Marie Hájková
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THE LATTICE OF BI-NUMERATIONS OF ARITHMETIC, II

Marie HÁJKOVÁ, Praha

This paper is a direct continuation of our [6]. The knowledge of [6] is presupposed. Similarly as in [6], in the whole paper $\mathcal{A} = \langle A, K \rangle$ denotes a fixed axiomatic theory with the following properties:

1. $\mathcal{A}$ is a primitive recursive set,
2. $\mathcal{A}$ is consistent,
3. $P \equiv A$ (P is the Peano's arithmetic).

Numbering of definitions and theorems in this paper begins with 3.1; references like 2.24 or 1.18 refer to definitions and theorems from [6].

III. Reducibility; a non-describability theorem

We shall now study the problem of reducibility of elements of $[\text{Bin}]$. We recall the definition:

3.1. Definition. An element $x$ of a lattice $M = \langle M, \leq, \cap, \cup \rangle$ is irreducible if, for each $x$, $y \in M$, $x \cup y = x$ implies $x = x$ or $y = x$.

3.2. Theorem. Let $\mathcal{A}$ be reflexive, let $\gamma, \beta \in \text{Bin}$ and suppose $\gamma \prec \mathcal{A} \beta$. Then there is a

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\[ \sigma \in \text{Bin} \quad \text{such that} \]
\[ (\star) \quad \left\{ \begin{array}{l}
\sigma \leq A \\ \gamma \end{array} \right\} \cup \{ \sigma \} = [\beta] . \]

The main idea of the proof: Let \( \alpha' \in \text{Bin} \) such that \( \alpha' < A \beta \). Put
\[ \sigma'(x) = \alpha'(x) \lor F_{m_{x}}^{(K)}(x) \land \]
\[ \lor \bigvee_{y \in x} \left[ P_{x} f_{\beta} (\beta, y) \land \bigwedge_{z \in y} \sim P_{x} f_{\gamma} (\beta, x) \right] . \]

Evidently, \( \sigma' \leq A \beta \) and \([ \gamma ] \cup [ \sigma' ] = [ \beta ] \). But it is not clear whether \( \sigma' \neq A \beta \). So we modify the definition of \( \sigma' \) and find a \( \sigma \) satisfying \((\star)\) in the form
\[ \sigma(x) = \alpha(x) \lor F_{m_{x}}^{(K)}(x) \land \bigvee_{y \in x} \left[ P_{x} f_{\beta} (\beta, y) \land \bigwedge_{z \in y} \sim P_{x} f_{\gamma} (\beta, x) \right] . \]

The following lemma gives a necessary and sufficient condition for the existence of a \( \sigma \in \text{Bin} \) with required properties \((\star)\).

3.3. Lemma. Let \( \beta, \gamma \in \text{Bin} \) and let \( \gamma < A \beta \). There exists a \( \sigma \in \text{Bin} \) satisfying \((\star)\) if and only if there exist a formula \( \alpha \in \text{Bin} \) and a formula \( \psi(\gamma) \) which is a PR-formula in \( A \) with exactly one free variable \( \gamma \) such that

\[ (1) \quad \vdash_{A} (\neg \text{Con}_{A} \land \text{Con}_{\gamma}) \rightarrow \bigvee_{\gamma} \psi(\gamma) , \]

\[ (2) \quad \vdash_{A} (\neg \text{Con}_{A} \land \text{Con}_{\alpha}) \rightarrow \bigvee_{\gamma} \psi(\gamma) . \]

Proof of Lemma 3.3. Let \( \sigma \in \text{Bin} \) satisfy the conditions \((\star)\). It suffices to put \([ \alpha ] = [ \gamma ] \cap [ \sigma ]\)
and $\psi (\eta) = \operatorname{Pr} f_\beta (\overline{\eta}, z)$.

Conversely, let $\psi (\eta)$ and $x \in \mathcal{B} \mathcal{M}$ satisfy the conditions (1) and (2). Put

$$\mathcal{F}(x) = \alpha (x) \lor \overline{\operatorname{Pr} f_\beta (\overline{\eta}, y)} \land \lor \psi (\eta) \land \overline{\operatorname{Pr} f_\beta (\overline{\eta}, \eta)}.$$

By (1) and the definition of $\mathcal{F}$, we have $\vdash_{\mathcal{K}} \overline{\operatorname{Con}}_\beta \leftrightarrow \overline{\operatorname{Con}}_\gamma$, i.e. $[\gamma] \cup [\mathcal{F}] = [\beta]$. By (2) and the definition of $\mathcal{F}$, we have $\vdash_{\mathcal{A}} \overline{\operatorname{Con}}_\gamma \rightarrow \overline{\operatorname{Con}}_\beta$, i.e. $\gamma \lessdot \beta$.

**Proof of Theorem 3.2.** By 2.11, we can assume

$$\vdash_{\mathcal{A}} \bigwedge (\mathcal{F} (x) \rightarrow \beta (x)).$$

Using the diagonal construction 1.9 and Lemma 1.1 determine $\eta$ such that

(1) $\vdash_{\mathcal{A}} \eta \leftrightarrow \bigwedge (\overline{\operatorname{Pr} f_\phi} (\overline{\eta}, y) \rightarrow \lor_{x < \eta} \overline{\operatorname{Pr} f_\beta} (\overline{\eta}, x))$.

We shall prove

(2) $\vdash_{\mathcal{A}} \eta$.

Let $\vdash_{\mathcal{A}} \eta$ and let $\alpha$ be a proof of $\eta$ in $\mathcal{A}$. Then

$\vdash_{\mathcal{A}} \overline{\operatorname{Pr} f_\beta} (\overline{\eta}, x)$, and therefore, by Lemma 3.1 [1], $\vdash_{\mathcal{A}} \sim \eta$, because $\beta$ bi-numerates $\mathcal{A}$. It is a contradiction and so we obtain $\vdash_{\mathcal{A}} \eta$.

Put

(3) $\psi (\eta) = \overline{\operatorname{Pr} f_\beta} (\overline{\eta}, x) \land \lor_{x < \eta} \overline{\operatorname{Pr} f_\beta} (\overline{\eta}, y)$.

Evidently, $\psi (\eta)$ is a PR-formula in $\mathcal{P}$ and $\operatorname{Fr} (\psi) = \eta$. We shall prove

(4) $\vdash_{\mathcal{A}} \sim \eta \rightarrow (\sim \overline{\operatorname{Con}}_\gamma \land \lor_{y} \psi (\eta))$.

In $\mathcal{A}$, suppose $\sim \eta$. Then $\lor_{y} [\overline{\operatorname{Pr} f_\beta} (\overline{\eta}, y) \land \lor_{x < \eta} \overline{\operatorname{Pr} f_\beta} (\overline{\eta}, x)]$ and consequently
The last formula is \( \sim \forall \psi (y) \). From the assumption \( \sim \eta \), we have \( 
abla x \phi (x) \). On the other hand, by 1.7, \( \sim \eta \) implies \( \forall x \phi (\sim \eta) \), because \( \sim \eta \) is an RE-formula in \( \mathcal{P} \). Consequently, we obtain \( \sim \text{Con}_\mathcal{P} \).

We shall now prove

(5) \( \vdash \mathcal{A} (\sim \text{Con}_\mathcal{P} \land \sim \forall \psi (y)) \rightarrow \sim \eta \).

In \( \mathcal{A} \), suppose \( \sim \text{Con}_\mathcal{P} \) and \( \sim \forall \psi (y) \). Then

\[
\forall x (\forall x \phi (\sim \eta, y) \rightarrow \forall x \phi (\sim \eta, x)), \forall x \phi (\sim \eta, y), \\
\forall x \phi (\sim \eta, y) \land \forall x \phi (\sim \eta, x)
\]

and consequently \( \sim \eta \).

