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THE LATTICE OF RADICAL FILTERS OF A COMMUTATIVE NOETHERIAN RING

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As it was shown by V. Dlab [2], there is a one-to-one correspondence between all radical filters and some sets of prime ideals of a commutative Noetherian ring (namely, the set of all prime ideals contained in \( \mathfrak{F} \) corresponds to any radical filter \( \mathfrak{F} \)). In this brief note, there is given a new one-to-one correspondence between all radical filters and some sets of prime ideals of a commutative Noetherian ring \( \Lambda \) and it is shown that the lattice \( \mathcal{L} \) of all radical filters of \( \Lambda \) is distributive. Further, some necessary and sufficient conditions for \( \Lambda \), under which the lattice \( \mathcal{L} \) is complementary, are given.

In what follows, \( \Lambda \) stands for an associative commutative Noetherian ring with unity. Recall that a (non-empty) family \( \mathfrak{F} \) of ideals of \( \Lambda \) is called a radical filter (commutativity is assumed!) if

1. \( I \in \mathfrak{F} \), \( I \subseteq J \Rightarrow J \in \mathfrak{F} \),

2. \( I \subseteq J, J \in \mathfrak{F} \) and \( (I : \lambda) \in \mathfrak{F} \) for any \( \lambda \in J \Rightarrow I \in \mathfrak{F} \), where \( (I : \lambda) = \{ \mu \in \Lambda \mid \mu \lambda \subseteq I \} \).

Let us denote by \( \mathcal{P} \) the set of all prime ideals of \( \Lambda \).
and by \( \mathcal{M} \) the set of all maximal ideals of \( \Lambda \).

We call a subset \( \mathcal{N} \) of \( \mathcal{P} \) a radical set, if any two elements of \( \mathcal{N} \) are incomparable (in the order of the inclusion). Let \( \mathcal{L} \) be any (non-empty) set of ideals of \( \Lambda \).

The maximal elements of the set of all prime ideals which are contained in some ideal from \( \mathcal{L} \) form a radical set - the radical set belonging to \( \mathcal{L} \).

**Lemma 1**: Let \( \mathcal{N} \subseteq \mathcal{P} \) be a radical set. Then the set \( \mathcal{L}_{\mathcal{N}} = \{ I, I \notin N, \forall N \in \mathcal{N}, I \text{ ideal in } \Lambda \} \) is the radical filter.

**Proof**: The property (1) is evident. Proving (2) indirectly we shall show

\[
\text{(3) } I \notin \mathcal{L}_{\mathcal{N}} = \bigvee J, J \in \mathcal{L}_{\mathcal{N}}, I \subseteq J , \text{ there exists } \lambda \in J \text{ with } (I : \lambda) \notin \mathcal{L}_{\mathcal{N}}.
\]

Let us suppose \( I \notin \mathcal{L}_{\mathcal{N}} \). Then there exists \( N \in \mathcal{N} \) with \( I \subseteq N \). For \( J \in \mathcal{L}_{\mathcal{N}} \) we have \( J \supseteq N \neq \emptyset \), hence we can take \( \lambda \in J \setminus N \). Then \( (I : \lambda) = \{ \mu \in \Lambda, \mu \lambda \in I \subseteq N \} \subseteq \subseteq (N : \lambda) \). But \( (N : \lambda) = N \) because \( N \) is a prime ideal and \( \lambda \notin N \) which finishes the proof of (3).

**Lemma 2**: Let \( \mathcal{N}_1, \mathcal{N}_2 \) be two radical sets. Then \( \mathcal{L}_{\mathcal{N}_1} \subseteq \mathcal{L}_{\mathcal{N}_2} \) if and only if to any \( N_2 \in \mathcal{N}_2 \) there exists \( N_1 \in \mathcal{N}_1 \) with \( N_2 \subseteq N_1 \). Consequently, \( \mathcal{L}_{\mathcal{N}_1} = \mathcal{L}_{\mathcal{N}_2} \) if and only if \( \mathcal{N}_1 = \mathcal{N}_2 \).

