

Josef Kolomý

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SOME MAPPING THEOREMS AND SOLVABILITY OF NONLINEAR
EQUATIONS IN BANACH SPACE

(Preliminary communication)

Josef KOLOMÍ, Praha

In this note we are concerned with some mapping theorems and solvability of nonlinear operator equations in Banach spaces. For the recent results in these topics see for instance Browder [1], Brezis [2], Petryshyn [3] and others.

Let X, Y be normed linear spaces. We use " \rightarrow ", " \xrightarrow{w} " to denote strong and weak convergence, respectively. A mapping $F: X \rightarrow Y$ is said to be

weakly continuous if

$$\mu_n, \mu \in X, \mu_n \xrightarrow{w} \mu \implies F(\mu_n) \xrightarrow{w} F(\mu);$$

demicontinuous if

$$\mu_n, \mu \in X, \mu_n \rightarrow \mu \implies F(\mu_n) \xrightarrow{w} F(\mu);$$

p -positively homogeneous if $F(t\mu) = t^p F(\mu)$

for each $\mu \in X, t \geq 0$, where $p > 0$.

We shall say that a functional φ is quasi-convex on a convex set $M \subset X$, if $\mu, \nu \in M, \lambda \in [0, 1] \implies \varphi(\lambda\mu + (1-\lambda)\nu) \leq \max(\varphi(\mu), \varphi(\nu))$. By $B_\varphi(\mu)$,

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$\partial B_{\sigma}(\mu)$ we denote an open ball centered about μ and with the radius $\sigma > 0$, its boundary, respectively.

Theorem 1. Let X be a reflexive Banach space, $F: X \rightarrow X$ a weakly continuous mapping so that $F(0) = 0$. Assume $E(a) = \{\mu \in X: \|F(\mu)\| \leq a\}$ is bounded for each $a > 0$. Let for some $\lambda > 0$, $\mu, \nu \in X$, $\mu \neq \nu \implies \|\mu - \nu - \lambda(F(\mu) - F(\nu))\| < \|\mu - \nu\|$. Then $F(X) = X$; i.e. for each $\eta \in X$ there exists at least one $\mu \in X$ so that $F(\mu) = \eta$.

Remark. The condition that $E(a)$ is bounded for each $a > 0$ is satisfied for instance when F is μ -positively homogeneous operator on X and $\|F(\mu)\| \geq m > 0$ for each $\mu \in X$, $\|\mu\| = \kappa$ (κ positive and fixed).

Theorem 2. Let X, Y be normed linear spaces, X reflexive, $F: X \rightarrow Y$ a weakly continuous mapping, $F(0) = 0$ and so that $E(a) = \{\mu \in X: \|F(\mu)\| \leq a\}$ is bounded for each $a > 0$. Let $H: X \rightarrow Y$ be a μ -positively homogeneous mapping of X onto Y . Suppose that for each point $\mu \in X$ there exist constants α_{μ} , σ_{μ} ($0 \leq \alpha_{\mu} < 1$, $0 < \sigma_{\mu}$) and a mapping $G_{\mu}: X \rightarrow Y$ so that $\nu \in B_{\sigma_{\mu}}(\mu) \implies \|F(\nu) - F(\mu) - G_{\mu}(\nu - \mu)\| \leq \alpha_{\mu} \|H(\nu - \mu)\|$. Assume there exists $R > 0$ and $\epsilon_{\mu} > 0$ so that $\nu \in B_R(0) \implies \|G_{\mu}(\nu) - H(\nu)\| \leq \epsilon_{\mu} \|H\nu\|$ for each $\mu \in X$. If $\epsilon_{\mu} + \alpha_{\mu} < 1$ for each $\mu \in X$, then $F(X) = Y$.

Corollary 1. Let X, Y be normed linear spaces,

$F: X \rightarrow Y$ a weakly continuous mapping so that $E(\alpha) = \{u \in X: \|F(u)\| \leq \alpha\}$ is bounded for each $\alpha > 0$ and $F(0) = 0$. Let G be μ -positively homogeneous mapping of X onto Y . Assume that for each point $u \in X$ there exist the constants α_u, σ_u ($0 \leq \alpha_u < 1, \sigma_u > 0$) so that $v \in B_{\sigma_u}(u) \Rightarrow \|F(v) - F(u) - G(v-u)\| \leq \alpha_u \|G(v-u)\|$. Then $F(X) = Y$.

