is separated.

(c) If $E'$ is a subspace of $E$, then $E' \otimes F$ is a topological vector subspace of $E \otimes F$.

(d) The topology $T(E)$ may be characterized by analogous properties as in Theorem 5.
3. Further properties of tensor topologies

The $W$-topology $T(\mathcal{B})$ (cf. [11]) on $E \otimes F$ (denoted also by $E \otimes^w F$), projectively generated by the system $\mathcal{B}$ of all continuous bilinear mappings from $E \times F$ into $G$, $G \in \mathcal{E}$, is, in general, coarser than $T(\mathcal{S})$ (denoted also by $E \otimes^s F$). It is not hard to show examples when $T(\mathcal{S})$ is strictly finer than $T(\mathcal{B})$. Take, e.g., a locally convex non-normable space $E$, $F$ means the topological dual of $E$. Since the separately continuous bilinear function $(\omega, x) \mapsto \langle \omega, x \rangle$ is not continuous on $E \times F$, we may conclude (similarly as in [3]) that the tensor topologies in question are different on $E \otimes F$. On the other hand, if $E$ and $F$ are both metrizable, $E$ $\gamma$-barrelled, then any separately continuous bilinear mapping into an arbitrary TVS is continuous on $E \times F$ (cf. [12]).

This implies:

Theorem 2. Let $E$ and $F$ be two metrizable TVS's, $E$ $\gamma$-barrelled. Then $T(\mathcal{B})$ and $T(\mathcal{S})$ coincide on $E \otimes F$.

Consider now two spectra $(E_m; m \in \mathbb{N})$ and $(F_m; m \in \mathbb{N})$ of TVS's with continuous injections $i_m$ and $j_m$, respectively. Denote $E = \lim \text{ind} E_m$ and $F = \lim \text{ind} F_m$. Then it holds:

Theorem 10. Under the stated conditions the inductive limit $\lim \text{ind} (E_m \otimes^s F_m)$ is topologically isomorphic with $E \otimes^s F$. If, in addition, all $E_m$ and $F_m$ are $F$-spaces, then $\lim \text{ind} (E_m \otimes^w F_m)$ is topologically isomorphic with $E \otimes^w F$.

- 757 -
Proof. Denote by $I$ the identical mapping from
\[ \lim \operatorname{ind} \left( E_n \otimes \mathbb{F} \right) \]
on $E_n \otimes \mathbb{F}$. Since the restriction of $I$ to each $E_n \otimes \mathbb{F}$ (equal to $i_n \otimes i_n$) is continuous, we derive from the definition of the inductive limit topology the continuity of $I$.

On the other hand, it suffices to prove the separate continuity of the bilinear mapping $(x, y) \mapsto x \otimes y$ from $E \times \mathbb{F}$ into $\lim \operatorname{ind} \left( E_n \otimes \mathbb{F} \right)$. Take therefore some $y \in \mathbb{F}$, for a certain $m$ we obtain $y \in \mathbb{F}_m$. The restriction $u_m$ of the linear mapping $u: x \mapsto x \otimes y$ to each $E_m$, $m \geq m$, is evidently continuous. But the same statement is true for the restriction of $u$ to $E$, $1 \leq n < m$. This implies the continuity of $u$ on $E$.

Similarly, one proves the continuity of $v: y \mapsto x \otimes y$.

Remark 7. From the precedent theorem we can derive an example when the statement of Remark 1 (b) (for TVS's) need not be true for infinite topological products. Consider now the spaces over the field of real numbers and take for any $i \in \mathbb{N}$ the real line $\mathbb{R}_i = \mathbb{R}$. Denoting by $E$ the topological product $\prod_{i \in \mathbb{N}} \mathbb{R}_i$ and by $\mathbb{F}$ the topological direct sum $\sum_{i \in \mathbb{N}} \mathbb{R}_i$, we obtain, as in [3], from Theorem 10, since $\sum_{i \in \mathbb{N}} \mathbb{R}_i = \lim \operatorname{ind} \left( \prod_{i \in \mathbb{N}} \mathbb{R}_i \right)$ that
\[ E \otimes \mathbb{F} = \lim \operatorname{ind} \left( E \otimes \sum_{i \in \mathbb{N}} \mathbb{R}_i \right) = \lim \operatorname{ind} \left[ \prod_{i \in \mathbb{N}} \mathbb{E}_i \right] = \prod_{i \in \mathbb{N}} \mathbb{E}_i , \]
where $\mathbb{E}_i$ is topologically isomorphic with $E$ for each $i \in \mathbb{N}$. On the other hand, it holds $\prod_{i \in \mathbb{N}} \left( \mathbb{R}_i \otimes \mathbb{F} \right) = \sum_{i \in \mathbb{N}} \mathbb{F}_i$, where $\mathbb{F}_i$ is topologically isomorphic with $F$ for any $i \in \mathbb{N}$.