**Proof**: At first, suppose that the condition holds. Then \( I \notin \mathcal{L}_{\mathcal{N}_1} \Rightarrow I \notin N, \forall N \in \mathcal{N}_1 \Rightarrow I \notin N, \forall N \in \mathcal{N}_2 \Rightarrow I \notin \mathcal{L}_{\mathcal{N}_2} \). Conversely, if there exists \( N \in \mathcal{N}_2 \) which is not contained in any \( N' \in \mathcal{N}_1 \), then \( N \in \mathcal{L}_{\mathcal{N}_1} \supsetneq \mathcal{L}_{\mathcal{N}_2} \). For the proof of the last part let us note that if \( \mathcal{L}_{\mathcal{N}_1} = \mathcal{L}_{\mathcal{N}_2} \), then to any \( N_2 \in \mathcal{N}_2 \) there exists \( N_1 \in \mathcal{N}_1 \) and,
further, $\mathcal{N}'_1 \subseteq \mathcal{N}_2$ with $\mathcal{N}_2 \subseteq \mathcal{N}_3 \subseteq \mathcal{N}'_2$. But $\mathcal{N}_2 = \mathcal{N}'_2$ for $\mathcal{N}_2$ being a radical set which implies $\mathcal{N}_2 \subseteq \mathcal{N}_3$. The inclusion $\mathcal{N}_1 \subseteq \mathcal{N}_2$ follows by symmetrical arguments.

**Theorem 1:** There is a one-to-one correspondence between all radical filters and all radical sets of prime ideals of $\Lambda$.

**Proof:** In view of Lemmas 1 and 2 it suffices to prove that to any radical filter $\mathcal{F}$, there exists a radical set $\mathcal{H}$ such that $\mathcal{F} = \mathcal{H}_\mathcal{F}$. Let $\mathcal{H}$ be the set of all maximal elements of the set of all ideals which do not belong to $\mathcal{F}$. It is easy to see that it suffices to show that $\mathcal{H}$ contains the prime ideals only. One can easily show that an ideal $I$ is prime if and only if $(I:\lambda) = I$ for any $\lambda \in \Lambda - I$. Let us take $I \in \mathcal{H}$ arbitrarily, and let us assume the existence of $\lambda \in \Lambda - I$ with $(I:\lambda) \not\subseteq I$. By hypothesis (maximality of $I$) it is $(I:\lambda) \in \mathcal{F}$ and $J = \{I, \lambda\} \in \mathcal{F}$ ( $J$ is the ideal generated in $\Lambda$ by $I$ and $\lambda$ ). Writing any element $\varphi \in J$ in the form $\varphi = \alpha I + \beta$, $\alpha \in \Lambda$, $\beta \in I$, we have $\mu \varphi = \alpha \mu I + \mu \beta \in I$ for any $\mu \in (I:\lambda)$, hence $(I:\lambda) \subseteq (I:\varphi)$ for any $\varphi \in J$. Then $I \in \mathcal{F}$ by (1) and (2), which contradicts our hypothesis. Theorem 1 is therefore proved.

It is easy to see that the intersection of any set of radical filters is a radical filter so that the radical filters form a (complete) lattice which we denote by $\mathcal{L}$.

**Theorem 2:** Let $\mathcal{N}_1$, $\mathcal{N}_2$ be two radical sets of prime
ideals. Then \( \mathcal{E}_{\mathcal{V}_1} \land \mathcal{E}_{\mathcal{V}_2} = \mathcal{E}_\mathcal{V} \), where \( \mathcal{V} \) is the radical set belonging to \( \mathcal{V}_1 \cup \mathcal{V}_2 \) and \( \mathcal{E}_{\mathcal{V}_1} \lor \mathcal{E}_{\mathcal{V}_2} = \mathcal{E}_\mathcal{V} \)

where \( \mathcal{V} \) is the radical set belonging to the set

\[ \mathcal{A} = \{ N_1 \cap N_2, N_1 \in \mathcal{V}_1, N_2 \in \mathcal{V}_2 \} \]

