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Introduction. The homomorphism graph is defined as a graph which arises in a natural way on the set of all endomorphisms of a graph. Here we are interested in the question, under which conditions a graph is isomorphic with its homomorphism graph.

We shall need the following definitions:
Let X be a set. Let M be a set of mappings of X into itself. The pair (X, M) is called a transformation monoid if the identity mapping of X belongs to M and the set M is closed under composition.

Two transformation monoids (X, M) and (Y, N) are isomorphic if there exists a 1-1 mapping $F : X \longrightarrow Y$ such that the mapping $\mathcal{F} : M \longrightarrow N$ defined by $\mathcal{F}(f)(F(x)) = F(f(x))$ is an algebraic isomorphism of monoids (M, N).

A transformation monoid (X, M) is called abstract if (X, M) is isomorphic with $(M, \mathbf{L}_M)$ where $\mathbf{L}_M = \{L_f \mid f \in M\}$ ($L_f : M \longrightarrow M$ is defined by: $L_f(g) = f \circ g$).

We shall use the following well-known lemma (see

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e.g. [4]).

Lemma. A transformation monoid (X, M) is abstract if and only if there exists $x_0 \in X$ such that for every $x \in X$ there exists exactly one $f \in M$ such that $f(x_0) = x$ (x_0 is called an exact source).

A graph (X, R) is a set X with relation $R \subset X \times X$. Concerning graphs we use the notations of [1]. Let us remark that the monoid $C(X, R)$ of all compatible mappings (homomorphisms) of the graph (X, R) into itself is understood here in its actual form as a transformation monoid.

All the graphs concerned here are finite.

The following definition was suggested by Z. Hedrlín.

Definition. Let (X, R) be a graph. Define the homomorphism graph $(C(X, R), M)$ of the graph (X, R) as follows:

$f, g \in C(X, R)$, then $(f, g) \in M \Leftrightarrow (f(x), g(x)) \in R$ for every $x \in X$. Note that this graph is one of the graphs related to tensor products, see [2]. We say that the graph (X, R) is homomorphism perfect if it is isomorphic to the graph $(C(X, R), M)$. The property of being homomorphism perfect is studied here in its relationship to the abstractness of the transformation monoid $(X, C(X, R))$.

Theorem 1. Let (X, R) be a homomorphism perfect graph. Then the transformation monoid $(X, C(X, R))$ is abstract.

Proof. Let F be an isomorphism of (X, R) onto $(C(X, R), M)$ - the homomorphism graph of (X, R).

Consider $C(C(X,R),M)$. Clearly, $L_f \in C(C(X,R),M)$ for every $f \in C(X,R)$ ($(f_i, f_j) \in M$ implies ${}^2L_f(f_i, f_j) = (f \circ f_i, f \circ f_j) \in M$ as f is compatible). Further, $f_i \neq f_j$ implies $L_{f_i} \neq L_{f_j}$.

Thus $\mathrm{card}\, C(X,R) = \mathrm{card}\, C(C(X,R),M) = \mathrm{card}\, L_M$. Since $L_M \subset C(C(X,R),M)$, $L_M = C(C(X,R),M)$ holds. Thus it remains to prove that the transformation monoids $(X, C(X,R))$ and $(C(X,R), C(C(X,R),M))$ are isomorphic.

We shall show that this isomorphism is carried by the mapping F, i.e. that the mapping $\mathcal{F}$ defined by $\mathcal{F}(f)(F(x)) = F(f(x))$ is an algebraic isomorphism.

First, we shall prove that $f \in C(X,R)$ implies $\mathcal{F}(f) \in C(C(X,R),M)$. Let $(f_i, f_j) \in M$. Then

$$(\mathcal{F}(f)(f_i), \mathcal{F}(f)(f_j)) = (\mathcal{F}(f)F(F^{-1}(f_i)),$$

$$\mathcal{F}(f)F(F^{-1}(f_j))) = (F(f(F^{-1}(f_i))), F(f(F^{-1}(f_j)))) \in M .$$

Evidently $\mathcal{F}(f \circ g) = \mathcal{F}(f) \circ \mathcal{F}(g)$ for every f, g. Further, $f \neq g$ implies $\mathcal{F}(f) = \mathcal{F}(g)$ and consequently $\mathcal{F}$ is 1-1. Q.E.D.

Theorem 1 does not give a sufficient condition for homomorphism perfect graphs. We construct a graph (even a class of graphs) possessing an abstract transformation monoid of homomorphisms into itself which is not a homomorphism perfect graph.