(4) and (5) imply

(6) \( \vdash \mathcal{A} (\sim \text{Con}_\mathcal{P} \land \text{Con}_\mathcal{P}) \rightarrow \forall \psi (y) \).

Put \( \mathcal{E} = \mathcal{A} \cup \{ \sim \eta \} \). The theory \( \mathcal{E} = \langle \mathcal{E}, K \rangle \) is consistent by (2). By (4), we have

(7) \( \vdash \mathcal{E} \sim \text{Con}_\mathcal{P} \).

Let \( \varepsilon (x) \) be a PR-formula in \( \mathcal{P} \) defined as follows: \( \varepsilon (x) = \forall (x) \lor x \vDash \sim \eta \). Evidently, \( \varepsilon (x) \) bi-numerates \( \mathcal{E} \). Using the diagonal construction 1.9, determine \( \mathcal{E} \) such that

(8) \( \vdash \mathcal{E} \phi \leftrightarrow \forall x (\forall x \phi (\bar{\phi}, x) \rightarrow \sim \text{Con}_\mathcal{P} \land \bar{x}) \).

Put \( \alpha (x) = \beta (x) \land \forall y < x \sim \forall x \phi (\bar{\phi}, y) \). Evidently, \( \alpha \in \text{Bin} \). Analogously as in the proof of 7.4 [1], one can prove

(8) \( \vdash \mathcal{E} \phi \).
(9) \( \vdash \alpha \sim \varphi \rightarrow C_{\alpha} \)

(7), (8) and (9) give

(10) \( \vdash \alpha \left( \sim C_{\alpha} \land C_{\alpha} \right) \rightarrow \eta \cdot \)

(10) and (4) give

(11) \( \vdash \alpha \left( \sim C_{\alpha} \land C_{\alpha} \right) \rightarrow \forall \psi (y) . \)

(11) and (6) show that the conditions of Lemma 3.3 are satisfiable.

3.4. Corollary. If \( \alpha \) is reflexive, then every element of \([\text{Bin}]\) is reducible.

Theorem 3.2 enables us to formulate a partial result on the "non-describability" of elements of \([\text{Bin}]\). First we define some notions and prove a lemma.

3.5. Definition. Let \( \varphi \in Fm_{\Lambda_1} \). \( \varphi \) is said to be a \( \Delta_0 \)-formula, \( \varphi \in \Delta_0 \), if it belongs to the least class containing all atomic formulas in \( \Lambda_1 \), closed under \( \land \) and \( \sim \) and which contains with every formula \( \varphi_1 \) also \( \forall w \left( u \leq w \leq v \land \varphi_1 \right) \), where \( u, v, w \) are distinct variables.

3.6. Definition. Let \( \varphi \in Fm_{\Lambda_1} \). \( \varphi \) is said to be a \( \Sigma_1 \)-formula, \( \varphi \in \Sigma_1 \), if either \( \varphi \in \Delta_0 \) or \( \varphi \) has the form \( \forall \omega_0 \ldots \forall \omega_{\kappa} \varphi_1 \), \( \varphi_1 \in \Delta_0 \) and \( \omega_0, ..., \omega_{\kappa} \) are distinct variables.

Remark. These definitions are analogous to the Lévy's definitions of \( \Delta_0 \)-formulas and \( \Sigma_1 \)-formulas of the set theory [4].

3.7. Lemma. Let \( M = \langle M, \leq, \cap, \cup \rangle \) be a lattice, let \( \varphi \in \Delta_0 \) and \( F\varphi (\varphi) = \{ \omega_0, ..., \omega_{\kappa-1} \} \). Suppo-
se $a, b \in M$ and $a \leq b$. Furthermore, let $a_0, \ldots, a_{k-1}$ be elements of $M$ such that $a \leq a_i \leq b$ for $i = 0, \ldots, k-1$. Then

$$M \models \varphi[a_0, \ldots, a_{k-1}]$$

if and only if

$$(a, b) \models \varphi[a_0, \ldots, a_{k-1}].$$

**Proof** by induction on formulas.

(a) If $\varphi$ is atomic then the assertion is obvious.

(b) Let $\varphi$ have the form $\psi_1 \land \psi_2$. For the sake of brevity of notation, suppose $Fv(\psi_1) = Fv(\psi_2) = \emptyset$. Then

$$M \models (\psi_1 \land \psi_2)[a_0, \ldots, a_{k-1}]$$

if

$$(M \models \psi_1[a_0, \ldots, a_{k-1}] \text{ and } M \models \psi_2[a_0, \ldots, a_{k-1}])$$

and

$$(a, b) \models \psi_1[a_0, \ldots, a_{k-1}] \text{ and } (a, b) \models \psi_2[a_0, \ldots, a_{k-1}].$$

(c) If $\varphi$ has the form $\neg \psi$, the induction step is trivial.

(d) Let $\varphi$ be $\forall \nu (\nu_\mu \leq \nu \leq \nu_\kappa \land \psi)$. We can suppose $\kappa \geq \mu, \kappa$. Suppose

$$M \models \varphi[a_0, \ldots, a_{k-1}].$$

Then there is an $e \in M$ such that $a \leq a_\mu \leq e \leq a_\kappa \leq b$ and

$$M \models \psi[a_0, \ldots, a_{k-1}, e].$$

By the induction hypothesis, $(a, b) \models \psi[a_0, \ldots, a_{k-1}, e]$ and consequently $(a, b) \models \forall \nu (\nu_\mu \leq \nu \leq \nu_\kappa \land \psi)[a_0, \ldots, a_{k-1}]$.

The converse implication is proved analogously.

**3.8. Definition.** Let $M = \langle M, \leq, \cap, \cup \rangle$ be a lattice and let $\langle a_0, \ldots, a_{k-1} \rangle \in M^k$. The $k$-tuple $\langle a_0, \ldots, a_{k-1} \rangle$ is said to be $\Sigma_1$-definable.
in $M$ if there is a $\Sigma_1$-formula $\varphi$ such that
\[ < a_0, ..., a_{k-1} > \]
is the unique $\mathcal{K}$-tuple satisfying $\varphi$ in $M$.

3.9. Theorem on $\Sigma_1$-non-definability. Let $\mathcal{K}$ be
reflexive. Then no $\mathcal{K}$-tuple of elements of $[\text{Bin}]$ is
$\Sigma_1$-definable in $[\text{Bin}]$. Moreover, if $\varphi \in \Sigma_1$, 
$\text{Pr}_\kappa (\varphi) = \{ \omega_0, ..., \omega_{k-1}, \langle \alpha_0, ..., \alpha_{k-1} \rangle \in [\text{Bin}] \}$ and if 
$[\text{Bin}] \models \varphi [\alpha_0, ..., \alpha_{k-1}]$, then there are $\alpha'_0, ..., \alpha'_{k-1} \in [\text{Bin}]$ such that $[\alpha'_0] \neq [\alpha'_1]$ for all
$i, j = 0, ..., k - 1$ and $[\text{Bin}] \models \varphi [\alpha'_0, ..., \alpha'_{k-1}]$.

Proof. Let $\varphi$ be a $\Sigma_1$-formula and let $[\text{Bin}] \models$
$\varphi [\alpha_0, ..., \alpha_{k-1}]$. We can suppose that $\varphi$
has the form $\bigvee_{\mathcal{K}} \psi (v_0, ..., v_{k-1})$, where $\psi \in \Delta_0$. 
It follows that there are $[\alpha'_0], ..., [\alpha'_{k-1}] \in [\text{Bin}]$
such that $[\text{Bin}] \models \psi [\alpha'_0, ..., \alpha'_{k-1}]$. Put $[\beta] =
\{ \alpha'_0 \cup ... \cup [\alpha'_{k-1}] \}$ and let $[\gamma] \leq [\beta]$, $[\alpha] \leq [\beta]$ 
$\leq [\alpha_0 \cap ... \cap [\alpha_{k-1}]$ (cf. 2.6). By Theorem 3.2, there
is a $[\sigma] \leq [\beta]$ such that $[\gamma] \cup [\sigma] = [\beta]$. 

