Commentationes Mathematicae Universitatis Carolinae

Miloš Zahradník

On the common fixed point for commuting Lipschitz functions

Commentationes Mathematicae Universitatis Carolinae, Vol. 12 (1971), No. 4, 713--721

Persistent URL: http://dml.cz/dmlcz/105379

Commentationes Mathematicae Universitatis Carolinae

12,4 (1971)

ON THE COMMON FIXED POINT FOR COMMUTING LIPSCHITZ FUNCTIONS

Miloš ZAHRADNÍK, Praha

Introduction. This note deals with the existence of solution of the equation $f(x) = g(x) = x$, where f, g are commuting and lipschitz functions.

Let f be a real-valued function defined on the set $M \subset E_1$ and $\alpha \geq 0$. f is said to be a lipschitz function on M with the constant α, if the inequality

$$|f(x) - f(y)| \leq \alpha |x - y|$$

holds for each $x, y \in M$.

Let f, g be two real-valued functions defined on the interval $I \subset E_1$ with values in I. f and g are said to be the commuting functions (we abbreviate $f \circ g = g \circ f$) if

$$f(g(x)) = g(f(x))$$

holds for each $x \in I$.

In [1] there was proved

Theorem A. Let f and g be two commuting lipschitz functions with the constants α and β, respectively, defined on $\langle 0,1 \rangle$ with values in $\langle 0,1 \rangle$.

Suppose that one of the following conditions holds:

AMS Primary: 47H10 Ref. Ž. 7.978.53
Secondary: 26A16

a) $\alpha > 1, \ \beta < \dfrac{\alpha + 1}{\alpha - 1}$,

b) $\alpha \leq 1, \ \beta \geq 0$.

Then there exists $x_0 \in \langle 0, 1 \rangle$ such that

$$f(x_0) = g(x_0) = x_0 \ .$$

In this note, the previous theorem will be proved for

$$\alpha > 1, \quad \beta = \frac{\alpha + 1}{\alpha - 1} \ .$$

Preliminary lemmas.

Lemma 1. Let f be the real-valued lipschitz mapping on an interval $I \subset E_1$ with a constant $\alpha \geq 0$. If there exist two points $x_0, y_0 \in I$, $x_0 < y_0$ such that $|f(x_0) - f(y_0)| = \alpha \, |x_0 - y_0|$ then f is a linear function on $\langle x_0 \ y_0 \rangle$.

Lemma 2. Let f, g be the real-valued functions defined on the interval I with values in I . Let $a \in (-\infty, \infty)$, $b \in (0, \infty)$. Denote $x^* = \dfrac{x - a}{b} = Tx$ for $x \in I$ and set

$$f^* = T \circ f \circ T^{-1} \ , \ g^* = T \circ g \circ T^{-1} \text{ on } I^* = T(I) \ .$$

The following assertions hold:

(I) $f(x) > x$ iff $f^*(x^*) > x^*$,

(II) $f(x) > g(x)$ iff $f^*(x^*) > g^*(x^*)$,

(III) $f \circ g = g \circ f$ on I iff $f^* \circ g^* = g^* \circ f^*$ on I^*,

(IV) $\dfrac{f(x)-f(y)}{x-y} = \dfrac{f^*(x^*)-f^*(y^*)}{x^*-y^*}$ for $x \neq y$.

(Proofs are obvious.)

Main theorem.

Theorem. Let f, g be two commuting mappings of any compact interval I into itself. Suppose that f and g are lipschitz functions with the constants α and β, respectively, on I.
Let $\alpha > 1$, $\beta = \dfrac{\alpha+1}{\alpha-1}$.

Then there exists $x_0 \in I$ such that

$$x_0 = f(x_0) = g(x_0) .$$

Proof. I. (This part of the proof and the next one are the same as a part of the proof from [1].) Suppose $\beta \geq \alpha$ and let f, g have not a common fixed point in I. Let $N_g = \{x \in I; g(x) = x\}$ and $N_f = \{x \in I; f(x) = x\}$.

It is obvious that $N_g \neq \emptyset$, $N_f \neq \emptyset$. Using the commutativity property of functions, we have $f(N_g) \subset N_g$ and $g(N_f) \subset N_f$.
Denote $a = \inf N_g$, $b = \sup N_g$. Then $a < b$ and since N_g is closed, $a, b \in N_g$. This fact implies $f(a) > a$ and $f(b) < b$.
Denote

$$x_0 = \sup\{x \in N_g ; f(x) > x\} ,$$

$$x_1 = \inf\{x \in N_g ; x > x_0, f(x) < x\} .$$

Then $x_0, x_1 \in N_g$ and

(1) $$(x_0, x_1) \cap N_g = \emptyset .$$

Evidently

(2) $$f(x_0) > x_0 , \quad f(x_1) < x_1 .$$

According to (1) we can suppose that

$$g(x) > x \quad \text{for} \quad x \in (x_0, x_1) .$$

Since $f(x_1) \in N_g - (x_0, x_1)$, $f(x_0) \in N_g - (x_0, x_1)$, we have

(3) $$f(x_1) \leq x_0 , \quad f(x_0) \geq x_1 .$$

(1,),(2) and (3) imply that the set $M = \{x \in (x_0, x_1) , f(x) = x\}$ is not empty and denote $s = \sup M$. Then $x_0 < s < x_1$ and $f(s) = s$.