<u>Example.</u> Let m be an even number. Define the graph (X_m, R) by $X_m = \{1, \dots, m\}$ and $R = \{(1,1), (2,2),$

$(3,3),\dots,(\frac{n}{2},\frac{n}{2}),(\frac{n}{2}+1,\frac{n}{2}+2),(\frac{n}{2}+3,\frac{n}{2}+3),\dots$

$\dots,(n-1,n),(n,\frac{n}{2}+1),(1,\frac{n}{2}+1),(2,\frac{n}{2}+2),\dots,(\frac{n}{2},n)\}$.

Evidently $C(X,R)=\{c_1,\dots,c_{\frac{n}{2}},id,f,f^2,\dots,f^{n-1}\}$, where $c_i(j)=i$ for all $j=1,\dots,n$; $f(i)=i+1$ for $i\neq\frac{n}{2},n$; $f(\frac{n}{2})=1$, $f(n)=\frac{n}{2}+1$. Clearly $f^n=id$. Let F be an isomorphism of (X,R) onto $(C(X,R),M)$. We have $(c_i,c_i)\in M$ for all $i=1,\dots,\frac{n}{2}$, thus $F\{1,\dots,\frac{n}{2}\}=\{c_1,\dots,c_{\frac{n}{2}}\}$. Thus there exists an i, $\frac{n}{2}\leq i\leq n$ such that $F(i)=id$, therefore $(c_{i-\frac{n}{2}},id)\in M$, i.e. $(i-\frac{n}{2},j)\in R$ holds for every $j=1,\dots,n$. This is a contradiction. (Evidently $C(X,R)$ is abstract monoid, any $i=\frac{n}{2}+1,\dots,n$ can serve as an exact source.)

In the following theorem we give a sufficient condition for a graph to be homomorphism perfect.

Theorem 2. Let (X,R) be a graph. If the transformation monoid $(X,C(X,R))$ is abstract and commutative, then the graph (X,R) is homomorphism perfect.

Proof. There exists an $x_0\in X$ which is not an exact source of $(X,C(X,R))$. Define the mapping $F:X\longrightarrow C(X,R)$ by $F(x)=f$, where $f(x_0)=x$ (such f is determined uniquely). We shall prove that F is an isomorphism of (X,R) onto $(C(X,R),M)$. Evidently F is 1-1.

Let $(x_1,x_2)\in R$ and let $F(x_i)=f_i$, $i=1,2$.

Then

$(F(x_1)(x), F(x_2)(x)) = (f_1(x), f_2(x)) = (f_1(F(x)(x_0))$,

$f_2(F(x)(x_0))) = (F(x)(f_1(x_0)), F(x)(f_2(x_0))) = (F(x)(x_1), F(x)(x_2))$

and as F is a compatible mapping $(F(x)(x_1), F(x)(x_2)) \in R$ for all $x \in X$. Thus $(F(x_1), F(x_2)) \in M$. Let $(f_1, f_2) \in M$. Putting $x = x_0$ we get $(x_1, x_2) \in R$,

Q.E.D.

A trivial consequence follows from the last part of our proof: Let (X, R) be a graph, $(C(X, R), M)$ its homomorphism graph. If the transformation monoid $(X, C(X, R))$ is abstract, then the graph $(C(X,R), M)$ is isomorphic with a spanning subgraph of (X, R) .

Now, the functional graphs will be studied. A graph (X, R) is called functional if for every $x \in X$ there exists at most one $y \in X$ such that $(x, y) \in R$ (see e.g. [3]).

Let k, m be integers, $1 \leq k \leq m$. Define the graph $G_{k,m} = (X_m, R)$ by $X_m = \{1, \dots, m\}$, $R = \{(1,2), (2,3), \dots, (m-1, m), (m, k+1)\}$. Evidently $G_{k,m}$ is functional.

<u>Theorem 3</u>. Let (X, R) be a functional graph. (X, R) is homomorphism perfect if and only if there are integers k and m $(1 \leq k \leq m)$ such that $(X, R) =$ $= G_{k,m}$.

<u>Proof</u>. We shall prove that $G_{k,m}$ is homomorphism perfect. Evidently $C(G_{k,m}) = \{id, f, f^2, \dots, f^{m-1}\}$,