Put $[\varepsilon] = [\gamma] \cap [\sigma]$. By 1.19 there exists an iso-
morphism $f$ of $< [\gamma]; [\beta] >$ and $< [\varepsilon]; [\sigma] >$.

By Theorem 3.7 we have $< [\gamma]; [\beta] > \models \psi [\alpha_0, ..., \alpha'_{k-1}]$, and putting $[\alpha'_i] = f([\alpha'_i]) (i = 0, ..., k - 1)$
we obtain $< [\varepsilon], [\sigma] > \models \psi [\alpha'_0, ..., [\alpha'_{k-1}]]$
by Theorem 1.20. Using again Theorem 3.7 we have $[\text{Bin}] \models$
$\varphi [\alpha'_0, ..., [\alpha'_{k-1}]]$, which implies $[\text{Bin}] \models$
$\varphi [\alpha'_0, ..., [\alpha'_{k-1}]]$. Since the intervals
\[ \langle \gamma \rangle ; \langle \beta \rangle \] and \[ \langle \varepsilon \rangle ; \langle \sigma \rangle \] are disjoint
we have \[ \langle \alpha_i \rangle \neq \langle \alpha'_j \rangle \] for \( i, j = 0, \ldots, \kappa - 1 \).

3.10. Remark. It can be easily seen from the proof
that we can obtain an infinite sequence of distinct \( \kappa \)
tuples of elements of \[ \langle Bin \rangle \] satisfying \( \varphi \).

IV. Relative complements in the lattice of bi-numerations of arithmetic

In this section we are going to study the problem of
existence of relative complements in the lattice \[ \langle Bin \rangle \].
Roughly speaking, we show that in every non-trivial interval
there are many elements having relative complement
(w.r.t. this interval) and many elements having no relative
complement (w.r.t. this interval).

We recall the definition.

4.1. Definition. Let \( M = \langle M, \leq, \cap, \cup \rangle \) be a
lattice and let \( a, \varrho, c, d \in M \). Suppose \( a \leq \rho \).
Then \( d \) is said to be a relative complement to \( c \)
with respect to \( a, \rho \) if \( c \cap d = a \) and \( c \cup d = \rho \).

4.2. Definition. Let \( M = \langle M, \leq, \cap, \cup \rangle \) be a
lattice, \( a, \rho, c \in M \) and suppose \( a \leq \rho \).
Then \( c \) is said to be complementible w.r.t. \( a, \rho \)
if there exists a \( d \in M \) which is a relative complement w.r.t.
\( a, \rho \).

The following lemma can be easily proved from the
axioms of the lattice theory.

4.3. Lemma. Let \( M = \langle M, \leq, \cap, \cup \rangle \) be a
lattice, \( a, \rho, c, d, d' \in M \) and suppose \( a \leq \rho \).
Then
(i) \( c \) is a relative complement to \( d \) w.r.t. \( a \), \( \mathcal{L} \) if and only if \( d \) is a relative complement to \( c \) w.r.t. \( a \), \( \mathcal{L} \);

(ii) if \( c \) is complementible w.r.t. \( a \), \( \mathcal{L} \), then \( a \leq c \leq \mathcal{L} \);

(iii) if \( \mathcal{M} \) is distributive and \( d \), \( d' \) are relative complements to \( c \) w.r.t. \( a \), \( \mathcal{L} \), then \( d = d' \).

4.4. Lemma. Let \( \mathcal{M} = (M, \leq, \cap, \cup) \) be a distributive lattice, \( a, a_1, \mathcal{L}, \mathcal{L}_1, c \in M \) and suppose \( a \leq a_1 < c < \mathcal{L}_1 \leq \mathcal{L} \). Then

(i) if \( c \) is complementible w.r.t. \( a \), \( \mathcal{L} \), then \( c \) is complementible w.r.t. \( a_1 \), \( \mathcal{L}_1 \);

(ii) if \( c \) is complementible w.r.t. \( a_1, \mathcal{L}_1 \) and both \( a_1 \) and \( \mathcal{L}_1 \) are complementible w.r.t. \( a \), \( \mathcal{L} \), then \( c \) is complementible w.r.t. \( a, \mathcal{L} \);

(iii) if \( a_1 \) and \( \mathcal{L}_1 \) be complementible w.r.t. \( a, \mathcal{L} \), then both \( a_1 \cup \mathcal{L}_1 \) and \( a_1 \cap \mathcal{L}_1 \) are complementible w.r.t. \( a, \mathcal{L} \).

Proof. (i) Let \( c \) be the relative complement to \( c \)

w.r.t. \( a, \mathcal{L} \). Put \( d' = (d \cap \mathcal{L}_1) \cup a_1 \). By elementary calculation, \( d' \cap c = a_1 \) and \( d' \cup c = \mathcal{L}_1 \).

(ii) Let \( d' \) be the relative complement to \( c \)

w.r.t. \( a_1, \mathcal{L}_1 \), let \( d_1 \) be the relative complement to \( a_1 \) w.r.t. \( a, \mathcal{L} \) and let \( d_2 \) be the relative complement to \( \mathcal{L}_1 \) w.r.t. \( a, \mathcal{L} \). Put \( d = (d_2 \cup d') \cap a_1 \). By elementary calculation, \( d \cup c = \mathcal{L} \) and \( d \cap c = a \).
(iii) Let \( c_1, d_1 \) be the relative complements to \( a_1, b_1 \) respectively w.r.t. \( a, b \). It can be easily shown that \( c_1 \cap d_1 \) is the relative complement to \( a_1 \cup b_1 \) w.r.t. \( a, b \) and that \( c_1 \cup d_1 \) is the relative complement to \( a_1 \cap b_1 \) w.r.t. \( a, b \).

4.5. Lemma. Let \( \alpha, \beta, \gamma, \sigma \in \text{Bin} \) and suppose \( \alpha \leq \gamma, \sigma \leq \beta \). Then

(i) \( [\gamma] \cup [\sigma] = [\beta] \) if and only if

\[ \neg \alpha \sim \text{Con}_\beta \land \text{Con}_\sigma \rightarrow \neg \text{Con}_\sigma ; \]

(ii) \( [\gamma] \cap [\sigma] = [\alpha] \) if and only if

\[ \neg \alpha \sim \text{Con}_\sigma \land \text{Con}_\alpha \rightarrow \text{Con}_\sigma ; \]

(iii) \( [\sigma] \) is a relative complement to \( [\gamma] \) w.r.t. \( [\alpha], [\beta] \) if and only if

\[ \neg \alpha \sim (\neg \text{Con}_\beta \land \text{Con}_\alpha) \rightarrow \neg (\text{Con}_\sigma \leftrightarrow \neg \text{Con}_\sigma) . \]

The lemma follows from Corollaries 2.20 and 2.22.

4.6. Lemma. Let \( \alpha, \beta, \gamma \in \text{Bin} \) and suppose \( \alpha \leq \gamma, \beta \). Then \( [\gamma] \) is complementible w.r.t. \( [\alpha], [\beta] \) if and only if there exists a formula \( \varphi (\gamma) \) which is a PR-formula in \( \mathcal{P} \) with exactly one free variable \( \gamma \) and such that

(1) \( \neg \alpha \sim (\neg \text{Con}_\beta \land \text{Con}_\alpha) \rightarrow (\text{Con}_\gamma \leftrightarrow \gamma \varphi (\gamma)) . \)

Proof. (i) Let \( [\sigma] \) be the relative complement to \( [\gamma] \) w.r.t. \( [\alpha], [\beta] \). Put \( \varphi (\gamma) = Prf_\sigma (0 \Rightarrow 1, \gamma) . \)

Evidently, \( \varphi (\gamma) \) is a PR-formula in \( \mathcal{P} \) and \( \text{Fv}(\varphi) = \{\gamma\} . \) (1) follows from Lemma 4.5 (iii).

