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Commentationes Mathematicae Universitatis Carolinae, Vol. 12 (1971), No. 1, 37--51

Persistent URL: <http://dml.cz/dmlcz/105326>

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SOME FIXED POINT THEOREMS IN METRIC AND BANACH SPACES

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§ 0. Introduction. This paper is devoted to the study of fixed points of some mappings in metric and normed spaces. Notations and terminology are described in Section 1. Section 2 contains some results near to those given by Kannan in [11] and Kirk in [13]. In Section 3 we study \mathcal{N} -m.c.l mappings and the relation between Fréchet differentiability and the measure of non-compactness. Section 4 is devoted to an application of a theorem of Browder [4].

§ 1. Notations and terminology. Let (X, d) and (Y, e) be two pseudometric spaces, C a subset of X and T a mapping of X into Y . Then T is said to be uniformly continuous on C with respect to X , if for each positive σ there is a positive ε such that if c is in C and x in X with $d(c, x) \leq \varepsilon$, then $e(T(c), T(x)) \leq \sigma$.

Let M be a subset of X and define

$Q(M) = \{ \varepsilon > 0 : M \text{ can be covered by a finite number of closed } \varepsilon\text{-balls in } X \}$

and the measure of non-compactness of the set M by $\chi(M) = \inf Q(M)$ (see Sadovskii [14]). For elementary properties of the measure of non-compactness and related topics

see [3],[8],[9],[15]. T is called a k -mcl mapping if $\chi(T(M)) \leq k \chi(M)$ for any subset M of X . T is called a strictly k -mcl mapping¹⁾ if $\chi(T(M)) < k \chi(M)$ for any non-precompact bounded subset M of X . In this terminology, T is concentrative if it is continuous and a strictly 1-mcl mapping. T is asymptotically regular (see [5]), if $e(T^n(x), T^{n+1}(x)) \rightarrow 0$ as $n \rightarrow +\infty$, for any x in X . It is easy to see that T is uniformly continuous on C with respect to X , respectively a k -mcl mapping, if it is k -Lipschitzian on C with respect to X (that is c in C and x in X implies that $e(T(c), T(x)) \leq k \cdot d(c, x)$ for some $k \geq 0$), respectively k -Lipschitzian on X .

Let (X, ρ) and (Y, ρ) be pseudonormed linear spaces and X_1 and Y_1 their closed unit balls at the origin. In what follows, " \rightharpoonup " and " \rightarrow " denote the convergence in the weak and strong (pseudonorm) topology, respectively. In [8] and [10] we computed the measure of non-compactness of X_1 : $\chi(X_1) = 0$ or 1 if $X/\rho^{-1}(0)$ has a finite or infinite dimension. If T is a linear mapping of X into Y , denote by $\chi(T)$ the number $\chi(T(X_1))$. It is easy to see that χ is a pseudonorm on the space of all linear bounded mappings from X into Y ; its kernel, that is the set $\chi^{-1}(0)$, consists of precompact linear mappings of X into Y . Clearly, $\chi(T) \leq \|T\|$ for any linear $T: X \rightarrow Y$.

1) k -mcl is the abbreviation of "Lipschitzian in the sense of the measure of non-compactness with constant k ".