**Proof:** The proof for intersection is direct and we shall omit it. Proving the part for join, let us have \( I \in \mathcal{E}_{\mathcal{V}_i}, i = 1, 2 \). Then \( I \not\subseteq N_i \) for any \( N_i \in \mathcal{V}_i, i = 1, 2 \) and therefore \( I \not\subseteq N \) for any \( N \in \mathcal{V} \) which denotes \( I \in \mathcal{E}_\mathcal{V} \) and hence \( \mathcal{E}_{\mathcal{V}_1} \lor \mathcal{E}_{\mathcal{V}_2} \subseteq \mathcal{E}_\mathcal{V} \). Conversely, let \( \mathcal{E}_\mathcal{V} \) be any radical filter containing \( \mathcal{E}_{\mathcal{V}_1} \lor \mathcal{E}_{\mathcal{V}_2} \). Then from \( \mathcal{E}_{\mathcal{V}_i} \subseteq \mathcal{E}_\mathcal{V}, i = 1, 2 \) and Lemma 2 it easily follows that to any \( N' \in \mathcal{V}' \) there exist \( N_i \in \mathcal{V}_i, i = 1, 2 \) with \( N' \subseteq N_1 \cap N_2 \). Hence \( N' \subseteq N \) for some \( N \in \mathcal{V} \) owing to the definition of \( \mathcal{V} \). Using Lemma 2 again, one gets \( \mathcal{E}_\mathcal{V} \subseteq \mathcal{E}_\mathcal{V}' \) as was to be shown.

**Theorem 3:** The lattice \( \mathcal{L} \) is distributive.

**Proof:** We shall prove the "cancellation form" of distributivity indirectly, namely \( \forall \mathcal{V}, \mathcal{V}' = \mathcal{V} \land \mathcal{V}' \Rightarrow \mathcal{V} \lor \mathcal{V}' \Leftrightarrow \mathcal{V} \lor c \). Let us suppose we have three radical filters \( \mathcal{E}_{\mathcal{V}_1}, \mathcal{E}_{\mathcal{V}_2}, \mathcal{E}_{\mathcal{V}_3} \) satisfying \( \mathcal{E}_{\mathcal{V}_2} \subseteq \mathcal{E}_{\mathcal{V}_3} \) and

\[(4) \quad \mathcal{E}_{\mathcal{V}_1} \land \mathcal{E}_{\mathcal{V}_2} = \mathcal{E}_{\mathcal{V}_1} \land \mathcal{E}_{\mathcal{V}_3} = \mathcal{E}_{\mathcal{V}} \]

Let us put

\[ \mathcal{V}_1' = \mathcal{V} \land \mathcal{V}_1 \]
\[ \mathcal{V}_2' = \mathcal{V}_2 \land \mathcal{V}_1 \]
\[ \mathcal{V}_3' = \mathcal{V}_3 \land \mathcal{V}_1 \]
\[ \mathcal{V}_1'' = \{ N \in \mathcal{V}_1, \exists M \in \mathcal{V}_2 ; N \not\subseteq M \} \]
\[ \mathcal{V}_2'' = \{ N \in \mathcal{V}_2, \exists M \in \mathcal{V}_1 ; N \not\subseteq M \} \]

- 56 -
One can easily see (by using Theorem 2 and (4)) that $\mathcal{H}_i'$ and $\mathcal{H}_i''$ are disjoint and $\mathcal{H}_i' \cup \mathcal{H}_i'' = \mathcal{H}_i$, $i = 1, 2, 3$.

In view of $\mathcal{H}_2 \neq \mathcal{H}_3$, two cases can arise:

a) There exists $N_2 \in \mathcal{H}_2$ incomparable (in the inclusion) with any $N_3 \in \mathcal{H}_3$;

b) there exists $N_2 \in \mathcal{H}_2$, $N_3 \in \mathcal{H}_3$ with $N_2 \neq N_3$.

(we omit the symmetrical two cases concerning $\mathcal{H}_2$ and $\mathcal{H}_3$).

Ad a): For $N_2 \in \mathcal{H}_2$, we have $N_2 \in \mathcal{H}_2' \subseteq \mathcal{H}_2$ - a contradiction. Hence $N_2 \in \mathcal{H}_2''$, i.e. there exists $M \in \mathcal{H}_1$, $N_2 \subseteq M$.