(ii) Let \( \varphi (\gamma) \) be a PR-formula in \( \mathcal{P} , \)

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For $\varphi = \langle \gamma \rangle$ and suppose (1). Put

$$\sigma'(x) = \alpha(x) \lor F_m \varphi(x) \land$$

$$\lor_{y_1, y_2 < x} \left( \varphi(y_1) \land E_{f_\beta} \left( \varphi(y, y_2) \right) \right).$$

Evidently, $\sigma' \in Bin$, $\alpha \leq A \sigma' \leq A \beta$ and

$$\vdash_A (\sim \text{Con}_\beta \land \text{Con}_\alpha) \rightarrow (\sim \text{Con}_{\sigma} \leftrightarrow \lor \varphi (y)).$$

Therefore, by Lemma 4.5 (iii), $[\sigma']$ is the relative complement to $[\gamma]$ w.r.t. $[\alpha], [\beta]$.

4.7. Theorem. Let $\alpha, \beta, \gamma \in Bin$ and suppose $\alpha \leq A \gamma \leq A \beta$. Then

(i) if $[\gamma]$ is completable w.r.t. $[\alpha], [\beta]$ then there exists an $m \in \omega$ such that

(1) $\vdash_A (\sim \text{Con}_\beta \land \text{Con}_\gamma) \rightarrow F_{[A \land m]} \text{Con}_\alpha \rightarrow \text{Con}_\gamma;$

(ii) if $A$ is reflexive and (1) holds then $[\gamma]$ is completable w.r.t. $[\alpha], [\beta]$; in fact, if we put

$$\sigma'(x) = \alpha(x) \lor F_m \varphi(x) \land$$

$$\lor_{y_1, y_2 < x} \left( F_{f_\beta \left[ A \land m \right]} \text{Con}_{\alpha} \rightarrow \text{Con}_{\gamma}, y \right) \land$$

then $[\sigma']$ is the relative complement to $[\gamma]$ w.r.t. $[\alpha], [\beta]$.

Proof. (i) Let $[\gamma]$ be completable w.r.t. $[\alpha], [\beta]$. By Lemma 4.6, there exists a formula $\varphi(y)$ with exactly one free variable $y$ such that

$$\forall y \varphi(y)$$

is an RE-formula in $P$ and

(2) $\vdash_A (\sim \text{Con}_\beta \land \text{Con}_\alpha) \rightarrow (\text{Con}_\gamma \leftrightarrow \lor y \varphi(y)).$

Therefore, there exists an $m_1 \in \omega$ such that

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(3) \[ \vdash \Pr_{\mathcal{L} \cup \mathcal{M}_1}((\sim Cq\beta \land Cq\alpha) \rightarrow (Cq\gamma \leftrightarrow \bigvee \varphi(y))). \]

Let \( \psi \) be an RE-formula such that

(4) \[ \vdash \Pr_{\mathcal{L}}(\psi \leftrightarrow \bigvee \varphi(y)). \]

Evidently, we can suppose \( \psi \in \mathcal{S}_{\mathcal{K}_0} \). Therefore, there exists an \( n_2 \in \omega \) such that

(5) \[ \vdash \Pr_{\mathcal{L} \cup \mathcal{M}_2}(\psi \leftrightarrow \bigvee \varphi(y)). \]

By Lemma 3.9 \([1]\) and Corollary 5.5 \([1]\), we have

(6) \[ \vdash \Pr_{\mathcal{L}}(\psi \rightarrow \Pr_{\mathcal{M}_1}(\psi)). \]

Hence, by (4), (5), (6) there exists an \( n_3 \in \omega \) such that

(7) \[ \vdash \Pr_{\mathcal{L} \cup \mathcal{M}_3}(\psi \rightarrow \Pr_{\mathcal{M}_1}(\psi)). \]

\( \sim Cq\beta \) is an RE-formula in \( \Pr \). We can prove that there exists \( n_4 \in \omega \) such that

(8) \[ \vdash \Pr_{\mathcal{L} \cup \mathcal{M}_4}(\sim Cq\beta) \]

analogously as (7).

Taking \( n = \max(n_1, n_3, n_4) \) we have:

\[ \vdash \mathcal{A}((\sim Cq\beta \land Cq\gamma) \rightarrow \bigvee \varphi(y)) \quad \text{(by (2) and the assumption } \alpha \leq \mathcal{A} \varphi \text{)}. \]

\[ \vdash \mathcal{A}((\sim Cq\beta \land Cq\gamma) \rightarrow \Pr_{\mathcal{L} \cup \mathcal{M}_1}(\bigvee \varphi(y))) \quad \text{(by (7))}, \]

\[ \vdash \mathcal{A}((\sim Cq\beta \land Cq\gamma) \rightarrow \Pr_{\mathcal{L} \cup \mathcal{M}_1}(\sim Cq\beta \land Cq\alpha \rightarrow Cq\gamma)) \quad \text{(by (2))}, \]

\[ \vdash \mathcal{A}((\sim Cq\beta \land Cq\gamma) \rightarrow \Pr_{\mathcal{L} \cup \mathcal{M}_1}(Cq\alpha \rightarrow Cq\gamma)) \quad \text{(by (8))}. \]
(ii) Let $\mathcal{A}$ be reflexive and let $\sigma'$ be as indicated. Suppose that (1) holds. Evidently, $\sigma' \in \text{Bin}$ and
\[ \vdash_{\mathcal{A}} (\sim \text{Con}_\beta \land \text{Con}_{\sigma'}) \rightarrow \sim \text{Con}_{\sigma'}. \]
It follows from Lemma 4.5 that it suffices to show that
\[ \vdash_{\mathcal{A}} (\sim \text{Con}_\beta \land \text{Con}_{\sigma'}) \rightarrow \sim \text{Pr}_{[\mathcal{A} \setminus \sigma]} (\text{Con}_{\sigma'} \rightarrow \text{Con}_{\sigma}). \]
If $\alpha = \beta$, then (9) is evident. Suppose $\alpha < \beta$. Then $\mathcal{A} \uplus \{ \sim \text{Con}_\beta \land \text{Con}_{\sigma'} \}$ is consistent and, by 5.8 (ii) [1], reflexive. Therefore $\vdash_{\mathcal{A}} \sim \text{Con}_\beta \land \text{Con}_{\sigma'} \rightarrow \text{Con}_{\text{Con}[(\mathbf{A} \uplus \{ \sim \text{Con}_\beta \land \text{Con}_{\sigma'} \}) \setminus \sigma]}$ for each $n \in \omega$. In particular, putting $n' = \max(n, \sim \text{Con}_\beta \land \text{Con}_{\sigma'})$, we have $\vdash_{\mathcal{A}} (\sim \text{Con}_\beta \land \text{Con}_{\sigma'}) \rightarrow \text{Con}_{\text{Con}[(\mathbf{A} \uplus \{ \sim \text{Con}_\beta \land \text{Con}_{\sigma'} \}) \setminus \sigma]}$, i.e.
\[ \vdash_{\mathcal{A}} (\sim \text{Con}_\beta \land \text{Con}_{\sigma'}) \rightarrow \sim \text{Pr}_{[\mathcal{A} \setminus \sigma]} (\text{Con}_{\sigma} \rightarrow \text{Con}_{\sigma'}). \]
Evidently,
\[ \vdash_{\mathcal{A}} \sim \text{Pr}_{[\mathcal{A} \setminus \sigma]} (\text{Con}_{\sigma} \rightarrow \text{Con}_{\sigma'}) \rightarrow \sim \text{Pr}_{[\mathcal{A} \setminus \sigma]} (\text{Con}_{\sigma} \rightarrow \text{Con}_{\sigma'}). \]
(10) and (11) show that (9) holds.

