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ON THE CONVERGENCE OF SEQUENCES OF LINEAR OPERATORS AND ADJOINT OPERATORS

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1. Introduction

Let X and Y be two Banach spaces with the norms $\|\cdot\|_X$ and $\|\cdot\|_Y$, respectively. X^* (resp. Y^*) denotes the adjoint space of all bounded linear functionals on X (resp. on Y). The pairing between $x^* \in X^*$ and $x \in X$ is denoted by $\langle x, x^* \rangle_X$ (analogously for $y^* \in Y^*$ and $y \in Y$). We shall use the symbols $\xrightarrow{X}$, $\xrightarrow{x}$ to denote the strong convergence in X and the weak convergence in X, respectively. If $\mathcal{L}(X, Y)$ is the space of all bounded linear operators from X into Y then the convergence of a sequence $(A_n) \subset \mathcal{L}(X, Y)$ can be considered in various meaning. We shall consider the following types.

Definition 1. Let $A \in \mathcal{L}(X, Y)$, $(A_n) \subset \mathcal{L}(X, Y)$. Then

(i) (A_n) is said to be converged to A if $A_n x \xrightarrow{Y} Ax$ for any $x \in X$.

(ii) (A_n) is said to be continuously converged to A if $A_n x_n \xrightarrow{Y} Ax$ for any $(x_n) \subset X$, $x_n \xrightarrow{X} x$.

(iii) $(A_n$ is said to be weakly converged to A if

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$A_n x \xrightarrow{Y} Ax$ for any $x \in X$.

(iv) (A_n) is said to be weakly continuously converged to A if $A_n x_n \xrightarrow{Y} Ax$ for any $(x_n) \subset X$, $x_n \xrightarrow{X} x$.

The convergence of (A_n) to A in the meaning of (i) or (ii),(iii),(iv) is denoted by $A_n \longrightarrow A$ or $A_n \xrightarrow{c} A$, $A_n \longrightarrow A$, $A_n \xrightarrow{c} A$, respectively.

The relations among these types of the convergence are examined in Section 2.

Let A^* denote the adjoint operator to $A \in \mathcal{L}(X, Y)$, i.e. A^* is such an element of $\mathcal{L}(Y^*, X^*)$ that $\langle Ax, y^* \rangle_Y = \langle x, A^* y^* \rangle_X$ for any $x \in X$, $y^* \in Y^*$. It can be shown that $A_n \longrightarrow A$ does not imply $A_n^* \longrightarrow A^*$ (see Example 1 in Section 2 or Yosida [4], Chap.VII,§ 1, Prop.1). In Proposition 2 and Theorem 1 we shall give the sufficient and necessary condition under that $A_n^* \longrightarrow A^*$. The special case of operators with norms equal to 1 is given in Theorem 2 and in its Corollary. Solving the problem when the convergence (i) implies the convergence (iv) for any sequence $(A_n) \subset \mathcal{L}(X, Y)$, we obtain a new characterization of Banach spaces with finite dimension (Theorem 3). The convergence of adjoint operators is important for instance in the case that $X = Y$ and (A_n) are projections (i.e. $A_n^2 = A_n$), $A = I$ (I denotes the identity operator) - see e.g. Browder [1]). Except rewriting the main results of Sections 2 we shall give the conditions for the convergence of adjoint projections in the sense of (i) in Definition 1,in Section 3.

2. <u>The relations among various types of the convergence</u>

Two relations are obvious, namely $A_m \longrightarrow A$ or $A_m \xrightarrow{c} A$ implies $A_m \longrightarrow A$.

<u>Proposition 1.</u> $A_n \longrightarrow A$ if and only if $A_m \xrightarrow{c} A$.

<u>Proof</u>. As from $A_n \xrightarrow{c} A$ it obviously follows that $A_m \longrightarrow A$, we have only to prove the necessary part. If $A_m \longrightarrow A$, then, by virtue of the Banach-Steinhaus theorem (see e.g. Yosida [4]), there exists a positive number K such that $\| A_m \| \leq K$ for any positive integer m . Let now $x_m \xrightarrow{X} x$. By the triangle inequality, we have

$\| A_m x_m - Ax \|_Y \leq K \| x_m - x \|_X + \| A_m x - Ax \|_Y$.

It follows that $A_m \xrightarrow{c} A$.

An analogous statement for the weak convergence does not hold as it will be shown in the sequel. The following two statements make clear the notion of weakly continuously converging sequences.

