

Ladislav Procházka

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CONCERNING ALMOST DIVISIBLE TORSION FREE ABELIAN GROUPS

Ladislav PROCHÁZKA, Praha

A torsion free group G (all groups here are supposed to be abelian) will be called almost divisible if the set of all positive primes p with $pG \neq G$ is finite. In this note we shall give some conditions that are necessary and sufficient for an almost divisible group G to be completely decomposable. In the paper [2] of D.K. Harrison (see Proposition 5.2) such necessary and sufficient conditions are formulated for the groups of finite rank. But it was shown later (see [3]) that these conditions are not sufficient in general. However, the remark following Theorem 2 shows that the Harrison's conditions are sufficient whenever the corresponding type set is linearly ordered.

If G is a torsion free group, then $\mathcal{I}(G)$ will denote the type set of all non zero elements in G ; G is said to be homogeneous of the type μ if $\mathcal{I}(G)$ consists of one element μ only. For a type μ and a prime p the relation $\mu(p) = \infty$ means that in any height belonging to μ the p -height is ∞ ; the symbols $G(\mu)$, $G^*(\mu)$ and $G^{**}(\mu)$ represent the subgroups of G defined in

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[1, §42]. The rank of a group G is denoted by $\kappa(G)$ and $\kappa_p(G)$ stands for its p -rank (see [5]). But in this note we shall use the relation $\kappa_p(G) = 0$ only; this last relation says that for any finite set $M \subseteq G$ the p -primary component of the torsion group $\{M\}_* / \{M\}$ is reduced ($\{M\}_*$ denotes here the least pure subgroup of G containing M).

First of all we shall prove the following helpful assertion.

Lemma. Let G be an almost divisible torsion free group and let, for a type $\mu \in \mathcal{V}(G)$, the following conditions be fulfilled:

(a) The group $G(\mu) / G^*(\mu)$ is torsion free and belongs to some Baer's class Γ_α ;

(b) for any prime p the inequality $\mu(p) \neq \infty$ implies

$$\kappa_p(G(\mu) / G^*(\mu)) = 0.$$

Then the group $G^*(\mu)$ is a direct summand of $G(\mu)$, $G(\mu) = G_\mu \dot{+} G^*(\mu)$, where G_μ is completely decomposable and homogeneous of the type μ , or $G_\mu = 0$.

Proof. If $G^*(\mu) = G(\mu)$, then $G_\mu = 0$, therefore we may suppose that $G^*(\mu) \neq G(\mu)$. The group $G(\mu)$ as a pure subgroup of G is likewise almost divisible and so is the factor group $\bar{G} = G(\mu) / G^*(\mu)$ as well. In view of (a), the group \bar{G} is torsion free and the type of any of its non zero elements is $\geq \mu$. Thus, if p is a prime with $\mu(p) = \infty$, then $p\bar{G} = \bar{G}$. But if $\mu(p) \neq \infty$, then by (b) $\kappa_p(\bar{G}) = 0$.

In the last case, each pure rank one subgroup of \bar{G} is

of zero μ -rank (see [4, Corollary 2]), therefore each non zero element of \bar{G} has a finite μ -height in \bar{G} (see [6, Lemma 6.1]). Now we deduce from the finiteness of the set of all primes μ with $\nu(\mu) \neq \infty$ that \bar{G} is homogeneous of the type μ . Thus the inequality $\mu \bar{G} \neq \bar{G}$ implies $\nu(\mu) \neq \infty$ and therefore $\nu_\mu(\bar{G}) = 0$. By (a), \bar{G} belongs to some Baer's class Γ_∞ and in view of [4, Corollary 4] \bar{G} is completely decomposable. Evidently, μ is the type of any element $g \in G(\mu) \div G^*(\mu)$, hence, according to the Baer's lemma [1, the note following Theorem 46.5] $G^*(\mu)$ is a direct summand in $G(\mu)$. Thus we have $G(\mu) = G_\mu \dot{+} G^*(\mu)$, $G_\mu \cong G(\mu) / G^*(\mu) = \bar{G}$, therefore G_μ is completely decomposable and homogeneous of the type μ .