where f is defined by $f(i) = i+1,\ i = 1,\dots, n-1;\ f(n) = k$. Since $(X_n, C(G_{k,n}))$ is abstract (1 is an exact source) and commutative, $G_{k,n}$ is a homomorphism perfect graph. Let (X, R) be a functional graph $(X,R) \neq G_{k,n}$ for any k, n. We shall prove that $(X, C(X,R))$ is not abstract, i.e. (X, R) is not homomorphism perfect. Suppose that (X,R) is abstract. Let x_0 be an exact source. Let $x_1, \dots, x_p$ be all points of X such that there does not exist $(x_i', x_i) \in R$, $i = 1,\dots, p$. Clearly $p \geq 2$. Evidently, $x_0 \in \{x_1, \dots, x_p\}$. For every $i = 1,\dots, p$ there exist k_i, n_i such that G_{k_i, n_i} is a subgraph of (X, R). ($G_{k_i,n_i} = (\{{}_ix_1, {}_ix_2, \dots$ $\dots, {}_ix_{n_i}\}, \{({}_ix_1, {}_ix_2), \dots, ({}_ix_{n_i-1}, x_{n_i}), ({}_ix_{n_i}, {}_ix_{k_i+1})\})$ where ${}_ix_1 = x_1$.) Evidently, $\{{}_ix_{k_i+1}, {}_ix_{k_i+2}, \dots, {}_ix_{n_i}\} = \overline{X}$ $(\mathit{card}\ \overline{X} = h)$ is the same set for all $i = 1,\dots, p$. (If the opposite holds then there exist two sets $A \subset X$, $B \subset X$ such that $A \cap B = \emptyset$, $A \cup B = X$ such that $(a, b) \notin R$ and $(b, a) \notin R$ for every $a \in A$, $b \in B$. Assume that $x_0 \in A$. Then $(B, R/B)$ must be rigid (see [1]). This is a contradiction.)

Denote the points of $\overline{X}$ by $\overline{x}_1, \dots, \overline{x}_h$. For every x_i $(i = 1,\dots, p)$ there exists t $(1 \leq t \leq h)$ such that ${}_ix_{k_i+1} = \overline{x}_t$. We say that x_i belongs to $\overline{x}_t$. Suppose that there exist $x_{p_1}, \dots, x_{p_m}$ $(m \geq$ $\geq 2,\ 1 \leq p_1 < p_2 < \dots < p_m \leq p$) such that $x_{p_1}, \dots$ $\dots, x_{p_m}$ belongs to the same $\overline{x}_t$. Let $k_r =$ $= \max(k_{p_1}, \dots, k_{p_m})$.

We shall prove $x_r \neq x_0$. Let us define f_0 as follows:

$$f_0({}_i x_{k_i}) = {}_r x_{k_r},\ f_0({}_i x_{k_i - 1}) = {}_r x_{k_r - 1}, \dots, f_0({}_i x_1) = {}_r x_{k_r - k_i + 1}$$

for all $i = p_1, \dots, p_m$ and $f_0 = id$ for the rest of X. Evidently, f_0 is an homomorphism and since $id(x_r) = x_r$, $f_0(x_r) = x_r$, x_r is not an exact source. Clearly there holds: $k_j < k_r$, $j \in \{p_1, \dots, p_m\}$ implies x_j is not an exact source as there is no homomorphism f such that $f(x_j) = x_r$. The mapping f_0 also shows that no x_j ($j \in \{1, \dots, p\} \setminus \{p_1, \dots, p_m\}$) is an exact source.

Hence it follows that for every $\overline{X}_t$ ($t = 1, \dots, h$) there exists at most one x_i ($i = 1, \dots, p$) belonging to $\overline{X}_t$. Suppose that x_j $j \in \{1, \dots, p\}$ is an exact source (i.e. $x_j = x_0$). Let $k_i = q_i h + r_i$ where q_i, r_i are positive integers $r_i < h$, $h = m_i - k_i$, $i = 1, \dots, p$. Define f' by:

$$f'({}_i x_{k_i}) = {}_i x_{m_i},\quad f'({}_i x_{k_i - 1}) = {}_i x_{m_i - 1},\ \dots,$$

$$f'({}_i x_{2k_i - m_i}) = {}_i x_{m_i},\ \dots,$$

$$\dots \qquad \dots \qquad \dots \qquad \dots \qquad \dots,$$

$$f'({}_i x_{q_i + 1\, k_i - q_i m_i}) = {}_i x_{m_i},\ \dots,\ f'({}_i x_1) = {}_i x_{m_i - r_i + 1}$$

for all $i \neq j$ and $f' = id$ for the rest of X. Evidently, f' is a homomorphism. Since $id(x_j) = x_j$, $f'(x_j) = x_j$, we have a contradiction. This proof holds for all $j = 1, \dots, p$, hence (X, R) is not abstract, Q.E.D.

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References

[1] Z. HEDRLÍN and A. PULTR: On rigid undirected graphs, Canad.J.Math.18(1966),1237-1242.

[2] A. PULTR: Tensor products in the category of graphs, Comment.Math.Univ.Carolinae 11(1970),619-639.

[3] M. YOELI and A. GINZBURG: On homomorphic images of transition graphs, J.Franklin Institute 278 (1964),291-296.

[4] Z. HEDRLÍN and P. GORALČÍK: O sdvigach pologrupp I, Periodičeskije i kvaziperiodičeskije preobrazovanija, Mat.časopis SAV 3(1968),161-176.

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