Now, let X and Y be normed linear spaces, C a subset of X and T a mapping of C into Y . Then T is said to be (a) demicontinuous if $x_n \rightarrow x_0$ in C implies $T(x_n) \rightarrow T(x_0)$ in Y ; (b) weakly continuous if $x_n \rightarrow x_0$ in C implies $T(x_n) \rightarrow T(x_0)$ in Y ; (c) convex if the functional $f(x) = \|x - T(x)\|$ and the set C are convex; (d) Fréchet differentiable at a point x in C (see [16]) if x is in the interior of C and $T(x+h) = T(x) + T'(x)h + \omega(x,h)$ ($h \in X \cap (C-x)^2$), where $T'(x)$, the Fréchet derivative of T at x , is a linear continuous mapping of X into Y and $\omega(x,h)$, the remainder of T at x , satisfies the condition: $\lim_{h \rightarrow 0} \frac{\|\omega(x,h)\|}{\|h\|} = 0$; (e) uniformly Fréchet differentiable on C (see [16]) if C is open, T is Fréchet differentiable at any x in C and $\lim_{h \rightarrow 0} \frac{\|\omega(x,h)\|}{\|h\|} = 0$ uniformly for x in C ; (f) feebly semicontractive if $Y = X$ a Banach space and there is a mapping V of $C \times C$ into X such that $T(x) = V(x,x)$ for all x in C , $\|V(x,z) - V(y,z)\| \leq \|x - y\|$ (x, y, z in C) and the map $x \rightarrow V(\cdot, x)$ is compact from C to the space of maps of C to X with the uniform metric. The kernel of C is the set $K(C) = \{x \in X: C \text{ is starshaped with respect to } x, \text{ that is, the closed segment } [x, z] \text{ is contained in } C \text{ for any } z \text{ in } C\}$.

§ 2. In this section we shall present some sufficient conditions on the existence of fixed points of some mappings in metric spaces. These results are related to those of

 2) $C - x$ denotes the set $\{c - x : c \in C\}$.

Kannan [11] and Kirk [13].

Theorem 1. Let (X, τ) be a non-empty compact space and d a non-negative real-valued symmetric function on $X \times X$ such that $d(x, y) = 0$ implies $x = y$ ($x, y \in X$). Suppose that T_1 and T_2 are mappings of X into itself satisfying the following conditions:

(1) if $T_1(x) = x = y = T_2(y)$ is not true, then

$$d(T_1(x), T_2(x)) < \frac{1}{2} [d(x, T_1(x)) + d(y, T_2(y))];$$

(2) the function $f(x, y) = d(x, T_1(x)) + d(y, T_2(y))$

is lower semi-continuous on $(X, \tau) \times (X, \tau)$.

Then the mappings T_1 and T_2 have a common fixed point which is the unique fixed point of each of T_1 and T_2 .

Proof. If x and w are fixed points of T_1 and T_2 respectively, with $x \neq w$, then by (1) we have $d(T_1(x), T_2(w)) < \frac{1}{2} [0 + 0] = 0$, a contradiction, proving the trivial part of the theorem.

Since $f(x, y)$ is a lower semi-continuous function on the (non-empty) compact space $(X, \tau) \times (X, \tau)$, there is a point (x, w) in $X \times X$ at which f attains its infimum.

If

$$(*) \quad T_1(T_2(w)) = T_2(w) = w$$

or

$$(**) \quad x = T_1(x) = T_2(T_1(x))$$

is true, then w or x is a common fixed point of T_1 and

T_2 . Hence it suffices to prove that at least one of $(*)$

and $(**)$ is satisfied. Suppose not. Then, by (1)

$$\begin{aligned} f(T_2(w), T_1(x)) &= d(T_2(w), T_1(T_2(w))) + d(T_1(x), T_2(T_1(x))) = \\ &= d(T_1(T_2(w)), T_2(w)) + d(T_1(x), T_2(T_1(x))) < \end{aligned}$$

$$\begin{aligned}
&< \frac{1}{2} [d(T_2(w), T_1(T_2(w))) + d(w, T_2(w))] + \\
&+ \frac{1}{2} [d(x, T_1(x)) + d(T_1(x), T_2(T_1(x)))] = \\
&= \frac{1}{2} [f(x, w) + f(T_2(w), T_1(x))],
\end{aligned}$$

that is, $f(T_2(w), T_1(x)) < f(x, w)$ - a contradiction to the minimality of f at the point (x, w) .

In the above theorem one can take, for instance, as d a metric on X . Proofs of the following corollaries are similar to those given in [7], [10]. We can obtain further assertions by taking $T_1 = T_2 = T$.

Corollary 1. Let (X, τ) be a non-empty compact space and d a non-negative real-valued lower semi-continuous function on $(X, \tau) \times (X, \tau)$. Suppose that T_1 and T_2 are continuous mappings of X into itself satisfying the condition (1) of Theorem 1. Then the conclusion of Theorem 1 remains valid.