One observes that $T(\mathcal{F})$ is the finest vector topo-
logy on $E \otimes F$ for which $(x, \psi) \rightarrow x \otimes \psi$ is separately continuous. Similarly, $T(\mathcal{B})$ is the maximal vector topology on $E \otimes F$ under which the canonical bilinear mapping $(x, \psi) \rightarrow x \otimes \psi$ is continuous. These properties suggest also the possibility of extending the theory of completions of tensor products to a particular case when the spaces in question are non-metrizable. Denote, therefore, by $\mathcal{V}_w(E)$ (by $\mathcal{V}_w'(E)$) the finest vector (the finest locally convex) topology on a vector space $E$. Putting $E_w = (E, \mathcal{V}_w(E))$, $E'_w = (E, \mathcal{V}_w'(E))$, we may now formulate in these terms:

**Theorem 11.** Let $E$ and $F$ be two vector spaces.

(a) For the topology $T(\mathcal{V})$ of $E_w \otimes F_w$ it holds

$$T(\mathcal{V}) = \mathcal{V}_w(E \otimes F).$$

(b) The space $E'_w \otimes F'_w$ under the topology $T(\mathcal{V}')$ is topologically isomorphic with $E \otimes F$ under $\mathcal{V}_w'(E \otimes F)$.

(c) If, in particular, $E$ and $F$ possess countable algebraic bases, then the space $E_w \otimes F_w$ under $T(\mathcal{V})$ is topologically isomorphic with $E'_w \otimes F'_w$ under $T(\mathcal{V}')$.

(d) The space $E_w \otimes F_w$ under $T(\mathcal{V})$ is complete. Similarly, $E'_w \otimes F'_w$ under $T(\mathcal{V}')$ is complete.

**Proof.** For any $\psi \in F$ the inverse image of the topology $\mathcal{V}_w(E \otimes F)$ defined by $\mu : x \rightarrow x \otimes \psi$ is a vector topology on $E$ coarser than $\mathcal{V}_w(E)$ for which $\mu$ is continuous. Hence $\mu$ is continuous on $E_w$. For symmetrical reasons $\psi \rightarrow x \otimes \psi$ is continuous on $F_w$ for any $x \in E$. Thus $T(\mathcal{V}) = \mathcal{V}_w(E \otimes F)$. Similarly, one proves (b).
If $E$ and $F$ are, moreover, with countable algebraic bases, then $E \otimes F$ possess also a countable algebraic basis, hence $\tau'_\omega (E \otimes F) = \tau'_\omega (E \otimes F)$. From (a) and (b) we may now derive (c).

The finest vector topology of a vector space $G$ coincides with the direct sum topology of a system of finite-dimensional $F$-spaces, hence $G_{\omega r}$ is complete (cf. [14]). This implies (d) by taking $G = E \otimes F$. The same arguments may be used in proving the second statement of (d).

Since the topology $T(G)$ (and consequently, $T(G')$) is identical under the conditions of Theorem 11 (c) with the finest vector topology $\tau'_\omega (E \otimes F)$, one is tempted to ask whether in this case also the locally convex projective topology $T(G')$ (in the sense of A. Grothendieck) does not coincide with $\tau'_\omega (E \otimes F)$. If it were so, then we could obtain, in view of the above stated equality, that $E_{\omega r} \otimes F_{\omega r}$ is complete under $T(G) = \tau'_\omega (E \otimes F)$ (hence under $T(G') = \tau'_\omega (E \otimes F)$). To prove our conjecture, it suffices to establish that, under the conditions of Theorem 11 (c), any separately continuous bilinear mapping of $E_{\omega r} \times F_{\omega r}$ into a locally convex space $G$ is continuous.

Suppose therefore that $\mu$ is a separately continuous mapping from $E_{\omega r} \times F_{\omega r}$ into a locally convex space $G$. If $\psi_i \in F$, $1 \leq i \leq n$, then for any neighborhood $U$ in $G$ there is a neighborhood $V$ in $E$ such that $\mu(x, \psi_i) \in U$ for all $x \in V$, $1 \leq i \leq n$. This implies for a finite convex combination $\sum \lambda_i = 1$, $\lambda_i \geq 0$, the inclusion $\mu(x, \sum \lambda_i \psi_i) = \sum \lambda_i \cdot \mu(x, \psi_i) \in U$. 