Let $g(s) = t$. Then $t \in N_f$, $t > s$ and $t > x_1$.

II. The next relations are valid:

$$t - x_0 = g(s) - g(x_0) \leq \beta(s - x_0) ,$$

$$s - x_0 \leq s - f(x_1) = f(s) - f(x_1) \leq \alpha(x_1 - s) .$$

$$s - x_0 = \frac{\alpha}{\alpha+1}(x_1 - x_0) ,$$

$$t - x_0 \leq f(t) - f(x_1) \leq \alpha(t - x_1) ,$$

$$\frac{\alpha}{\alpha-1}(x_1 - x_0) \leq t - x_0 ,$$

$$t - x_0 \leq \frac{\alpha\beta}{\alpha+1}(x_1 - x_0) \leq t - x_0 .$$

The last inequality implies

$$t - x_0 = g(s) - g(x_0) = \beta(s - x_0) ,$$
$$s - x_0 = f(s) - f(x_1) = \alpha(x_1 - s) ,$$

$$t - x_0 = f(t) - f(x_1) = \alpha(t - x_1) \,,$$

$$s = \frac{\alpha}{\alpha+1}(x_1 - x_0) + x_0 \,, \quad t = \frac{\alpha}{\alpha-1}(x_1 - x_0) + x_0 \,,$$

$$t - x_1 = g(s) - g(x_1) = \beta(x_1 - s) \,.$$

Hence, using Lemma 1, we have:

(4) $g(x) = \beta(x - x_0) + x_0$ for $x \in \langle x_0 , s \rangle$,

(5) $g(x) = \beta(x_1 - x) + x_1$ for $x \in \langle s , x_1 \rangle$,

(6) $f(x) = \alpha(s - x) + s$ for $x \in \langle s , x_1 \rangle$,

(7) $f(x) = \alpha(x - x_1) + x_0$ for $x \in \langle x_1 , t \rangle$.

III. We can suppose (without loss of generality) - see Lemma 2) that $s = -\alpha$ and $x_1 = \beta$.
Then $x_0 = -\alpha^2\beta$, $t = \beta^2\alpha$ and

(8) $g(x) = \beta(x + \alpha^2\beta) - \alpha^2\beta$ for $x \in \langle -\alpha^2\beta , -\alpha \rangle$,

(9) $g(x) = \beta(\beta - x) + \beta$ for $x \in \langle -\alpha , \beta \rangle$,

(10) $f(x) = \alpha(-\alpha - x) - \alpha$ for $x \in \langle -\alpha , \beta \rangle$,

(11) $f(x) = \alpha(x - \beta) - \alpha^2\beta$ for $x \in \langle \beta , \beta^2\alpha \rangle$.

Using (8), we have $g(-2\alpha - \beta) = \beta$.

The next relations are valid:

$$f(g(-2\alpha-\beta)) = -\alpha^2\beta \ ,$$

(12) $$g(f(-2\alpha-\beta))-g(-\alpha) = f(g(-2\alpha-\beta))-g(-\alpha) = = -\alpha^2\beta - \beta^2\alpha \ ,$$

$$|g(f(-2\alpha-\beta)) - g(-\alpha)| \leq \beta|f(-2\alpha-\beta)+\alpha| \ .$$

Using $f(-\alpha) = -\alpha$, we obtain

(13) $$|f(-2\alpha-\beta)-f(-\alpha)| \geq \alpha\beta + \alpha^2 \ .$$

But

(14) $$|f(-2\alpha-\beta)-f(-\alpha)| \leq \alpha|\alpha+\beta| \ .$$

From (13) and (14) we obtain:

f is a linear function on $\langle -2\alpha-\beta, -\alpha \rangle$ and $|f(x)-f(-\alpha)| = \alpha|x+\alpha|$.