Theorem 3. Let X, Y be normed linear spaces, X reflexive, $M \subset X$ open, $F: M \rightarrow Y$ a weakly continuous map on M , $G: X \rightarrow Y$ μ -positively homogeneous mapping onto Y so that for each $u_1, u_2 \in M \Rightarrow \|F(u_1) - F(u_2) - G(u_1 - u_2)\| \leq \alpha \|G(u_1 - u_2)\|$, where $0 \leq \alpha < 1$. Suppose that for each point $u_0 \in M$ there exist the numbers $\sigma_{u_0} > 0, a_{u_0} > 0$ so that $\overline{B_{\sigma_{u_0}}(u_0)} \subset M$ and $u \in \partial B_{\sigma_{u_0}}(u_0) \Rightarrow \|F(u_0) - F(u)\| \geq a_{u_0} \|u - u_0\| = a_{u_0} \sigma_{u_0}$. Then $F(M)$ is open.

Let X, Y be normed linear spaces. Following [4] a mapping $K: X \rightarrow Y$ is said to be relatively open. if for each open $E \subset X$ the set $K(E)$ is open in $K(X)$. A linear (i.e. additive and homogeneous) mapping $K: X \rightarrow Y$ is relatively open \iff if there exists a constant $M > 0$ so that for each $y \in K(X)$ there exists $x \in X$ such that

$$(1) \quad y = Kx \quad \text{and} \quad \|x\| \leq M \|y\|.$$

A mapping $G: B_r(0) \rightarrow Y, B_r(0) \subset X$, is said to be closed on $B_r(0)$ if $u_n, u \in B_r(0)$,

$u_n \rightarrow u, G(u_n) \rightarrow v \implies v = G(u)$. The following theorem generalizes the results of Graves [5] and Bartle [6].

Theorem 4. Let X be a Banach space, Y a normed linear space, $G: B_r(0) \rightarrow Y$ a closed mapping of $B_r(0) \subset X$ into Y so that $G(0) = 0$. Assume $K: X \rightarrow Y$ is linear and relatively open so that $\|G(u) - G(v) - K(u-v)\| \leq \alpha \|u-v\|$, ($\alpha > 0$), for each $u, v \in B_r(0)$, where $\alpha M < 1$ (here M is a constant from (1)). Then the equation $G(u) = \varphi$ has a solution u in $B_r(0)$ provided $\|\varphi\| < \rho = r(1 - M\alpha)/M$.

In comparison with Graves and Bartle we need not assume that Y is complete, G, K are continuous (or G is Fréchet differentiable) and that K is onto Y . Theorems 3,4 are connected with openness of nonlinear operators.

Some other and rather general results concerning this matter will be published in [7].

The following theorems are related to those of Beluce-Kirk [8], Kirk [9] and Daneš [10].

Theorem 5. Let X be a normed linear space, $M \subset X$ a convex subset of X containing 0 , $F: M \rightarrow M$ a mapping so that $u, v \in M, u \neq v \implies \|F(u) - F(v)\| < \|u - v\|$. Assume that for some $c > 0$ the set $\{u \in M: \|u - F(u)\| \leq c\}$ is non-void and weakly compact and that $\varphi(u) = \|u - F(u)\|$ is quasi-convex on M . Then there exists a unique point $u^* \in M$.

that $F(u^*) = 0$.

Theorem 6. Let X, Y be normed linear spaces, $M \subset X$ a convex open subset of X , $0 \in M$, $F: M \rightarrow Y$ a map such that $E(c) = \{u \in M: \|F(u)\| \leq c\}$ is non-void and weakly compact for some $c > 0$. Suppose $G: X \rightarrow Y$ is μ -positively homogeneous mapping of X onto Y . Let for each point $u \in M$ there exist constants α_u, σ_u ($0 \leq \alpha_u < 1, \sigma_u > 0$) so that $B_{\sigma_u}(u) \subset M$ and $v \in B_{\sigma_u}(u) \Rightarrow \|F(v) - F(u) - G(v-u)\| \leq \alpha_u \|G(v-u)\|$. If either a) F is weakly continuous on M , or b) F is demicontinuous on M , and $\psi(u) = \|F(u)\|$ is quasi-convex on M , then there exists $u^* \in M$ so that $F(u^*) = 0$.

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Matematicko-fyzikální fakulta
 Karlova universita
 Sokolovská 83, Praha 8
 Československo

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