- 760 -
for all $x \in Y$. But, taking for $\psi_i$ linearly independent points of $E_m = \bigoplus_{i \in I} R_i$ (the notation is the same as in Remark 7) we may observe that $\mu$ is hypoequicontinuous in the sense of A. Grothendieck, i.e., with respect to the family of all bounded subsets in $E$ and $F$. But $E'_\nu$ and $F'_\nu$ are DF-spaces. From the well-known theorem of A. Grothendieck (cf.[41]) we conclude that $\mu$ is continuous on $E \times F$. Thus we proved:

**Theorem 12.** Let $E$ and $F$ be two vector spaces with countable algebraic bases. Then on $E'_\nu \otimes F'_\nu$ it holds

$$T(\mathcal{B}) = T(\mathcal{B}') = \tau'_\nu(E \otimes F) = \tau'_\nu(E \otimes F).$$

Especially, the space $E'_\nu \otimes F'_\nu$ under $T(\mathcal{B})$ (under $T(\mathcal{B}')$, respectively) is complete.

An open question concerning the topologies $T(\mathcal{Y})$ and $T(\mathcal{B})$ is the separatedness of $E \otimes F$. Our hypothesis is that $E \otimes F$ under $T(\mathcal{B})$ (hence under $T(\mathcal{Y})$) is separated whenever $E$ and $F$ are separated. In any case, if there are separated locally convex topologies coarser than the original on $E$ and $F$, then $T(\mathcal{B})$ is separated on $E \otimes F$. Such situation occurs, e.g., for the spaces $L^\nu$, $0 < \nu < 1$.

We shall now note an example which may serve as a prototype for the decision of the separatedness of $E \otimes F$ in certain cases.

**Example.** Suppose that $\Sigma$ is separated and that $\mathcal{F} = \mathcal{F}(X)$ is a space of bounded scalar-valued functions on $X$ with the topology of uniform convergence on a system $\mathcal{T} = \{B\}$ of subsets in $X$, $\cup B = X$. Denote furt-
her, by \( \mathcal{F}_E(X) \), a vector space of vector-valued functions on \( X \) with values in \( E \), bounded on each \( \mathcal{B} \in \mathcal{T} \) and such that \( a \cdot f \in \mathcal{F}_E(X) \) for all \( f \in \mathcal{F}(X) \), \( a \in E \). The topology of \( \mathcal{F}_E(X) \) is that of uniform convergence on \( \mathcal{T} \). The bilinear mapping \( \mu : (a, f) \rightarrow a \cdot f \) of \( E \times \mathcal{F} \) into \( \mathcal{F}_E(X) \) is evidently continuous, hence it defines a linear continuous mapping \( \tilde{\mu} \) of \( E \otimes \mathcal{F} \) under \( T(\mathcal{B}) \) into \( \mathcal{F}_E(X) \). Further, one may verify that \( \tilde{\mu} \) is one-to-one (by Lemma 1.1 of [9]). Since \( \mathcal{F}_E(X) \) is separated, the space \( E \otimes \mathcal{F} \) under \( T(\mathcal{B}) \) is also separated.

4. The problem of topologies

As it was established in [11], the space \( \mathcal{B}(E, F; G) \) of all continuous bilinear mappings from \( E \times F \) into \( G \) is algebraically isomorphic with the space \( \mathcal{L}(E \otimes F; G) \) of all continuous linear mappings from \( E \otimes F \) into \( G \). It is natural to ask under what conditions this algebraic isomorphism is a topological one \((\mathcal{B}(E, F; G) \text{ is taken with the topology of bi-bounded convergence; } \mathcal{L}(E \otimes F; G) \text{ is under the strong topology).})

If \( E \) and \( F \) are locally convex, then any bounded set in \( E \otimes_w F \) is also bounded under the projective locally convex topology. Consequently, the stated correspondence will be a topological isomorphism \((G \text{ locally convex) whenever it is a topological isomorphism for the locally convex projective topology. Such situation occurs, e.g., if } E \text{ and } F \text{ are DF-spaces.}

- 762 -
Recall that a TVS is said to be locally bounded if it contains a bounded neighborhood of the origin (cf. [7]). In [1] it was established that the topology of such a space is generated by a quasi-norm \( x \rightarrow \| x \| \) satisfying, especially, the Bourgin's multiplier property \( \| x + y \| \leq \beta (\| x \| + \| y \| ) \), for a certain multiplier \( \beta \geq 1 \).