4.8. Corollary. Let $\alpha, \beta, \gamma, \sigma' \in \text{Bin}$ and suppose $\alpha < \beta$.

(i) If $[\sigma']$ is the relative complement to $[\gamma]$ w.r.t. $[\alpha], [\beta]$, then there exists an $n \in \omega$ such that
\[\begin{align*}
(1) \quad & \gamma =_A \alpha (x) \land \text{Pr}_{\mathcal{A} \setminus \sigma'} (\text{Con}_{\sigma'} \rightarrow \text{Con}_{\sigma'}, \text{Pr}_{f \beta} (0 \not\equiv 1, \nu_2)), \\
(2) \quad & \sigma' =_A \alpha (x) \land \text{Pr}_{\mathcal{A} \setminus \sigma'} (\text{Con}_{\sigma'} \rightarrow \text{Con}_{\gamma}, \nu_1) \land \text{Pr}_{f \beta} (0 \not\equiv 1, \nu_2)).
\end{align*}\]
and, moreover,

\[(3) \vdash A (\sim \text{Con}_\beta \land \text{Con}_\alpha) \rightarrow (\text{Pr}_{[A \Delta n]} (\text{Con}_\alpha \rightarrow \text{Con}_\gamma) \lor \text{Pr}_{[A \Delta n]} (\text{Con}_\alpha \rightarrow \text{Con}_\delta)) \]

(ii) if \( A \) is reflexive and (1), (2), (3) hold, then \([\sigma]\) is the relative complement to \([\gamma]\) w.r.t. \([\alpha]\), \([\beta]\).

4.9. Theorem. Let \( \alpha, \beta, \xi \in \text{Bin} \) and let \( \alpha <_A \beta \). Put \( \xi = \alpha + \{\sim \text{Con}_\beta \land \text{Con}_\alpha \} \) and \( \varepsilon(x) = \xi(x) \lor x \equiv \sim \text{Con}_\beta \land \text{Con}_\alpha \). Let \( \gamma \) be defined as follows:

\[\gamma(x) = \alpha(x) \lor F_{\text{fin}}(\xi)(x) \lor (\sim R_{\eta}(\eta_1) \land R_{\eta}(0 \equiv 1, \eta_2)).\]

Then \([\gamma]\) is complementible w.r.t. \([\alpha], [\beta]\) if and only if

(1) \( \vdash \xi \sim \text{Con}_\xi \), i.e. if and only if

(1)' \( \vdash A (\sim \text{Con}_\beta \land \text{Con}_\alpha) \rightarrow \text{Pr}_A (\sim \text{Con}_\alpha) \).

Proof. Note that \( \gamma \in \text{Bin}, \alpha <_A \gamma <_A \beta \) (cf. Theorem 2.12) and

(2) \( \vdash A (\sim \text{Con}_\beta \land \text{Con}_\alpha) \leftrightarrow (\text{Con}_\gamma \leftrightarrow \xi_\text{E}) \).

(i) Let \([\gamma]\) be complementible w.r.t. \([\alpha], [\beta]\).

By Theorem 4.7, there exists an \( n \in \omega \) such that

(3) \( \vdash A (\sim \text{Con}_\beta \land \text{Con}_\gamma) \rightarrow \text{Pr}_{[A \Delta n]} (\text{Con}_\alpha \rightarrow \text{Con}_\gamma) \).

Hence

(4) \( \vdash A (\sim \text{Con}_\beta \land \text{Con}_\gamma) \rightarrow \text{Pr}_{[A \Delta n]} (\sim \text{Con}_\beta \land \text{Con}_\alpha \rightarrow \text{Con}_\gamma) \).

(2) gives
(5) \[ \neg \alpha \rightarrow \neg \phi \left( \overline{\text{Con}_\gamma} \leftrightarrow \overline{\phi} \right). \]

(4) and (5) show that \[ \neg \alpha \rightarrow \left( \neg \text{Con}_\beta \land \neg \text{Con}_\gamma \right) \rightarrow \neg \phi \left( \overline{\text{Con}_\gamma} \right) \] and therefore

(6) \[ \neg \alpha \rightarrow \left( \neg \text{Con}_\beta \land \neg \text{Con}_\gamma \right) \rightarrow \neg \phi \left( \overline{\text{Con}_\gamma} \right). \]

By (2), \[ \neg \alpha \rightarrow \left( \neg \text{Con}_\gamma \land \neg \text{Con}_\alpha \right) \rightarrow \neg \phi \left( \overline{\text{Con}_\gamma} \right). \]
Hence

(7) \[ \neg \alpha \rightarrow \left( \neg \text{Con}_\gamma \land \neg \text{Con}_\alpha \right) \rightarrow \neg \phi \left( \overline{\text{Con}_\gamma} \right). \]

(6) and (7) give \[ \neg \alpha \rightarrow \left( \neg \text{Con}_\beta \land \neg \text{Con}_\alpha \right) \rightarrow \neg \phi \left( \overline{\text{Con}_\gamma} \right). \]

(ii) Let \[ \neg \gamma \rightarrow \neg \phi \left( \overline{\text{Con}_\gamma} \right). \]

Put
\[ \delta \left( x \right) = \alpha \left( x \right) \lor \text{Fm}^{(x)} \left( \gamma \right) \land \bigwedge_{x < \gamma} \left( \neg \phi \left( \overline{\text{Con}_\gamma} \right) \land \neg \phi \left( \overline{\text{Con}_\beta} \right) \right) \land \left( P \left( \phi \left( \overline{\text{Con}_\gamma} \right) \right) \land \text{Fm}^{(x)} \left( \gamma \right) \right). \]
Evidently, \[ \delta \in \text{Bin} \] and \[ \alpha \leq \alpha \leq \delta \leq \beta \]. We have

\[ \neg \gamma \rightarrow \neg \phi \left( \overline{\text{Con}_\gamma} \right) \rightarrow \neg \left( \neg \phi \left( \overline{\text{Con}_\gamma} \right) \lor \phi \left( \overline{\text{Con}_\gamma} \right) \right) \]

and it follows that \[ \neg \alpha \rightarrow \left( \neg \text{Con}_\beta \land \neg \text{Con}_\alpha \right) \rightarrow \left( \neg \phi \left( \overline{\text{Con}_\gamma} \right) \lor \phi \left( \overline{\text{Con}_\gamma} \right) \right). \]
Hence, by Lemma 4.5, \[ \gamma \] is complemen-
tible w.r.t. \[ \alpha \], \[ \beta \].

4.10. Corollary. Let \[ \alpha , \beta , \gamma_1 , \gamma_2 \in \text{Bin} \] and let \[ \alpha \leq \alpha \leq \gamma_1 \leq \beta \]. Suppose that both \[ \gamma \] are complemen-
tible w.r.t. \[ \alpha \], \[ \beta \].

Then there exists a \[ \gamma \in \text{Bin} \] such that

(i) \[ \gamma_1 \leq \gamma \leq \gamma_2 \]

(ii) \[ \gamma \] is complemen-
tible w.r.t. \[ \alpha \], \[ \beta \].

Proof. It suffices to take \[ \gamma \] from Theorem 4.9, where we replace \[ \alpha \] by \[ \gamma_1 \], \[ \beta \] by \[ \gamma_2 \] and \[ \xi \] by \[ \gamma \].

The assertion follows from Lemma 4.4.
4.11. **Corollary.** Let \( \alpha, \beta \in \text{Bin} \) , \( \alpha \leq \beta \).
Denote by \( \text{Comp} (\alpha, \beta) \) the set of all \( [\gamma] \) such that

(i) \( \alpha \leq [\gamma] \leq [\beta] \),

(ii) \([\gamma]\) is complementible w.r.t. \([\alpha], [\beta]\).

Then the structure \( \langle \text{Comp} (\alpha, \beta), \leq, \cap, \cup \rangle \) is an atomless (denumerable) Boolean algebra. (Note that it is known that all such algebras are isomorphic.)

We shall now be interested in non-complementible elements.