Corollary 2. Let X be a non-empty weakly compact subset of a normed linear space, T_1 and T_2 weakly continuous mappings of X into itself satisfying the condition (1) of Theorem 1 with $d(x, y) = \|x - y\|$. Then the conclusion of Theorem 1 remains valid.

Corollary 3. Let X be a non-empty weakly compact convex subset of a normed linear space, T_1 and T_2 demicontinuous mappings of X into itself satisfying the condition (1) of Theorem 1 with $d(x, y) = \|x - y\|$. Let the function f (see Theorem 1) be convex on $X \times X$. Then the conclusion of Theorem 1 remains valid.

Corollary 4. Let X, T_1, T_2 and d be as in Corollary

3. Suppose that $I - T_1$ and $I - T_2$ are convex. (I denotes the identity mapping on X .) Then the conclusion of Theorem 1 remains valid.

Theorem 2. Let (X, d) be a complete metric space, C a non-empty compact subset of X and T a (not necessarily continuous) mapping of X into itself which is uniformly continuous on C with respect to X . Let $\alpha(T, x)$ be a subset of X , for any $x \in X$. Suppose that:

$$(1) \inf_{x \in X} d(x, T(x)) = 0;$$

$$(2) \overline{\alpha(T, x)} \cap C \neq \emptyset \quad \text{for each } x \text{ in } X;$$

$$(3) d(y, T(y)) \leq \kappa(d(x, T(x))) \quad \text{for each } y \in \alpha(T, x), x \in X, \text{ where } \kappa(t) \text{ is a function defined on } (0, +\infty) \text{ with } \kappa(\varepsilon) \rightarrow 0 \text{ as } \varepsilon \rightarrow 0+.$$

Then T has a fixed point in X (even in C).

Proof. Let $\varepsilon > 0$ be given. Then, by (1), there exists a point x in X such that $d(x, T(x)) < \varepsilon$; by (2), there are y in $\alpha(T, x)$ and c in C with $d(y, c) < \varepsilon$. Thus, by (3), we have

$$\begin{aligned} d(c, T(c)) &\leq d(c, y) + d(y, T(y)) + d(T(y), T(c)) \leq \\ &\leq \varepsilon + \kappa(\varepsilon) + \sigma(\varepsilon) = \eta(\varepsilon), \end{aligned}$$

where $\sigma(\varepsilon) = \sup\{d(T(x), T(w)) : x \in X, w \in C, d(x, w) \leq \varepsilon\}$ is the modul of uniform continuity of T on C with respect to X . The fact $\eta(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0+$ implies that

$\inf_{c \in C} d(c, T(c)) = 0$. The continuity of T on the non-empty compact subset C ensures the existence of a point x_0 in C such that $d(x_0, T(x_0)) = \inf_{c \in C} d(c, T(c)) = 0$, and x_0 is a fixed point of T .

Remark. The condition (1) of Theorem 2 is satisfied if (X, d) is a bounded complete subset of a normed linear space and T is a nonexpansive mapping of X into itself and the kernel of X intersects the range of T , $K(X) \cap R(T) \neq \emptyset$ (see [10], Proposition 4), or if T is asymptotically regular, $d(T^n(x), T^{n+1}(x)) \rightarrow 0$ as $n \rightarrow +\infty$, for any x in X . In many cases we can take $\alpha(T, x) \subset \{T^n(x)\}_{n=0}^{\infty}$, or $\alpha(T, x) \subset \{T^n(x) : n = 0, 1, \dots\}$, if X is a subset of a linear space (cf. Kirk [13], Cor. 2.1).

§ 3. k-mcL mappings and Fréchet differentiable mappings.

Proposition 1. Let (X, ρ) and (Y, ρ) be pseudonormed linear spaces and T a linear mapping of X into Y . Then:

- (1) T is continuous if and only if $\chi(T) < +\infty$;
- (2) T is precompact (that is, it maps bounded subsets of X into precompact subsets of Y) if and only if $\chi(T) = 0$;
- (3) if T is continuous then it is a $\chi(T)$ -mcL mapping;
- (4) if T is not precompact, then T is not a k -mcL mapping for any $k < \chi(T)$.