**Theorem 13.** Let \( E, F \) and \( G \) be locally bounded spaces. Then \( \mathfrak{B}(E, F; G) \) is topologically isomorphic with \( \mathcal{L}(E \odot F; G) \).

**Proof.** If \( A \) and \( B \) are bounded in \( E \) and \( F \), then also \( A \odot B \) is bounded in \( E \odot F \), hence the canonical isomorphism of \( \mathcal{L}(E \odot F; G) \) onto \( \mathfrak{B}(E, F; G) \) is continuous.

Conversely, let \( C \) be a bounded subset in \( E \odot F \) and let \( W \) be a bounded neighborhood of the origin in \( G \). It will be supposed that \( W \) is defined by means of a quasi-norm as the subset of all \( x \) in \( G \) with \( \| x \| \leq 1 \).

The bounded neighborhoods in \( E \) and \( F \) are denoted by \( U_o \) and \( V_o \), respectively. Since \( C \) is bounded in \( E \odot F \), there is \( \lambda > 0 \) such that
\[
\lambda C \subseteq \bigcup_{n=1}^{\infty} k \cap (U_o \odot V_n) ,
\]
where \( U_m = \frac{1}{2^m} \cdot U_o \), \( V_n = \frac{1}{2^n} \cdot V_o \), \( n \in \mathbb{N} \) (cf. [11]).

Further, we take a number \( m \) such that for the multiplier \( \beta \) of the quasi-norm \( x \rightarrow \| x \| \) it holds \( \beta \leq 2^m \).

Put now
\[
W^* = W + \frac{1}{2^1} \cdot W + \frac{1}{2^2} \cdot W + \ldots + \frac{1}{2^{m-1}} \cdot W .
\]

Since \( W^* \) is bounded in \( G \), we obtain \( W^* \leq \alpha \cdot W \) for a fixed \( \alpha \geq 1 \). Consider now the set
Evidently, \( \mu(\Omega) \subseteq W^{**} \) whenever the bilinear mapping \( \mu \) satisfies \( \mu(U_0, V_o) \subseteq W \). By an elementary and immediate calculation one verifies that if \( z_i \in W^*, 1 \leq i \leq k, k \) arbitrary, then

\[
\left| \frac{1}{2^m} z_1 + \ldots + \frac{1}{2^m} z_k \right| \leq \alpha \left( \frac{1}{2^m} + \frac{1}{2^m} + \ldots + \frac{1}{2^m} \right).
\]

Hence, if \( z \in W^{**} \), then \( z = x + y, x \in W^* \),

\[
y \in \sum_{n=1}^{k} \frac{1}{2^m} \cdot W^*
\]

for some \( \lambda \in \mathbb{N} \), consequently

\[
\|z\| \leq \beta \left( \|x\| + \|y\| \right) \leq 2 \alpha \beta,
\]

i.e., \( W^{**} \subseteq 2 \alpha \beta \cdot W \).

Thus we have proved: To any \( \mu \in \mathcal{B}(E, F; G) \) with \( \mu(U_0, V_o) \subseteq \frac{\lambda}{2 \alpha \beta} W \), the corresponding linear extension \( \tilde{\mu} \) satisfies \( \tilde{\mu}(C) \subseteq W \). This property ensures the continuity of the canonical mapping from \( \mathcal{B}(E, F; G) \) into \( \mathcal{L}(E \odot F; G) \).

By carrying through the same type of arguments used in proving Theorem 13 and taking into account, for essential simplification of the procedure, that \( W^{**} \subseteq 2 \cdot W \) whenever \( W \) is locally convex, one may prove the following statement:

**Theorem 14.** Let \( E \) and \( F \) be two locally bounded spaces, \( G \) locally convex. Then \( \mathcal{B}(E, F; G) \) is topologically isomorphic with \( \mathcal{L}(E \odot F; G) \).

**Remark 8.** Consider now locally convex spaces and denote by \( E \odot F \) the tensor product \( E \odot F \) under the projective tensor topology in the sense of A. Grothendieck. The identical mapping of \( \mathcal{L}(E \odot F; G) \) onto
\( \mathcal{L}(E \otimes_w F ; G) \) is evidently continuous. The space \( \mathcal{L}(E \otimes_e F ; G) \) being isomorphic with \( \mathcal{B}(E, F ; G) \), is also topologically isomorphic with \( \mathcal{L}(E \otimes_w F ; G) \) in the class of DF-spaces and in other special cases (e.g., it holds for \( L^1(\mu) \otimes E, E \) an F-space; cf. [3]).

Hence, the following question is of some interest: Under what conditions \( \mathcal{L}(E \otimes_w F ; G) \) and \( \mathcal{L}(E \otimes_e F ; G) \) are topologically isomorphic?