Now we are in a position to prove a theorem concerning almost divisible groups with the linearly ordered type set (in natural order of the types).

Theorem 1. Let G be an almost divisible torsion free group with the linearly ordered type set $\mathcal{T}(G)$. Then G is completely decomposable if and only if for any $\mu \in \mathcal{T}(G)$ the condition (a) together with the condition

(b*) $\nu_\mu(G / G^*(\mu)) = 0$ whenever $\nu(\mu) \neq \infty$ are fulfilled.

Proof. If G is completely decomposable and $G = \sum_{i \in I} J_i$ is a complete decomposition of G , then $\mathcal{T}(G)$ coincides also with the set of the types of all rank one groups J_i ($i \in I$). Thus for any $\mu \in \mathcal{T}(G)$ the torsion free group $G(\mu) / G^*(\mu)$ is completely decomposable and homogeneous of the type μ ; evidently, $G(\mu) / G^*(\mu) \in$

$\in \Gamma_\alpha$ ($1 \leq \alpha \leq 2$). The group $G / G^*(\mu)$ is completely decomposable as well and the types of its direct summands are $\leq \mu$. Hence, if $\mu(\rho) \neq \infty$, then $G / G^*(\mu)$ is ρ -reduced and in view of [4, Corollary 1] we have $\kappa_\rho(G / G^*(\mu)) = 0$. Thus in this case the conditions (a), (b*) are fulfilled.

Now, let us suppose that G satisfies (a) and (b*); we shall show that G is completely decomposable. From the hypothesis it follows immediately that $\mathcal{T}(G)$ is finite. Let us put $\mathcal{T}(G) = \{\mu_1 < \dots < \mu_m\}$. Then we shall prove the complete decomposability of G by induction on $n = \text{card } \mathcal{T}(G)$.

For $n = 1$ the group G is homogeneous of the type μ_1 and $G^*(\mu_1) = 0$. Then the inequality $\rho G \neq G$ for a prime ρ implies $\mu_1(\rho) \neq \infty$ and in view of (b*) we have $0 = \kappa_\rho(G / G^*(\mu_1)) = \kappa_\rho(G)$. Hence, by [4, Corollary 4], G is completely decomposable.

Thus, suppose $n \geq 2$ and let our assertion hold whenever the cardinality of the corresponding type set is $n - 1$. Since $G(\mu_1) = G$, we can apply our Lemma to G for $\mu = \mu_1$ and we get

$$(1) \quad G = H \dot{+} G^*(\mu_1),$$

where the group H is completely decomposable. If we put $G^*(\mu_1) = G_1 = G(\mu_2)$, then by (1) G_1 is also almost divisible and $\mathcal{T}(G_1) = \{\mu_2 < \dots < \mu_m\}$. We shall now verify that G_1 fulfils (a) and (b*) for all types of $\mathcal{T}(G_1)$. In fact, if $\mu \in \mathcal{T}(G_1)$, then $\mu_1 < \mu_2 \leq \mu$ and hence $G(\mu) \subseteq G(\mu_2) = G_1$, which implies

$G_1(\mathcal{U}) = G(\mathcal{U})$; analogously, we obtain $G^*(\mathcal{U}) \cong \cong G^*(\mathcal{U}_1) = G_1^*$, therefore $G^*(\mathcal{U}) = G_1^*(\mathcal{U})$. Thus we have $G_1(\mathcal{U}) / G_1^*(\mathcal{U}) = G(\mathcal{U}) / G^*(\mathcal{U})$, which means that G_1 fulfils (a) for each $\mathcal{U} \in \mathcal{Z}(G_1)$. By (1), we can write for any $\mathcal{U} \in \mathcal{Z}(G_1)$

$$(2) \quad G / G^*(\mathcal{U}) = (H \dot{+} G_1) / G^*(\mathcal{U}) \cong H \dot{+} G_1 / G^*(\mathcal{U}) = \\ = H \dot{+} G_1 / G_1^*(\mathcal{U});$$