Proof. (1) and (2) follow at once from the definition of $\chi(T)$ and Lemma 1, (2) and (3) in [9]. The same considerations as in the proof of Theorem 8 in [10] prove (3). The part (4) of the theorem is a consequence of the equality $\chi(T) \equiv \chi(T(X_1)) = \chi(T) \cdot \chi(X_1)$. (Note that $\chi(T) > 0$ implies that the dimension of the quotient space $X/\rho^{-1}(0)$ is infinite and $\chi(X_1) = 1$, cf. Proposition 6 in [10].)

Proposition 2. Let (X, d) and (Y, e) be pseudometric

spaces and $\{T_n\}_{n=1}^{\infty}$ a sequence of \mathcal{K} -m.c.L mappings of X into Y which converges, uniformly on bounded subsets of X , to a mapping T of X into Y . Then T is a \mathcal{K} -m.c.L mapping.

Proof. Let $\varepsilon > 0$ be given and let M be a bounded subset of X . Then there exists n_0 such that $e(T_{n_0}(x), T(x)) \leq \varepsilon$ for all x in M . Hence the Hausdorff distance (with respect to e) of $T_{n_0}(M)$ and $T(M)$ is not greater than ε and, using [31, § 3, Lemma, or [8], Theorem 1.11, respectively [9], Lemma 1, (8), we obtain that $|\chi(T_{n_0}(M)) - \chi(T(M))| \leq \varepsilon$. Hence $\chi(T(M)) \leq \chi(T_{n_0}(M)) + \varepsilon \leq \mathcal{K} \cdot \chi(M) + \varepsilon$. Since $\varepsilon > 0$ was arbitrary, we have $\chi(T(M)) \leq \mathcal{K} \chi(M)$.

Theorem 3. Let X and Y be normed linear spaces, C an open non-empty subset of X and T a mapping of C into Y possessing the Fréchet derivative at a point x of C . Then $\lim_{\varepsilon \rightarrow 0+} \frac{\chi(T(x + \varepsilon X_1))}{\varepsilon}$ exists and equals to $\chi(T'(x))$.

Proof. There is an $\varepsilon_0 > 0$ such that the closed ε_0 -ball at x is contained in C . We can write

$$T(x+h) = T(x) + T'(x)h + \omega(x, h) \quad (\|h\| \leq \varepsilon_0, h \in X),$$

where $d(\varepsilon) = \sup \left\{ \frac{\|\omega(x, h)\|}{\|h\|} : h \in X, 0 < \|h\| \leq \varepsilon \right\}$ converges to 0 as ε tends to 0. Further,

$$T(x + \varepsilon X_1) \subset T(x) + T'(x)(\varepsilon X_1) + \omega(x, \varepsilon X_1) \quad (0 < \varepsilon \leq \varepsilon_0),$$

$$T'(x)(\varepsilon X_1) \subset T(x) - T(x + \varepsilon X_1) + \omega(x, \varepsilon X_1),$$

hence

$$\frac{T(z + \varepsilon X_1)}{\varepsilon} \subset \frac{T(z)}{\varepsilon} + T'(z)(X_1) + \frac{\omega(z, \varepsilon X_1)}{\varepsilon}$$

$$T'(z)(X_1) \subset \frac{T(z)}{\varepsilon} - \frac{T(z + \varepsilon X_1)}{\varepsilon} + \frac{\omega(z, \varepsilon X_1)}{\varepsilon} \quad (0 < \varepsilon \leq \varepsilon_0)$$

Thus

$$\frac{T(z + \varepsilon X_1)}{\varepsilon} \subset \frac{T(z)}{\varepsilon} + T'(z)(X_1) + \sigma(\varepsilon)X_1 \quad (0 < \varepsilon \leq \varepsilon_0)$$

$$T'(z)(X_1) \subset \frac{T(z)}{\varepsilon} - \frac{T(z + \varepsilon X_1)}{\varepsilon} + \sigma(\varepsilon)X_1,$$

that is

$$\left| \frac{\chi(T(z + \varepsilon X_1))}{\varepsilon} - \chi(T'(z)) \right| \leq \sigma(\varepsilon) \quad (0 < \varepsilon \leq \varepsilon_0),$$

and the theorem follows.