<u>Proposition 2</u>. If $A^*_m \longrightarrow A^*$ then $A_m \xrightarrow{c} A$.

<u>Proof</u>. Let (x_m) be such a sequence of elements of X that $x_m \xrightarrow{X} x$ and let $y^* \in Y^*$. Then

$\langle A_m x_m , y^* \rangle_Y = \langle x_m , A^*_m y^* \rangle_X \longrightarrow \langle x, A^* y^* \rangle_X = \langle Ax, y^* \rangle_Y$

because $A^*_m y^* \xrightarrow{X^*} A^* y^*$. Therefore $A_m \xrightarrow{c} A$.

<u>Theorem 1</u>. Let X be a separable and reflexive Banach space and let Y be a Banach space. Then from $A_m \xrightarrow{c} A$ it follows that $A^*_m \longrightarrow A^*$.

<u>Proof</u>. According to Kadec [3] there exists a norm $|\!|\!| \cdot |\!|\!|_{X^*}$ which is equivalent to the norm $\| \cdot \|_{X^*}$ generated by the norm $\| \cdot \|_X$ in X and which has the following

property

(P) If $x_n^* \xrightarrow{X^*} x^*$, $|||x_n^*|||_{X^*} \longrightarrow |||x^*|||_{X^*}$ then $x_n^* \xrightarrow{X^*} x^*$.

The norm $|||\cdot|||_{X^*}$ on X generates the new norm $|||\cdot|||_X$ on X by the relation

$$|||x|||_X = \sup_{|||x^*|||_{X^*}\le 1} |\langle x, x^*\rangle_X| \qquad \text{for } x \in X .$$

The norm $|||\cdot|||_X$ on X is also equivalent to the previous norm $\|\cdot\|_X$. Let now $A_n \xrightarrow{c} A$. Then $A_n \longrightarrow A$ and therefore, by using the reflexivity of X, also $A_n^* \longrightarrow A^*$. If $y^* \in Y^*$ then

$$|||A^* y^*|||_{X^*} \le \liminf_{n\to\infty} |||A_n^* y^*|||_{X^*} .$$

By virtue of the Hahn-Banach theorem, there exists a sequence (x_n) of elements of the sphere $S = \{x \in X, |||x|||_X = 1\}$ such that

$$|||A_n^* y^*|||_{X^*} = \langle x_n, A_n^* y^*\rangle_X = \langle A_n x_n, y^*\rangle_Y .$$

As X is reflexive, the sphere S is relatively weakly sequentially compact (see e.g. Day [2]) and therefore there exists a subsequence (x_{n_k}) of (x_n) such that $x_{n_k} \xrightarrow{X} x \in X$. Putting $z_{n_k} = x_{n_k}$ and $z_n = x$ for $n \neq n_k$ we have

$$|||A^* y^*|||_{X^*} \le \liminf_{n\to\infty} |||A_n^* y^*|||_{X^*} =$$

$$= \liminf_{n\to\infty} \langle A_n x_n, y^*\rangle_Y \le$$

$$\leq \liminf_{k\to\infty} \langle A_{n_k} x_{n_k}, y^* \rangle_Y = \lim_{n\to\infty} \langle A_n x_n, y^* \rangle_Y =$$

$$= \langle Ax, y^* \rangle_Y = \langle x, A^* y^* \rangle_X \leq$$

$$\leq |||\, x \,|||_X \cdot |||\, A^* y^* \,|||_{X^*} \leq |||\, A^* y^* \,|||_{X^*} \quad .$$

Hence

$$|||\, A^* y^* \,|||_{X^*} = \liminf_{n\to\infty} |||\, A_n^* y^* \,|||_{X^*} \quad .$$

We shall now prove that also

$$|||\, A^* y^* \,|||_{X^*} = \limsup_{n\to\infty} |||\, A_n^* y^* \,|||_{X^*} \quad .$$

If it is not this case, i.e. $\limsup_{n\to\infty} |||\, A_n^* y^* \,|||_{X^*} >$ $> |||\, A^* y^* \,|||_{X^*}$ then there exists such a subsequence (n_k) of positive integers that

$$|||\, A^* y^* \,|||_{X^*} < \lim_{k\to\infty} |||\, A_{n_k}^* y^* \,|||_{X^*} \quad .$$

By the same manner as above we get the contradiction.