At first, $N_2 = M \cap N_2$, $M \in \mathcal{H}_1$, $N_2 \in \mathcal{H}_2$ implies $N_2 \neq \mathcal{H}_1 \cap \mathcal{H}_2$ by Theorem 2. Secondly, $N_2 \subseteq M_1 \cap M_2$, $M_1 \in \mathcal{H}_1$, $M_2 \in \mathcal{H}_2$ implies $N_2 \subseteq M_1$, $M_2 \in \mathcal{H}_3$ - a contradiction proving $N_2 \neq \mathcal{H}_1 \cap \mathcal{H}_2$.

Ad b): It is easy to see that $N_2 \in \mathcal{H}_3'$ gives $N_2 = N_3$ - a contradiction.

Hence $N_3 \in \mathcal{H}_3''$, i.e. there exists $M \in \mathcal{H}_1$ satisfying $N_3 \subseteq M$. For $N_3 \subseteq M_1 \cap M_2$, $M_1 \in \mathcal{H}_1$, $M_2 \in \mathcal{H}_2$ we have $N_3 \subseteq M_2$ - a contradiction. Hence $N_3 \neq \mathcal{H}_1 \cap \mathcal{H}_2$.

Finally, $N_3 = M \cap N_3$, $M \in \mathcal{H}_1$ gives rise to $N_3 \neq \mathcal{H}_1 \cap \mathcal{H}_2$, which completes the proof of Theorem 3.

**Theorem 4:** An element $\mathcal{H}_1$ has a complement in $\mathcal{L}$ if and only if

a) $\mathcal{H}_1$ contains the maximal ideals only,

b) for any prime ideal $P$ the set $\mathcal{H}_P$ of all ideals
from \( \mathcal{M} \) containing \( P \) satisfies either \( \mathcal{M}_P \subseteq \mathcal{N} \) or \( \mathcal{M}_P \cap \mathcal{N} = \emptyset \).

**Proof:** It is clear that the unit element of \( \mathcal{L} \) is \( \mathcal{L}_\phi \) and the zero element is \( \mathcal{L}_\emptyset \). Let us suppose that the conditions a) and b) are satisfied and let \( \mathcal{N}' = \mathcal{M} \setminus \mathcal{N} \).

Then \( \mathcal{L}_\mathcal{N} \cap \mathcal{L}_\mathcal{N}' = \mathcal{L}_\emptyset \) by Theorem 2 and \( \mathcal{L}_\mathcal{N} \cap \mathcal{L}_\mathcal{N}' = \mathcal{L}_\phi \) by b) and Theorem 2.

Conversely, let \( \mathcal{L}_\mathcal{N} \) have a complement \( \mathcal{L}_\mathcal{N}' \) in \( \mathcal{L} \). If \( \mathcal{N} \) contains an ideal \( N \) which is not in \( \mathcal{M} \), then there exists \( M \in \mathcal{M} \) with \( N \subseteq M \). For \( M \in \mathcal{N}' \) we have \( N \in \mathcal{L}_\emptyset \setminus \mathcal{L}_\mathcal{N} \cap \mathcal{L}_\mathcal{N}' \), by Theorem 2 and for \( M \in \mathcal{N}' \) we have \( M \in \mathcal{L}_\mathcal{N} \cap \mathcal{L}_\mathcal{N}' \setminus \mathcal{L}_\emptyset \) - a contradiction proving a). Finally, \( \mathcal{N}' \) must be a complement of \( \mathcal{N} \) in \( \mathcal{M} \) (intersection). If there exists \( P \subseteq M \cap M' \), \( P \) prime, \( M \in \mathcal{N} \), \( M' \in \mathcal{N}' \), then \( P \in \mathcal{L}_\mathcal{N} \setminus \mathcal{L}_\mathcal{N} \cap \mathcal{L}_\mathcal{N}' \) - a contradiction proving b).

**Theorem 5:** The lattice \( \mathcal{L} \) is complementary if and only if any prime ideal in \( \mathcal{A} \) is maximal.

**Proof:** If \( \mathcal{L} \) is complementary, then by a) Theorem 4 and Lemma 1 any prime ideal in \( \mathcal{A} \) is maximal. The converse follows immediately from Theorem 4.

**References**


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