Remark. A direct consequence of the proof is that if T is uniformly Fréchet differentiable on C , then

$\frac{\chi(T(z + \varepsilon X_1))}{\varepsilon}$ converges to $\chi(T'(z))$ as $\varepsilon \rightarrow 0$, uniformly for z in C .

Corollary 1. Let X and Y be normed linear spaces, C an open non-empty subset of X and T a mapping of C into Y possessing the Fréchet derivative at a point z in C . If T is a \mathfrak{K} -m.c.L. mapping, then so is its Fréchet derivative $T'(z)$, that is $\chi(T'(z)) \in \mathfrak{K}$.

Proof. The proof is a direct consequence of Theorem 3 and [10], Proposition 6, respectively [8], Theorem 1.7.

Lemma 1. Let X and Y be normed linear spaces, C a non-empty bounded subset of X which is starshaped with respect to the origin of X and T an α -homogeneous mapping of C into Y for some $\alpha \leq 1$ (that is $T(tx) = t^\alpha T(x)$ if $t > 0$ and $x, tx \in C$) and a \mathfrak{K} -m.c.L. map-

ping on $C \cap X_1$ for some $k \geq 0$. Then T is a (strictly) k -m.c.L mapping on C .

Proof. We can restrict our consideration to the case when T is a k -m.c.L mapping on $C \cap X_1$. Let M be a bounded subset of C and denote $M_1 = M \cap X_1$ and $M_2 = M \cap (X \setminus X_1)$. Then there is a $t > 1$ such that $t^{-1}M_2$ is contained in X_1 . Then $\chi(T(M_2)) = \chi(t^{-1}T(t^{-1}M_2)) = t^{-k}\chi(T(t^{-1}M_2)) \leq t^{-k} \cdot k \cdot \chi(M_2)$. Therefore

$$\chi(T(M)) = \chi(T(M_1) \cup T(M_2)) = \max \{ \chi(T(M_1)),$$

$$\chi(T(M_2)) \} \leq \max \{ k \cdot \chi(M_1), k \cdot \chi(M_2) \} = k \cdot \chi(M).$$

§ 4. An application of a Browder's theorem. Recently, Browder [4] has proved the following important theorem:

Let X be a Banach space, C a closed bounded convex subset of X having the origin of X in its interior, T a mapping of C into X such that for each x in the boundary of C , $Tx \neq \lambda x$ for any $\lambda > 1$. Suppose that for a given constant $k \leq 1$ and a mapping V of $C \times C$ into X , $T(x) = V(x, x)$ for all x in C while

$$\|V(x, z) - V(y, z)\| \leq k \|x - y\| \quad (x, y \in C)$$

and the map $x \rightarrow V(\cdot, x)$ is compact from C to the space of maps from C to X with the uniform metric. Then:

(a) If $k < 1$, T has a fixed point in C .

(b) If $k \leq 1$ and $(1 - T)(C)$ is closed in X , then T has a fixed point in C .

By means of this theorem, Browder [4] derived a fixed point theorem for semicontractive mappings in uniformly convex Banach spaces, and Kirk [12] made this for strongly

semicontractive mappings in reflexive Banach spaces. Our purpose in this section is to give a fixed point theorem for concentrative feebly semicontractive mappings in Banach spaces. In the part (b) of the Browder's theorem, the problem is to prove that $(I - T)(C)$ is closed in X .

Lemma 2. Let X be a normed linear space, C a complete subset of X and T a concentrative mapping of C into X . Then the mapping $I - T$ maps bounded closed subsets of C into bounded closed subsets of X (I denotes the identity mapping of C into C).