thus for $\kappa(\rho) \neq \infty$ it is $\kappa_\rho(G / G^*(\mathcal{U})) = 0$ and hence by (2)

$$\kappa_\rho(H \dot{+} G_1 / G_1^*(\mathcal{U})) = 0.$$

Following [4, Corollary 2], we get $\kappa_\rho(G_1 / G_1^*(\mathcal{U})) = 0$, therefore the condition (b*) is satisfied by G_1 . Under the inductive hypothesis G_1 and in view of (1) G is completely decomposable as well. Thus the proof of our theorem is finished.

If the group G is torsion free of finite rank and H any of its pure subgroups, then $\kappa_\rho(G) = \kappa_\rho(H) + \kappa_\rho(G/H)$ for every prime ρ (see [6, Theorem 6]). In particular, we obtain that $\kappa_\rho(G) = 0$ implies $\kappa_\rho(G/H) = 0$ for each pure subgroup H of G . We shall use this last fact in the proof of the following theorem. Let us recall that if G is torsion free and ρ any prime, then $G[\rho^\infty]$ will denote the greatest ρ -divisible subgroup of G . Evidently, $\kappa_\rho(G[\rho^\infty]) = \kappa(G[\rho^\infty])$ (see [5, Theorem 1]).

Theorem 2. Let G be an almost divisible torsion free group of finite rank with the linearly ordered type set $\mathcal{Z}(G)$. Then G is completely decomposable if and only if $\kappa_\rho(G) = \kappa(G[\rho^\infty])$ for every prime ρ .

Proof. If G is completely decomposable, then for any prime π , $G = G[\pi^\infty] \dot{+} G_1$ where G_1 is also completely decomposable and π -reduced. Then, by [4, Corollary 1] $\kappa_\pi(G_1) = 0$. Since $\kappa_\pi(G) = \kappa_\pi(G[\pi^\infty]) + \kappa_\pi(G_1)$ (see [6, Theorem 6]), we get $\kappa_\pi(G) = \kappa_\pi(G[\pi^\infty]) = \kappa(G[\pi^\infty])$.

To prove the converse consider $\kappa_\pi(G) = \kappa(G[\pi^\infty])$ for all primes π . For the proof of complete decomposability of G it suffices to show that G fulfills (b*) only, (a) being trivial. Let $\mathcal{Z}(G) = \{\mu_1 < \dots < \mu_n\}$, take $\mu \in \mathcal{Z}(G)$ and suppose $\mu(\pi) \neq \infty$ for some prime π . In order to prove the relation $\kappa_\pi(G/G^*(\mu)) = 0$ we shall distinguish two cases: $G[\pi^\infty] = 0$ and $G[\pi^\infty] \neq 0$. If $G[\pi^\infty] = 0$, then $\kappa_\pi(G) = \kappa_\pi(G[\pi^\infty]) = 0$ and in view of the preceding remark we have $\kappa_\pi(G/G^*(\mu)) = 0$. If $G[\pi^\infty] \neq 0$, then there exists an integer $j \leq n$ with $\mu_j(\pi) = \infty$; since $\mu_j \leq \mu$ and $\mu(\pi) \neq \infty$, it is certainly $1 < j$. Let i denote the smallest integer with $\mu_i(\pi) = \infty$; we shall show that $G[\pi^\infty] = G(\mu_i)$. The relation $\mu_i(\pi) = \infty$ implies the inclusion $G(\mu_i) \subseteq G[\pi^\infty]$. But if $0 \neq g \in G[\pi^\infty]$ and $\mu_{\pi} = \text{type}(g)$, then $\mu_{\pi}(\pi) = \infty$, therefore $i \leq \mu$. Hence we conclude $\mu_i \leq \mu_{\pi}$ and $g \in G(\mu_i)$. Thus we have shown that $G[\pi^\infty] = G(\mu_i)$ and also $G[\pi^\infty] = G^*(\mu_{i-1})$ ($2 \leq i$). By [6, Theorem 6] we have