Summarizing, we have $A_n^* y^* \xrightarrow{X^*} A^* y^*$ and $|||\, A_n^* y^* \,|||_{X^*} \longrightarrow |||\, A^* y^* \,|||_{X^*}$ and thus, by the validity of Property (P), we obtain that $|||\, A_n^* y^* - A^* y^* \,|||_{X^*} \longrightarrow 0$. By using the equivalence property of the norms $|||\cdot|||_{X^*}$, $\|\cdot\|_{X^*}$, it is $A_n^* \to$ $\longrightarrow A^*$ which was to be proved.

<u>Remark 1</u>. We have heard that S.L. Trojanski proved that the Kadec theorem takes place in the case that X is reflexive and not necessarily separable (to appear in Studia Mathematica, vol.37). Therefore, the assumption of separability can be omitted in Theorem 1. We shall use this remark in the sequel.

Corollary 1. Let X and Y be two Banach spaces and let X be a reflexive space. Then the following conditions are equivalent:

(i) $A_m^* \longrightarrow A^*$.

(ii) $A_m^* \xrightarrow{c} A^*$.

(iii) $A_m \xrightarrow{c} A$.

(iv) $(\| A_m \|)$ is a bounded sequence and $A_m^* y^* \xrightarrow{X^*} A^* y^*$ for $y^* \in \mathfrak{D}^*$ where the linear hull of $\mathfrak{D}^*$ is dense in Y^*.

Proof. The equivalence of (i) and (ii) is stated in Proposition 1, the equivalence of (i) and (iii) is proved in Proposition 2, Theorem 1 and Remark 1, the equivalence of (i) and (iv) is the Banach-Steinhaus theorem. .

Corollary 2. Let X and Y be two Banach spaces and let Y be a reflexive space. Then the following conditions are equivalent:

(i) $A_m \longrightarrow A$.

(ii) $A_m \xrightarrow{c} A$.

(iii) $A_m^* \xrightarrow{c} A^*$.

(iv) $(\| A_m \|)$ is a bounded sequence and $A_m x \xrightarrow{Y} Ax$ for $x \in \mathfrak{D}$, where the linear hull of $\mathfrak{D}$ is dense in X .

The following example shows that there is no relation between $A_m \longrightarrow A$ and $A_m^* \longrightarrow A^*$ even in the case of projections on Hilbert spaces.

Example 1. Let $X = \ell^2$, $e_n = (\delta_{in})_i$ (δ_{in} is the Kronecker symbol) and P_n be the orthogonal projection onto $X_n = Lin(e_1, \dots, e_n)$ (Lin stands here for the linear hull). For $x = \sum_{i=1}^{+\infty} \xi_i e_i \in \ell^2$ we put $T_n x = \xi_{n+1} e_1 + \dots + \xi_{2n} e_n$ and $Q_n = P_n + T_n$. Then $Q_n^2 = Q_n$, i.e. Q_n is a projection onto X_n, $Q_n \longrightarrow I$ (I denotes the identity operator). From $e_n \xrightarrow{X} \theta$ and $Q_n e_{n+1} = e_1$ we see that $Q_n \overset{c}{\not\longrightarrow} I$ and, by virtue of Proposition 2, the sequence (Q_n^*) does not converge to I^*. One can easily show that $Q_n^* x \longrightarrow x$ if and only if $x = \theta$.

Remark 2. Example 1 shows that $A_n \longrightarrow A$ does not imply $A_n \xrightarrow{c} A$. By the same manner ($Q_n^* \xrightarrow{c} I^*$ as it follows from Corollary 2) $A_n \xrightarrow{c} A$ does not imply $A_n \longrightarrow A$. Especially, $A_n \longrightarrow A$ does not imply $A_n \xrightarrow{c} A$.

Theorem 2. Let X be a reflexive Banach space. Let the norm $\|\cdot\|_{X^*}$ on X^* generated by the norm $\|\cdot\|_X$ on X have Property (P) (see the proof of Theorem 1). Let $(A_n) \subset \mathcal{L}(X, X)$ be such that $\|A_n\| = 1$ and $A_n \longrightarrow I$. Then $A_n^* \longrightarrow I^*$.