4.12. **Theorem.** Let \( \mathcal{A} \) be reflexive, \( \alpha, \beta \in \text{Bin} \) and suppose \( \alpha \leq [\beta] \). Then there exists a \( \gamma \in \text{Bin} \) such that

(i) \( \alpha \leq [\gamma] \leq [\beta] \),

(ii) \([\gamma]\) is non-complementible w.r.t. \([\alpha], [\beta]\).

**Proof.** Let \( E = A \cup \{ \sim \text{Con}_{[\beta]} \land \text{Con}_{[\alpha]} \} \), put \( \varepsilon_1 (x) = \alpha (x) \lor x \sim \text{Con}_{[\beta]} \land \text{Con}_{[\alpha]} \) and let \( \mathcal{E} = \langle E, K \rangle \). Evidently, \( \mathcal{E} \) is consistent and reflexive (cf. Theorem 5.8 [1]) and \( \varepsilon_1 (x) \) is a PR-formula in \( \mathcal{P} \) bi-numerating \( E \). Using the diagonal construction 5.1 [1], determine a \( \varphi \) such that

\[ \vdash_{\mathcal{E}} \varphi \leftrightarrow \bigwedge_{\sim} (P \varphi \varepsilon_1 (\overline{\alpha}, x) \rightarrow \sim \text{Con}_{[\varepsilon_1 \cap \alpha]}). \]

Suppose \( \not\vdash_{\mathcal{E}} \varphi \). Then for some \( m \), we would have \( \vdash_{\mathcal{E}} \sim \text{Con}_{[\varepsilon_1 \cap \alpha]} \), which would make \( \mathcal{E} \) inconsistent. Hence
Define $\xi$, $\varepsilon$, $\eta$ as follows:

$$\xi(x) = \alpha(x) \land \land \land \sim \forall x \forall y < x \exists P_x f_{\xi_1}(\overline{y}, \overline{y}),$$

$$\varepsilon(x) = \xi(x) \lor x \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim 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(i) Non-complementible elements are dense in 
\(<\{\alpha\};\{\beta\}>\); i.e., for every \(\sigma', \tau \in \text{Bin}\) such that 
\(\alpha \leq_{A} \sigma' \leq_{A} \tau \leq_{A} \beta\) there is a non-complementible \(\gamma\) 
such that \(\sigma' \leq_{A} \gamma \leq_{A} \tau\).

(ii) Non-complementible elements are not closed w.r.t.
the operations \(\cup, \cap\); in fact, for every \(\gamma \in \text{Bin}\) 
such that \(\alpha \leq_{A} \gamma \leq_{A} \beta\) there are \(\sigma', \tau \geq_{A} \alpha\) 
such that \([\sigma] \cup [\tau] = [\gamma]\) and \([\sigma'], [\tau]\) are 
non-complementible. Similarly, for every \(\sigma \in \text{Bin}\) such 
that \(\alpha \leq_{A} \sigma' \leq_{A} \beta\) there are \(\sigma', \tau \leq_{A} \beta\) such 
that \([\sigma'] \cap [\tau] = [\gamma]\) and \([\sigma'], [\tau]\) are non-complementible.

(Consequently, the interval \(<\{\alpha\};\{\beta\}>\) is generated 
by its non-complementible elements.)

**Proof.** (i) follows from Theorem 4.12 and Lemma 4.4 (i).

(ii) Let \(\alpha \leq_{A} \gamma \leq_{A} \beta\). By Corollary 4.10 
there are \(\sigma'_1, \tau'_1 \in \text{Bin}\) such that \(\alpha \leq_{A} \sigma'_1\), 
\(\tau'_1 \leq_{A} \gamma\) and \([\sigma'_1] \cup [\tau'_1] = [\gamma]\). It follows from 
the part (i) of this corollary that we can define non-complementible \(\sigma', \tau\) such that \(\sigma'_1 \leq_{A} \sigma \leq_{A} \gamma\) and 
\(\tau'_1 \leq_{A} \tau \leq_{A} \gamma\). Evidently, \([\sigma] \cup [\tau] = [\gamma]\).

The second part of the assertion can be proved analogously.

The following theorem shows that the dual theorem to 
Theorem 3.2 does not hold.

4.14. **Theorem.** Let \(A\) be \(\omega\)-consistent and let 
\(\alpha \in \text{Bin}\). Then there exists a \(\gamma \in \text{Bin}\) such that 

(i) \(\alpha \leq_{A} \gamma\),

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(ii) \([\gamma]\) is non-complementible w.r.t. \([\alpha],[\beta]\) for any \(\beta \not\leq A \gamma\); in other words

(iii) there is no \(\delta \not\leq A \alpha\) for which \([\gamma] \cap \delta \not\leq A \gamma\).  

Proof. Note that the proof will only be a deeper analysis (formalization) of the proof of 7.5 [1]. 

Let \(D = A + \{ \sim P_n\alpha (\sim Con\alpha) \}\). To show that \(D\) is consistent, we shall show that 

\[ \vdash_A P_n\alpha (\sim Con\alpha) \] 

Let \(\vdash_A P_n\alpha (\sim Con\alpha)\), i.e. 

\[ \vdash A \cup P_n f_\alpha (\sim Con\alpha, \eta) \] 

It follows from \(\omega\)-consistency of \(A\) that there exists an \(m \in \omega\) such that 

\[ \vdash A \sim P_n f_\alpha (\sim Con\alpha, m) \] 

The formula 

\[ P_n f_\alpha (\sim Con\alpha, m) \] 

is a PR-formula in \(P\), and therefore decidable. Consequently, there exists an \(m \in \omega\) such that 

\[ \vdash A P_n f_\alpha (\sim Con\alpha, m) \] 

Hence 

\[ \vdash A \sim Con\alpha \] 

since \(P_n f_\alpha\) bi-numerates \(P_n f_\alpha\). 

On the other hand, \(\vdash A \sim Con\alpha\), since \(A\) is \(\omega\)-consistent. Hence, 

\[ \vdash A P_n\alpha (\sim Con\alpha) \] 

Put \(\xi(x) = \alpha(x) \vee x \equiv Con\alpha\). Evidently, 

\[ (1) \quad \vdash_D Con\xi \] 

Using the diagonal construction 5.1 [1], we can construct a \(\nu_\xi \in FM_{\kappa_0}\) such that 

\[ \vdash D \nu_\xi \leftrightarrow \sim \cup P_n f_\xi (\Delta_\xi) \] 

It follows from 5.6 [1] that 

\[ (2) \quad \vdash A \nu_\xi \rightarrow Con\xi \] 

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Hence, by (1), we have

\[ \neg \vdash \gamma \vdash_2 \gamma, \quad \text{i.e.} \quad \neg \vdash P\nu_2 (\overline{\gamma_2}). \]

Put

\[ \gamma(x) = \alpha(x) \lor \nu_{\kappa}(x) \land \nu_{\gamma}(x, y). \]

Evidently, \( \gamma \in \text{Bin} \) and

\[ \neg \vdash \text{Con}_\gamma \rightarrow \nu_2. \]

Hence there exists an \( m_0 \in \omega \) such that for every \( m \geq m_0 \)

\[ \neg \vdash \text{Pr}_{[\alpha \land m]} (\text{Con}_\gamma \rightarrow \nu_2). \]

Since \( \neg \vdash \text{Pr}_{[\alpha \land m]} (\text{Con}_\alpha \rightarrow \nu_2) \rightarrow \text{Pr}_2 (\nu_2 \land x), \) we have,

by (1),

\[ \neg \vdash \text{Prop}_{[\alpha \land m]} (\text{Con}_\alpha \rightarrow \nu_2) \quad \text{for every } m \in \omega. \]

(5) and (6) give

\[ \neg \vdash \text{Prop}_{[\alpha \land m]} (\text{Con}_\alpha \rightarrow \text{Con}_\gamma) \quad \text{for every } m \geq m_0 \]

and therefore for every \( m \in \omega \).