$$\kappa_\pi(G) = \kappa_\pi(G[\pi^\infty]) + \kappa_\pi(G/G[\pi^\infty]);$$

since $\kappa_\pi(G[\pi^\infty]) = \kappa(G[\pi^\infty]) = \kappa_\pi(G)$, we get

$$(3) \quad 0 = \kappa_\pi(G/G[\pi^\infty]) = \kappa_\pi(G/G^*(\mu_{i-1})).$$

From $\kappa(\rho) \neq \infty$ it follows $\mu \leq \mu_{i-1}$ and hence $G^*(\mu_{i-1}) \subseteq G^*(\mu)$. Thus we have

$$G/G^*(\mu) \cong (G/G^*(\mu_{i-1})) / (G^*(\mu)/G^*(\mu_{i-1}))$$

and by (3) $\kappa_\rho(G/G^*(\mu)) = 0$. This means that G fulfills (b*) and Theorem 2 is proved.

Remark. The preceding theorem may be likewise formulated in the following way (see [2, Proposition 5.2]; for the definition of the regularity of a group see also [2, § 5]): Let G be an almost divisible torsion free group of finite rank with the linearly ordered type set $\mathcal{T}(G)$. Then the group G is completely decomposable if and only if it is regular.

Till now we have considered groups with the linearly ordered type set $\mathcal{T}(G)$ only. In order to investigate the general case we shall use [1, Theorem 48.6]. Thus we get the following assertion:

Theorem 3. An almost divisible torsion free group G is completely decomposable if and only if the conditions (a), (b) and

$$(c) \quad G^*(\mu) = G(\mu) \cap G^{**}(\mu)$$

are fulfilled for each type $\mu \in \mathcal{T}(G)$.

Proof. Firstly, assume that G is completely decomposable and that $G = \sum_{\lambda \in \Lambda} J_\lambda$ is one of its complete decompositions. Denote by $T(G)$ the set of all types of the groups J_λ ($\lambda \in \Lambda$); evidently, $T(G) \subseteq \mathcal{T}(G)$. For $\mu \in T(G)$ let A_μ denote the direct sum of all groups J_λ of the type μ ; certainly, it is $G(\mu)/G^*(\mu) \cong A_\mu$. If $\kappa(\rho) \neq \infty$, then A_μ is a ρ -reduced completely

decomposable group and in view of [4, Corollary 1] $\theta = \kappa_{\mu}(A_{\mu}) = \kappa_{\mu}(G(\mu)/G^*(\mu))$. Thus for $\mu \in T(G)$ the conditions (a) and (b) are fulfilled. But if $\mu \in \mathcal{Z}(G) \setminus T(G)$, then $G(\mu) = G^*(\mu)$ and the conditions (a), (b) are trivial. The condition (c) follows from [1, Theorem 48.6].

Further, suppose that G fulfils the conditions (a), (b), (c), and prove that G is completely decomposable. If $\mu \in \mathcal{Z}(G)$, then by Lemma there exists a direct decomposition of the form $G(\mu) = G_{\mu} \dot{+} G^*(\mu)$ where the group G_{μ} is completely decomposable and homogeneous of the type μ . Now, the proof proceeds in the same way as that of sufficiency in [1, Theorem 48.6]. Thus, firstly, it may be shown that the subgroups G_{μ} ($\mu \in \mathcal{Z}(G)$) generate their direct sum $\sum_{\mu} G_{\mu}$, and then we should get $G = \sum_{\mu} G_{\mu}$. The last relation is proved in [1, Theorem 48.6] under the assumption that $\mathcal{Z}(G)$ satisfies the maximum condition, but in our case $\mathcal{Z}(G)$ is finite, G being almost divisible. Since each G_{μ} is completely decomposable, so is the group $G = \sum_{\mu} G_{\mu}$ as well, which finishes the proof of the theorem.

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Matematicko-fyzikální fakulta
Karlova universita
Praha 8, Sokolovská 83
Československo

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