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DIFFERENTIAL OPERATORS AND G-STRUCTURES OF HIGHER ORDER

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0. Introduction. A differential operator D of order r on the manifold M of dimension n (manifolds and maps are assumed to be C^{∞}-differentiable) can be expressed in every admissible coordinate system (c.s.) $(x^1 \dots x^n)$ on M in the form:

$$D = \sum_{i=1}^{n} \alpha_i(x) \frac{\partial}{\partial x^i} + \dots + \sum_{i_1, \dots, i_r} \alpha_{i_1 \dots i_r} \frac{\partial^r}{\partial x^{i_1} \dots \partial x^{i_r}}$$

where $\alpha_i, \alpha_{i_1 i_2}, \dots$ are differentiable functions on M. Changing the coordinate system, we obtain the expression of D using another functions $\alpha'_{i \dots j}$. We ask if the coordinates can be changed in such a way that the functions α_i expressed in the new c.s. are constant. We try to solve this problem only locally. The coefficients of the differential operator of order r are transformed in such a way that we use the derivatives up to the order r of the change of coordinates only. Thus the differential operator of order r determines a geometrical object of order r and determines a mapping of the principal fibre bundle of r-frames $H^r(M)$ into the standard r-tangent space $F^r = T_0^r(R^n)$.

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The differential operator having in some c.s. constant coefficients,

(i) it maps $H^{k}(M)$ onto an orbit of the group L^{k}_{m} in F^{k} (see §3),

(ii) it determines a class of conjugate G-structures of order k on M ,

(iii) the G-structures are integrable (see § 4) on the coordinate domain.

Conversely, if the conditions (i) - (iii) are fulfilled, the considered differential operator has constant coefficients.

Guillemin and Singer in [3] solved this problem for coefficients at the highest derivatives transforming it into a problem of integrability of G-structures of order 1 . This paper generalizes the solution on of this problem to all coefficients. These conditions will be examined in more detail in the next paper.

1. Let us denote (following [1]) by L^{k}_{m} the Lie group of all invertible k-jets of maps from R^{n} into R^{km} with source and target in $0 \in R^{m}$. The composition of such two elements $\xi = j^{k}_{0} f$ and $\eta = j^{k}_{0} g$ is given by $\xi \circ \eta = j^{k}_{0}(f \circ g)$.

If M is a differentiable manifold (of class C^{∞}) and $m \in M$, we denote by

$A(m)$ the set of all differentiable functions defined on some neighbourhood of m ;

$A_0(m) \subset A(m)$ the subset of all functions which have the value 0 at the point m ;

$A_c(m) \subset A(m)$ the subset of all functions which are constant on some neighbourhood of m ;

$A_0^{k+1}(m) \subset A(m)$ the subset of all multiples of $k+1$ elements of $A_0(m)$.

The k-tangent space $T_m^k(M)$ of the manifold M at the point m may be defined as a vector space whose elements are linear functionals

(1) $$X : A(m) \longrightarrow R$$

such that $A_0^{k+1}(m) \cup A_c(m) \subset Ker\, X$ (see [1]).

If $h = (x^1 \dots x^n)$ is a coordinate system in a neighbourhood of $m \in M$, then $X \in T_m^k(M)$ can be expressed uniquely in the form

(2) $$X = \sum_{i=1}^{n} X^i \frac{\partial}{\partial x^i} + \dots + \sum_{1 \le i_1 \le \dots \le i_k \le n} \frac{\partial^k}{\partial x^{i_1} \dots \partial x^{i_k}} .$$

For $k = 1$, we obtain the ordinary tangent space of the manifold.

The space $T^k(M) = \bigcup_{m \in M} T_m^k(M)$ can be endowed with a structure of a differentiable manifold such that $T^k(M)$ is a vector bundle over M . A section of the bundle $T^k(M)$ is a map $D : U \longrightarrow T^k(M)$ such that $D(x) \in T_x^k(M)$ for all $x \in U$, and (2) shows that D is a differentiable operator of order $\le k$ on U .

2. Let us denote by $F^k = T_0^k(R^n)$ the k-tangent space at $0 \in R^n$. If $(x^1 \dots x^n)$ is a fixed c.s. on R^n , then the k-vectors

(3) $\partial_{i_1 \dots i_p} = \left(\frac{\partial^p}{\partial x^{k_1} \dots \partial x^{k_p}} \right)_0 \quad 1 \le i_1 \le \cdot \le k_p \le n, \quad p \le k$

form a basis of F^k .