Let \( \beta \models \gamma \) and let \( [\gamma] \) be complementible w.r.t. \([\alpha],[\beta]\). By Theorem 4.7, there exists an \( m \in \omega \) such that

\[ \neg \vdash \text{Prop}_{[\alpha \land m]} (\text{Con}_\beta \land \text{Con}_\gamma) \rightarrow \text{Prop}_{[\alpha \land m]} (\text{Con}_\alpha \rightarrow \text{Con}_\gamma). \]

Hence, by (7) and (8), we have

\[ \neg \vdash \text{Prop}_{[\alpha \land m]} (\text{Con}_\beta \land \text{Con}_\gamma) \rightarrow \text{Prop}_{\alpha} (\text{Con}_\gamma). \]

On the other hand, \( \neg \vdash \text{Prop}_{\alpha} (\text{Con}_\gamma) \rightarrow \text{Con}_2 \).
and therefore, by (2) and (4),

\[(10) \quad \vdash \alpha \mathcal{D}_{\alpha} \left( \sim \mathsf{Con}_{\alpha} \right) \rightarrow \sim \mathsf{Con}_{\gamma}. \]

But (9) and (10) show that \( \vdash \alpha \sim \mathsf{Con}_{\beta} \rightarrow \sim \mathsf{Con}_{\gamma} \), which is a contradiction with the assumption \( \gamma \triangleleft_\alpha \beta \).

4.15. Theorem. Let \( \mathcal{A} \), \( \alpha, \beta, \gamma, \phi, \tau \in \mathsf{Bin} \) and \( \alpha \triangleleft_\mathcal{A} \tau \triangleleft_\mathcal{A} \gamma \triangleleft_\mathcal{A} \phi \triangleleft_\mathcal{A} \beta \). Suppose that \([\gamma]\) is not complementible w.r.t. \([\alpha]\), \([\beta]\). Then there exist \( \gamma_1, \gamma_2 \in \mathsf{Bin} \) such that

(i) \( \tau \triangleleft_\mathcal{A} \gamma_1 \triangleleft_\mathcal{A} \gamma \triangleleft_\mathcal{A} \gamma_2 \triangleleft_\mathcal{A} \phi \),

(ii) if \( \gamma_1 \triangleleft_\mathcal{A} \gamma' \leq_\mathcal{A} \gamma_2 \), then \([\gamma']\) is not complementible w.r.t. \([\alpha]\), \([\beta]\).

Proof. Let

\[ E_1 = A \cup \{ \sim \mathsf{Con}_{\beta} \land \mathsf{Con}_{\gamma} \} \quad \text{and} \quad E_2 = A \cup \{ \sim \mathsf{Con}_{\gamma} \land \mathsf{Con}_{\phi} \} \]

and \( \mathcal{L}_2 = \langle E_2, K \rangle \). Evidently, \( \mathcal{L}_1 \) bi-numerates \( E_i \) \((i = 1, 2)\) and \( \mathcal{L}_i \) \((i = 1, 2)\) is consistent.

Using the diagonal construction 5.1[1], determine \( \varphi \) such that

\[ \vdash \varphi \leftrightarrow \bigwedge_{\gamma} \left[ \mathsf{Pr}_f_{\mathcal{L}_1} (\bar{\varphi}, \gamma) \lor \mathsf{Pr}_f_{\mathcal{L}_2} (\bar{\varphi}, \gamma) \right] \rightarrow \sim \mathsf{Con}_{\alpha} \land \mathsf{Con}_{\gamma} \land \sim \mathsf{Con}_{\phi}. \]

Suppose \( \vdash \varphi \). Then for some \( n \), we would have

\[ \vdash \mathcal{L}_1 \sim \mathsf{Con}_{\alpha} \land \mathsf{Con}_{\gamma} \land \sim \mathsf{Con}_{\phi}, \quad \text{i.e.} \]

\[ \vdash \sim \mathsf{Con}_{\beta} \land \mathsf{Con}_{\gamma} \rightarrow \mathsf{Pr}_{\mathcal{L}_1} (n) (\sim \mathsf{Con}_{\alpha} \rightarrow \mathsf{Con}_{\gamma}). \]
But \( \varphi \) is not completable w.r.t. \( [\alpha] \), \( [\beta] \) and therefore, by Theorem 4.7,

\[
\vdash A (\neg \text{Con}_\beta \land \text{Con}_\varphi) \rightarrow \text{Pr} [A \land m] (\text{Con}_\alpha \rightarrow \text{Con}_\varphi).
\]

Hence we have proved

\( (1) \) \( \vdash \varphi \).

Suppose \( \vdash \varphi \). Then for some \( n \), we would have

\[
\vdash \varphi \land \text{Con}_\alpha \land \exists x \in \exists \neg \text{Con}_\alpha \land \exists \neg \text{Con}_\varphi.
\]

Let \( n' = \max (n, \text{Con}_\alpha \land \exists \neg \text{Con}_\varphi) \). Then

\[
\vdash \varphi \land \text{Con} [\varphi \land n'].
\]

On the other hand, from reflexivity of \( A \), we have \( \vdash \varphi \). Hence

we have proved

\( (2) \) \( \vdash \varphi \).

Put \( \xi' (x) = \alpha (x) \land \bigwedge \psi \leq \alpha \neg \text{Pr} \varphi \xi \psi (\varphi, \psi) \land \neg \text{Pr} \varphi \xi \varphi (\varphi, \varphi) \).

Evidently, \( \xi' \in \text{Bin} \). Analogously as in the proof of Theorem 4.12, we can show

\( (3) \) \( \neg \varphi \land \varphi \rightarrow \text{Con} \xi' \land \exists x \in \exists \neg \text{Con}_\alpha \land \exists \neg \text{Con}_\varphi \),

\( (4) \) \( \neg \varphi \land \varphi \rightarrow \bigwedge \bigwedge (\xi' (x) \leftrightarrow \alpha (x) \land x \leq x) \).

Let \( \mu_1, \alpha \) be defined w.r.t. the theories

\( A + \{ \neg \text{Con}_\beta \land \text{Con}_\varphi \} \), \( A + \{ \neg \text{Con}_\beta \land \text{Con}_\varphi \land \neg \varphi \} \)

and \( A + \{ \neg \text{Con}_\varphi \land \text{Con}_\alpha \land \neg \varphi \} \)

(cf. Definition 1.16). Further let \( \mu_2, \alpha \) be defined w.r.t. the theories \( A + \{ \neg \text{Con}_\beta \land \text{Con}_\varphi \land \neg \varphi \land \mu_1, \alpha \} \)

and \( A + \{ \neg \text{Con}_\varphi \land \text{Con}_\alpha \land \neg \varphi \land \mu_1, \alpha \} \).

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Put

\( (5) \quad \xi(x) = \xi'(x) \lor \\
\lor y < x [M \rightarrow (x) \land x \equiv (C(x) \rightarrow C(y) \land x \equiv y)] \lor \\
\lor y < x [M \rightarrow (x) \land x \equiv (C(x) \land \neg C(y) \land y \equiv y)] \),

\( (6) \quad \gamma_1(x) = \pi(x) \lor \\
\lor Fm_{\gamma}(x) \lor y < x \lor \left( F_i(x) (C_{\gamma} \rightarrow C_{\gamma}) \land \\
\lor F_{\gamma}(0 \approx 1, y_2) \right) \),

\( (7) \quad \gamma_2(x) = \gamma(x) \lor \\
\lor Fm_{\gamma}(x) \lor y < x \lor \left( F_i(x) (C_{\gamma} \rightarrow C_{\gamma}) \land \\
\lor F_{\gamma}(0 \approx 1, y_2) \right) \).

Evidently, \( \xi, \gamma_1, \gamma_2 \in \text{Bin} \).