Proof. Let M be a closed and bounded subset of X . Since T is concentrative, we have $\chi(T(M)) \leq \chi(M) < +\infty$ and hence $T(M)$ is bounded. Now, the inclusion $(I - T)(M) \subset M - T(M)$ implies the boundedness of $(I - T)(M)$. Let $\{\psi_n\}_{n=1}^{\infty}$ be a sequence in $(I - T)(M)$ converging (strongly) to a point ψ_0 in X . Then there are points x_n in M such that $x_n - T(x_n) = \psi_n$. Denote $A = \{x_n : n = 1, 2, \dots\}$ and $B = \{\psi_n : n = 1, 2, \dots\}$. Then, clearly, $A \subset T(A) + B$ and $T(A) \subset A - B$. Thus, B being precompact (the underlying set of a convergent sequence), we have $\chi(A) \leq \chi(T(A)) + \chi(B) = \chi(T(A)) \leq \chi(A) + \chi(B) = \chi(A)$, that is, $\chi(T(A)) = \chi(A)$, and hence A is precompact. Then \bar{A} is a compact subset of C . There exists a subsequence $\{x_{m_k}\}$ of $\{x_n\}$ such that $x_{m_k} \rightarrow x_0$ for some x_0 in C . We have $T(x_{m_k}) \rightarrow T(x_0)$ since T is continuous. Hence $x_0 - T(x_0) = \psi_0$ and ψ_0 is in $(I - T)(C)$ which proves the lemma.

Lemma 3. Let X be a normed linear space, C a complete subset of X and T a concentrative mapping of C into X .

If $x_m \rightarrow x_0$ and $y_m = x_m - T(x_m) \rightarrow y_0$ for some $\{x_m\} \subset C$, $x_0 \in C$ and $y_0 \in X$, then $y_0 = x_0 - T(x_0)$.

Proof. Denoting $A = \{x_m\}$ and $B = \{y_m\}$ and using $A \subset T(A) + B$, $T(A) \subset A - B$, we have, by the same argument as in the proof of the preceding lemma, $\chi(A) = 0$. Hence \bar{A} is compact and $x_m \rightarrow x_0$ in \bar{A} implies $x_m \rightarrow x_0$. Therefore, $y_0 = x_0 - T(x_0)$.

Theorem 4. Let X be a Banach space, C a closed bounded convex subset of X having the origin of X in its interior, T a contractive feebly semicontractive mapping of C into X satisfying the Leray-Schauder condition: for each x in the boundary of C and for each $\lambda > 1$, $Tx \neq \lambda x$. Then T has a fixed point in C .

Proof. By Lemma 2, $(I - T)(C)$ is closed, and using the Browder's theorem mentioned at the beginning of this section, our theorem follows.

Corollary 1. Let X and C be as in the theorem. Let T be a contractive nonexpansive mapping of C into X satisfying the Leray-Schauder condition (see Theorem 4). Then T has a fixed point in C .

Corollary 2. Let X and C be as in the theorem. Let T be the sum of a contractive nonexpansive mapping and a compact mapping of C into X . Suppose that T satisfies the Leray-Schauder condition (see Theorem 4). Then T has a fixed point in C .

Lemma 4. Let X be a normed linear space and $\{x_n\}$ a sequence in X weakly converging to x_0 and let ε be a real number greater than $\chi(\{x_n : n = 1, 2, \dots\})$. Then there is n_0 such that for each $n \geq n_0$, x_n lies in the

2ε -ball at x_0 .

Proof. Suppose not. Then there is a subsequence $\{x_{m_n}\}$ of $\{x_n\}$ which is disjoint from the 2ε -ball at x_0 . Now, $\{x_{m_n}\}$, and hence $\{x_{m_{n_j}}\}$, is covered by a finite number of closed ε -balls. Hence there exist a point z in X and a subsequence $\{x_{m_{n_j}}\}$ of $\{x_{m_n}\}$ contained in the closed ε -ball at z . Since the closed ε -ball at z is convex and $x_{m_{n_j}} \rightarrow x_0$, the point x_0 lies in the closed ε -ball at z . Thus, $\{x_{m_{n_j}}\}$ being contained in the closed ε -ball at z , it is contained in the closed 2ε -ball at x_0 , a contradiction.

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(Oblatum 15.7.1970)