Every differentiable mapping φ of the manifold M into the manifold N induces a differentiable map

$$\varphi_*^k : T^k(M) \longrightarrow T^k(N)$$

given by

(4) $\varphi_*^k(X)f = X(f \circ \varphi), \quad X \in T_m^k(M), \quad f \in A_M(\varphi(m))$.

If φ is a diffeomorphism, then φ_*^k is also a diffeomorphism for every k and it holds

$$(\varphi \circ \psi)_*^k = \varphi_*^k \circ \psi_*^k \quad .$$

Now we define the action ρ of the group L_n^k on F^k. Every $\alpha \in L_n^k$ is given by a map $f : R^n \longrightarrow R^n$ such that $f(0) = 0$. The mapping f determines the mapping f_*^k : : $T_0^k(R^n) \longrightarrow T_0^k(R^n)$ depending on $\alpha = j^k \circ f$ only; it does not depend on f . Let us denote this induced mapping by $\tilde{\alpha}$ and define $\rho(\alpha) : F^k \longrightarrow F^k$ by

(5) $\rho(\alpha) X = (\tilde{\alpha}^{-1}) X = (f^{-1})_*^k X$

for $X \in F^k \quad \alpha = j^k \circ f$.

If $\beta = j^k \circ g$, then $\rho(\alpha \cdot \beta) X = \rho(\beta)(\rho(\alpha) X)$.

<u>Lemma 1</u> [1] . L_n^k acts on F^k effectively from the left.

3. Let $H^k(M)$ be the principal fibre bundle of k-frames on M and let $\pi : H^k(M) \longrightarrow M$ be the

canonical projection.

The element $p \in H^k(M)$ is the k-jet at $0 \in R^n$ of a local diffeomorphism F of a neighbourhood of $o \in \in R^n$ into M satisfying $F(o) = \pi(p)$.

The mapping F^{-1} is a c.s. defined on a neighbourhood of $\pi(p)$ and it determines an isomorphism

$$(6) \qquad (F^{-1})^k_* : T^k_{\pi(p)}(M) \to F^k$$

which depends on p only and which does not depend on F.

Let us denote, for every $X \in T^k_{\pi(p)}$, by $\Phi_X(p) = = (F^{-1})^k_* X$ the element of F^k obtained in the above described way, and denote by

$$(7) \qquad \Phi_X : \pi^{-1}(\pi(p)) \to F^k$$

the obtained mapping.

If D is a differential operator of order k on an open set $U \subset M$, then D can be treated as a field of k-vectors on U, and we can construct the map

$$(8) \qquad \Phi_D : H^k(U) \to F^k$$

such that $\Phi_D(p) = \Phi_{D(\pi(p))}(p)$.

The Lie group L^k_n acts on $H^k(U)$ and also on F^k. Let us examine $\Phi_D(p \cdot \alpha)$.

Let $p = j^k_o F$, $\alpha = j^k_o f$, then $p \cdot \alpha = j^k_o(F \circ f)$, $(F \circ f)^{-1} = f^{-1} \circ F^{-1}$ and

$[(F \circ f)^{-1}]^k_* X = (f^{-1})^k_* (F^{-1})^k_* X = \rho(\alpha)\Phi_D(p)$.

<u>Lemma 2</u>. The mapping Φ_D is a differentiable mapping of $H^k(U)$ into F^k, commuting with the action of the group L^k_n, e.g.,

$$\Phi_{D}(p\cdot\alpha) = \rho(\alpha)\,\Phi_{D}(p) \ .$$

Proof. The first assertion follows directly expressing Φ_D in local coordinates. The second assertion has been shown above.

Remark. Choosing a c.s. on R^n and a base of F^k in the sense of (3), Φ_D is determined by the system of functions $\varphi^i(p), \dots \varphi^{k\dots i_k}(p)$

$$\Phi_D(p) = \sum_{i=1}^{n} \varphi^i(p) \frac{\partial}{\partial x_i}(0) + \dots + \sum_{1 \le k \le \dots \le i_k \le} \varphi^{i\dots i_n}(p) \frac{\partial^k}{\partial x \dots \partial x}$$

this equation corresponding to the expression of D in the coordinate system (more accurately, in the germ of c.s.) given by F^{-1} at the point $\pi(p)$ $(p = j^k \circ F)$.

With any c.s. $h = (x^1 \dots x^n)$ on the neighbourhood U we can canonically associate a section $\tilde{h}$ of the principal fibre bundle $H^k(M)$ over U in the following way:
Let us denote by t_z , $R^n \longrightarrow R^n$ the map defined by

(10) $\qquad t_z(x) = (x + z)$ $\qquad\qquad$ (translation).