After a simple calculation we obtain that $f(-2\alpha-\beta) = = -\alpha^2\beta$ is not possible. Thus

(15) $$\begin{cases} f(x) = \alpha(-\alpha-x)-\alpha \text{ for } x \in \langle -2\alpha-\beta, -\alpha \rangle & \text{and} \\ f(-2\alpha-\beta) = \alpha^2 + \alpha\beta - \alpha & . \end{cases}$$

According to (12) we have

(16) $$\alpha^2\beta + \beta^2\alpha = -(g(\alpha^2+\alpha\beta-\alpha) - g(-\alpha)) \ ,$$

and

(17) $$|g(\alpha^2+\alpha\beta-\alpha)-g(-\alpha)| \leq \beta|\alpha^2+\alpha\beta| \ .$$

Hence, using Lemma 1, it is

(18) $g(x) = \beta(-\alpha - x) + \beta^2\alpha$ for $x \in \langle -\alpha, \alpha^2 + \alpha\beta - \alpha \rangle$.

Similarly as in (15),(18), we obtain

(19) $g(x) = -\beta(x - \beta) + \beta$ for $x \in \langle \beta, 2\beta + \alpha \rangle$,

(20) $f(x) = \alpha(\beta - x) - \alpha^2\beta$ for $x \in \langle -\beta^2 - \alpha\beta + \beta, \beta \rangle$.

IV. In the previous parts of this proof we proved under assumption f and g have not a common fixed point that the relations (8) - (20) are valid. In the next step we show that it is not possible.

Suppose, for example $\beta > 3$.

Then $\beta - \alpha\beta - \beta^2 < -\alpha^2\beta$ and

(20)implies $f(-\alpha^2\beta) = \alpha^3\beta + \alpha\beta - \alpha^2\beta$,

(19) implies $g(2\beta + \alpha) < -\alpha^2\beta$,

$$g(\alpha^3\beta + \alpha\beta - \alpha^2\beta) = g(f(-\alpha^2\beta)) =$$
$$= f(g(-\alpha^2\beta)) = \alpha^3\beta + \alpha\beta - \alpha^2\beta$$

and thus

$$\alpha^3\beta + \alpha\beta < (g(\alpha^3\beta + \alpha\beta - \alpha^2\beta) - g(2\beta + \alpha)) =$$
$$= \beta\,|\alpha^3\beta + \alpha\beta - \alpha^2\beta - 2\beta - \alpha|,$$
$$\alpha(\alpha - 1) < |\alpha^2 - 2|.$$

The last inequality is not true for $\beta > 3$.

Suppose $2 \leq \alpha \leq 3$, $2 \leq \beta \leq 3$. The relations (11) and (18) imply

$$f(\alpha^2 + \alpha\beta - \alpha) = \alpha^3 - \alpha^2 - \alpha\beta,$$

$$g(\alpha^3-\alpha^2-\alpha\beta) = -\beta\alpha + 2\beta^2\alpha + \beta\alpha^2 - \beta\alpha^3 .$$

Thus

$$f(-\alpha^2\beta) = f(g(\alpha^2+\alpha\beta-\alpha)) = g(f(\alpha^2+\alpha\beta-\alpha) =$$

$$= -\beta\alpha + 2\beta^2\alpha + \beta\alpha^2 - \beta\alpha^3 ,$$

(21) $|f(-\alpha^2\beta)-f(\beta-\alpha\beta-\beta^2)| = \beta\alpha(\alpha^2+1-\alpha-\beta)$.

(22) $|f(-\alpha^2\beta)-f(\beta-\alpha\beta-\beta^2)| \leq \alpha|\alpha^2\beta-\alpha\beta+\beta-\beta^2|$.

From (11),(22) and Lemma 1 we have

(23) $f(x) = -\alpha(x+\alpha\beta+\beta^2-\beta)+\beta^2\alpha$

for $x \in \langle -\alpha^2\beta, \beta-\alpha\beta-\beta^2 \rangle$

and similarly

(24) $g(x) = \beta(x-\alpha^2-\alpha\beta+\alpha)-\alpha^2\beta$

for $x \in \langle \alpha^2+\alpha\beta-\alpha, \beta^2\alpha \rangle$.

It is easy to show that under assumption that the relations (8),(9),(10),(11),(15),(18),(19),(20),(23),(24) are valid, f, g are not commuting.

The proof is completed.

Remarks: P. Huneke in [2] proved that in the case $\alpha = \beta > 3 + \sqrt{6}$ the problem about common fixed point for the commuting and lipschitz functions has no solution in general.

I wish to thank S. Fučík for calling my attention to this problem and for his help with the solution.

R e f e r e n c e s

[1] R. de MARR: A common fixed point for commuting mappings, Amer.Math.Monthly 70(1963),535-537.

[2] P. HUNEKE: On common fixed points of commuting functions on an interval, Trans.Amer.Math.Soc. 139(1969),371-381.

Matematicko-fyzikální fakulta
Karlova universita
Praha 8, Sokolovská 83
Československo

(Oblatum 15.3.1971)