(i) The inequalities \( \xi \leq_{\alpha} \gamma_1 \leq_{\alpha} \gamma \leq_{\alpha} \gamma_2 \leq_{\alpha} \theta \)

are evident. We have (cf. Theorem 1.18)

\( (8) \quad \vdash_{\alpha} (~ C_{\nu} \land C_{\nu}) \rightarrow \mu_{1, \alpha} \).

It is clear that

\( (9) \quad \vdash_{\alpha} \sim \mu_{1, \alpha} \rightarrow F_i(x) (C_{\nu} \rightarrow C_{\nu}) \),

\( (10) \quad \vdash_{\alpha} \sim C_{\nu} \land F_i(x) (C_{\nu} \rightarrow C_{\nu}) \rightarrow \sim C_{\nu} \).

and therefore

\( (11) \quad \vdash_{\alpha} \sim C_{\nu} \land C_{\nu} \rightarrow \mu_{1, \alpha} \).

(8) and (11) immediately give

\( (12) \quad \vdash_{\alpha} C_{\nu} \rightarrow C_{\nu} \),

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i.e. we have proved \( \Downarrow_A \gamma \leq_A \gamma' \).

We have (cf. Theorem 1.18)

(13) \( \Downarrow_A (\neg \text{Con}_\gamma \land \text{Con}_\alpha \land \neg \varphi \land \mu_{1, \alpha} \rightarrow \neg \mu_{2, \alpha} \).

Evidently, we have

(14) \( \Downarrow_A (\mu_{1, \alpha} \land \mu_{2, \alpha}) \rightarrow \bigwedge_x (\xi'(x) \leftrightarrow \xi(x)) \)

and therefore, by 4.4, we have

(15) \( \Downarrow_A (\sim \varphi \land \mu_{1, \alpha} \land \mu_{2, \alpha}) \rightarrow \bigvee_x (\xi(x) \leftrightarrow \alpha(x) \land x \leq x). \)

We know that

(16) \( \Downarrow_A \text{Con}_\alpha \rightarrow \sim \text{Pr}_\alpha (\overline{\text{Con}_\alpha}) \),

since \( \Downarrow_A \text{Con}_\alpha \rightarrow \text{Con}_\alpha \) and \( \Downarrow_A \text{Con}_\alpha \rightarrow \sim \text{Pr}_\alpha (\overline{\text{Con}_\alpha}) \)

(cf. Theorem 5.6 [1]). (15) and (16) give

(17) \( \Downarrow_A (\text{Con}_\alpha \land \mu_{1, \alpha} \land \mu_{2, \alpha} \land \neg \varphi) \rightarrow \sim \text{Pr}_\delta (\overline{\text{Con}_\alpha}) \)

and therefore

(18) \( \Downarrow_A (\text{Con}_\alpha \land \mu_{1, \alpha} \land \mu_{2, \alpha} \land \neg \varphi) \rightarrow \text{Con}_{\gamma_1} \)

since \( \Downarrow_A \sim \text{Pr}_\delta (\overline{\text{Con}_\alpha}) \rightarrow \sim \text{Pr}_\delta (\overline{\text{Con}_\alpha \land \sim \text{Con}_\gamma}) \) and

\( \Downarrow_A (\text{Con}_\alpha \land \sim \text{Pr}_\delta (\overline{\text{Con}_\alpha \land \sim \text{Con}_\gamma})) \rightarrow \text{Con}_{\gamma_1} \), (13) and

(18) imply

(19) \( \Downarrow_A \text{Con}_{\gamma_1} \rightarrow \text{Con}_\gamma \),

i.e. we have proved \( \Downarrow_A \gamma_1 \leq_A \gamma' \).

(ii) Let \( \gamma_1 \leq_A \gamma' \leq_A \gamma_2 \) and let \([\gamma']\) be complementible w.r.t. \([\alpha],[\beta]\). Then there exists an \( n \in \omega \) such that \( \Downarrow_A (\sim \text{Con}_\beta \land \text{Con}_{\gamma'}) \rightarrow \text{Pr}_k \land \text{Con}_\alpha \rightarrow \text{Con}_{\gamma'} \) (cf. Theorem 4.7) and
therefore there exists an $m \in \omega$ such that

\[(20) \vdash \neg (\neg \text{Con}_{\beta} \land \text{Con}_{\gamma}) \rightarrow \text{Pr}_{\text{can}} (\text{Con}_{\alpha} \rightarrow \text{Con}_{\gamma}).\]

We shall show that it is impossible.

We have (cf. Theorem 1.18)

\[(21) \vdash \neg (\neg \text{Con}_{\beta} \land \text{Con}_{\gamma} \land \neg \varphi \land \mu_{1,\alpha}) \rightarrow \mu_{2,\alpha}.\]

It is clear that

\[(22) \vdash \varphi \sim \mu_{2,\alpha} \rightarrow \text{Pr}_{\tilde{f}} (\text{Con}_{\alpha} \land \sim \text{Con}_{\gamma}) \text{ and in particular}\]

\[(23) \vdash \varphi \sim \mu_{2,\alpha} \rightarrow \text{Pr}_{\tilde{f}} (\sim \text{Con}_{\gamma}).\]

On the other hand, we have from (22)

\[(24) \vdash \varphi \sim \mu_{2,\alpha} \rightarrow \text{Pr}_{\tilde{f}} (\text{Pr}_{\tilde{f}} (\text{Con}_{\alpha} \land \sim \text{Con}_{\gamma})),\]

since $\text{Pr}_{\tilde{f}} (\text{Con}_{\alpha} \land \sim \text{Con}_{\gamma})$ is an RE-formula in $\mathcal{P}$ (cf. 1.7).

(6), (23) and (24) show that

\[(25) \vdash \varphi \sim \mu_{2,\alpha} \rightarrow \text{Pr}_{\tilde{f}} (\sim \text{Con}_{\gamma}).\]

By (3) and (5),

\[(26) \vdash \varphi (\neg \varphi \land \mu_{1,\alpha}) \rightarrow \sim \text{Pr}_{\tilde{f}} (\text{Con}_{\alpha} \rightarrow \text{Con}_{\gamma}) \text{ and therefore}\]

\[(27) \vdash \varphi (\neg \varphi \land \mu_{1,\alpha}) \rightarrow \sim \text{Pr}_{\tilde{f}} (\sim \text{Con}_{\alpha}).\]

On the other hand, by (26) and (7)

\[(28) \vdash \varphi (\text{Con}_{\gamma} \land \neg \varphi \land \mu_{1,\alpha}) \rightarrow \text{Con}_{\gamma}.\]

Using (21), (25) and (28) we can easily show

\[(29) \vdash \varphi (\neg \text{Con}_{\beta} \land \neg \text{Con}_{\gamma} \land \text{Pr}_{\tilde{f}} (\sim \text{Con}_{\gamma}) \rightarrow \text{Pr}_{\tilde{f}} (\sim \text{Con}_{\alpha}).\]
On the other hand, using (20), we have

\[(30) \vdash \sim \alpha \sim \operatorname{Con}_{\beta} \land \operatorname{Con}_{\beta} \land \mathcal{P}_{\gamma} (\sim \operatorname{Con}_{\delta}) \rightarrow \mathcal{P}_{\delta} (\sim \operatorname{Con}_{\xi}),\]

since \( \vdash \mathcal{P}_{\xi} (\operatorname{Con}_{\alpha} \rightarrow \operatorname{Con}_{\beta}) \rightarrow \mathcal{P}_{\gamma} (\operatorname{Con}_{\alpha} \rightarrow \operatorname{Con}_{\delta}) \)

and \( \vdash (\mathcal{P}_{\xi} (\operatorname{Con}_{\alpha} \rightarrow \operatorname{Con}_{\beta}) \land \mathcal{P}_{\delta} (\sim \operatorname{Con}_{\xi})) \rightarrow \mathcal{P}_{\delta} (\sim \operatorname{Con}_{\xi}). \)

This completes the proof.

References


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