5. Embedding theorems and concluding remarks

In conclusion, we note certain embedding theorems of tensor products into spaces of linear mappings and bilinear functions.

First, consider two TVS's \( E \) and \( F \). Under \( E^* \) and \( F^* \) we shall understand the strong topological duals of \( E \) and \( F \). If \( x \) is fixed in \( E \), \( \varphi \) in \( F \), then

\[ \mu_{x, \varphi} : (f, \varphi) \rightarrow \langle f, x \rangle \cdot \langle \varphi, \varphi \rangle \]

is a continuous bilinear function on \( E^* \times F^* \), hence an element of \( \mathcal{B}(E^*, F^*) \); it will be assumed \( \mathcal{B}(E^*, F^*) \) to be under the bi-bounded topology. One verifies immediately

\textbf{Theorem 15}. Suppose \( E \) and \( F \) to be locally bounded TVS's.

(a) The bilinear mapping \( (x, \varphi) \rightarrow \mu_{x, \varphi} \) induces a continuous linear mapping \( \mu \) of \( (E \otimes F, T(B)) \) into \( \mathcal{B}(E^*, F^*) \).

In particular, if \( (E, E^*) \) and \( (F, F^*) \) are dual pairs, then \( \mu \) is, moreover, an algebraic isomorphism.
(b) The spaces $E^*, F^*$ are locally bounded and $\mathcal{B}(E^*, F^*)$ is topologically isomorphic with $(E^* \odot_w F^*)^*$.

**Remark 9.** If we eliminate the hypothesis that $E$ and $F$ are locally bounded, then, similarly as in Theorem 15, we could embed $E \otimes F$ into the space $\mathcal{Y}(E^*, F^*)$ of all separately continuous bilinear forms on $E^* \times F^*$. If $\mathcal{Y}(E^*, F^*)$ is under the bi-equicontinuous topology (cf.[3]), then it is not hard to observe that the mapping $(x, y) \mapsto \mu_{x,y}$ is separately continuous, hence that its linear extension is also continuous on $E \otimes F$ under the topology $T(\mathcal{Y})$.

Each $x \otimes y \in E \otimes F$ induces also a continuous linear mapping from $E^*$ into $F$ by formula

$$\nu_{x,y}: f \mapsto \langle x, f \rangle \cdot y.$$ 

Similarly, to any $f \otimes y \in E^* \otimes F$ there corresponds a continuous linear mapping $\nu_{f,y}: x \mapsto \langle x, f \rangle \cdot y$ from $E$ into $F$.

In these terms it is easy to establish:

**Theorem 16.** If $E$ and $F$ are locally bounded, then:

(a) the mapping $(x, y) \mapsto \nu_{x,y}$ induces a linear continuous mapping $\nu$ from $E \otimes_w F$ into $\mathcal{L}(E^*, F)$;

(b) the mapping $(f, y) \mapsto \nu_{f,y}$ induces a linear continuous mapping $\nu$ from $E^* \otimes_w F$ into $\mathcal{L}(E, F)$;

(c) if $(E, E^*)$ is, moreover, a dual pair, then $\nu$ and $\nu$ are one-to-one.

**Remark 10.** Assuming $E^* \otimes_w F^*$ to be separated, we may form the completion $E^* \hat{\otimes}_w F$. If $\mathcal{L}(E^*, F^*)$ is now complete ($E$ and $F$ being locally bounded and separated),
then there exists a continuous linear transformation of the nuclear type \( \tilde{\omega} \) from \( E^* \otimes_w F \) into \( \mathcal{L}(E;F) \). Note that \( \mathcal{L}(E;F) \) is complete in this discussed case whenever \( F \) is complete (cf. [13]).

**Remark 11.** There are still other possibilities to topologize the tensor product \( E \otimes F \); we could determine (as in locally convex spaces) topologies on \( E \otimes F \) lying between \( T(\mathcal{A}) \) and \( T(\mathcal{G}) \). Such a topology may be defined, e.g., as that projectively generated by the system of all hypo-equicontinuous bilinear mappings on \( E \times F \).

**Remark 12.** If \( E \) and \( F \) are both locally bounded, then it is easy to prove that \( E \otimes F \) under \( T(\mathcal{A}) \), or under \( T(\mathcal{G}) \), is \( M \)-bornological. The author does not know whether or not the taking of such tensor products preserves the class of \( M \)-barreled, or of \( M \)-bornological spaces.

**References**


Vysoká škola strojní a textilní
Liberec
Československo

(Oblatum 22.12.1969)