For $\bar{y}_0 \in U$, $h(\bar{y}_0) = y_0$, we have $\bar{y}_0 = h^{-1}(y_0) = = h^{-1}(t_{y_0}(0)) = h^{-1} \circ t_{y_0}(0)$, and $\tilde{h}(\bar{y}_0) = j_0^k(h^{-1} \circ t_y)$ is well defined and it is an element of $H_y^k(M)$.

If there exists a c.s. on the neighbourhood U such that D has constant coefficients with respect to it, then Φ_D is constant on the section $\tilde{h}(U)$, and according to Lemma 2 we have

Lemma 3. If there exists a c.s. on M in the neighbourhood V such that D expressed in it has constant coefficients, then it holds:

(11) $\Phi_D(H^k(V))$ is contained in one orbit of the action of the group L^k_n on F^k.

Hence $\Phi_D(H^k(V))$ fills out the whole orbit.

Condition (11) is therefore a necessary condition for the existence of a c.s. in a neighbourhood of $m \in M$ such that D has constant coefficients in it.

4. Our next task is to produce necessary and sufficient conditions. We solve this problem locally.

Let us assume that M is a neighbourhood of $o \in R^n$, D a differential operator of order k on M satisfying (11), and denote by $\mathcal{O}$ the orbit $\Phi(H^k(M))$. Let us choose $p \in H^k(M)$, write $\Phi_D(p_o) = z \in \mathcal{O}$ and let $G_z \subset L^k_m$ be the isotropic group of z with respect to the action, e.g.,

$$G_z = \{\alpha \in L^k_n,\ \varphi(\alpha)z = z\}.$$

The following lemma is obvious.

Lemma 3. G_z is a closed subgroup of L^k_n and thus it is a Lie subgroup of L^k_n.

Now, the main result is given by the following.

Proposition 1. Let D be a differential operator of order k on a neighbourhood M of the point $0 \in R^n$ satisfying the condition (11). Let us choose a $z \in \mathcal{O} = \Phi_D(H^k(M))$. Then $\mathcal{P}_z = \Phi_D^{-1}(z)$ is a G_z-structure on M, where G_z is the isotropic group of z. If z' is another element of the orbit, then $\mathcal{P}_z$ and $\mathcal{P}_{z'}$ are conjugate structures on M.

Proof. The proposition follows from the bundle structure theorem for Lie groups.

Remark. A differential operator of order r satisfying the conditions of Prop. 1 on M determines a system of conjugate structures of order r on M.

Let us now define the notions of equivalence and integrability for G-structures of order r. Every diffeomorphism $g : M \to N$ determines a diffeomorphism $g^r : H^r(M) \to H^r(M)$. For $p = j_0^r F$ define $g^r(p) = j_0^r (g \circ F)$.

If G is a Lie subgroup of L_n^r and $\mathcal{P}_1 \to M$, $\mathcal{P}_2 \to M$ are two G-structures of order r on M and N, resp. then we say that $\mathcal{P}_1$ and $\mathcal{P}_2$ are equivalent if there exists a diffeomorphism $g : M \to M$ such that $g^r(\mathcal{P}_1) = \mathcal{P}_2$.

By the standard G-structure of order r on an open set $\mathcal{U} \subset R^n$ we mean a G-structure $\mathcal{P}_G$ given by

$$\mathcal{P}_G = \{\tilde{h}(x) \cdot \alpha, \ x \in \mathcal{U}, \ \alpha \in G\},$$

h being a fixed global c.s. and $\tilde{h}$ being the canonical section determined by h on $\mathcal{U}$.

The G-structure on M is called integrable if it is equivalent to a standard G-structure of order r on R^n. Similarly as in the case $r = 1$ we may define the notions of the local equivalence and the local integrability.

Lemma 4. The G-structure $\mathcal{P}$ of order r on M is integrable if and only if there is a c.s. h on M such that $\tilde{h}(M)$ is a section of $\mathcal{P}$

Proposition 2. Under the assumptions of Proposition 1 the following statements are equivalent:

(i) $\mathcal{P}_Z$ is an integrable G_Z-structure on M for some (and for all) $Z \in \mathcal{O}$.

(ii) There is a c.s. on M such that D has constant coefficients in it.

Example. It can be easily shown that D being an operator of the first order on $M \subset R^n$, e.g. D being a vector field on M, there is such a c.s. if and only if the field is nonzero everywhere on M.

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