Proof. By the reflexivity of X, from $A_n \longrightarrow I$ it follows that $A_n^* \longrightarrow I^*$. It is

$$\|x^*\|_{X^*} \le \liminf_{n\to\infty} \|A_n^* x^*\|_{X^*} \le \limsup_{n\to\infty} \|A_n^* x^*\|_{X^*} \le \|x^*\|_{X^*}$$

for all $x^* \in X^*$. Therefore $\|A_n^* x^*\|_{X^*} \longrightarrow \|x^*\|_{X^*}$ and, by virtue of Property (P), $A_n^* \longrightarrow I^*$.

Corollary 3. Let X be a reflexive Banach space. Suppose that the norms $\|\cdot\|_X$ and $\|\cdot\|_{X^*}$ have Property

(P). Let (A_n) be a sequence of elements of $\mathcal{L}(X, X)$ such that $\| A_n \| = 1$. Then the following conditions are equivalent:

(i) $A_n \xrightarrow{c} I$. (ii) $A_n \longrightarrow I$.

(iii) $A_n \overset{c}{\rightharpoonup} I$. (iv) $A_n \rightharpoonup I$.

(v) $A_n^* \xrightarrow{c} I^*$. (vi) $A_n^* \longrightarrow I^*$.

(vii) $A_n^* \overset{c}{\rightharpoonup} I^*$. (viii) $A_n^* \rightharpoonup I^*$.

<u>Remark 3.</u> Example 1 shows that there exists a sequence (A_n) such that $A_n \longrightarrow A$ and $A_n \overset{c}{\not\longrightarrow} A$. Since in the space ℓ^1 the notion of the strong convergence and the weak convergence of sequences are the same, we see (from Proposition 1) that for any Banach space Y and any $(A_n) \subset \subset \mathcal{L}(\ell^1, Y)$, $A_n \rightharpoonup A$, it is $A_n \overset{c}{\rightharpoonup} A$. The next Theorem 3 says that in the case $\mathcal{L}(X, Y)$ where X is a separable and reflexive Banach space, this is not possible.

<u>Theorem 3</u>. Let X be a separable and reflexive Banach space and let Y be a Banach space. Suppose that for any sequence $(A_n) \subset \mathcal{L}(X, Y)$ such that $A_n \longrightarrow A$ it is $A_n \overset{c}{\rightharpoonup} A$. Then X is a finite dimensional space.

<u>Proof</u>. Suppose that X is an infinite dimensional space. Let $(x_n) \subset X$ be such a sequence that $Lin(x_1, \dots)$ is dense in X. We denote $X_n = Lin(x_1, \dots, x_n)$. Without loss of generality we can suppose that $X_n \subsetneqq X_{n+1}$ for any positive integer n. It is easy to see that $\overline{\bigcup_{n=1}^{+\infty} X_n} = X$. We define by induction a sequence (e_n) such that $\| e_n \|_X = 1$

and $e_1, \dots, e_n$ is a basis for X_n and $\| e_n - y \|_X \geq \frac{1}{2}$ for each $y \in X_{n-1}$ and any integer n. (The last inequality can be guaranted for instance by using the F. Riesz theorem - see Yosida [4], Chap. III, § 2.)

According to one corollary of the Hahn-Banach theorem there exists a sequence $(f_n) \subset X^*$ such that $\langle e_i, f_n \rangle_X = \delta_{in}$, $i = 1, \dots, n$ and $\| f_n \|_{X^*} \leq 2$. It is easy to see that $\langle x, f_n \rangle_X \longrightarrow 0$ for any $x \in X$. We put $y_0 \in Y$, $y_0 \neq \theta$ and we define

$$A_n : x \longrightarrow \langle x, f_n \rangle_X \, y_0$$

for any $x \in X$ and any positive integer n. Then $A_n \longrightarrow \Theta$ (Θ denotes the null operator). By the assumptions of the reflexivity, there exists a subsequence (e_{n_k}) such that $e_{n_k} \xrightarrow{X} x_0$. But $A_{n_k} e_{n_k} = \langle e_{n_k}, f_{n_k} \rangle_X y_0 = y_0 \not\longrightarrow \theta$. It shows that $A_n \overset{c}{\not\longrightarrow} \Theta$, which contradicts the assumption.

Remark 4. From the discussion of this proof we can conclude that the statement of Theorem 3 is true if X is a normed linear space with a separable and reflexive subspace of the infinite dimension and on which there exists a bounded projection P. For, if E is such a subspace, we define (A_n) on E as above. We put $B_n x = A_n P x$ for $x \in X$. Then $B_n \longrightarrow \Theta$ and $B_n \overset{c}{\not\longrightarrow} \Theta$. Unfortunately, we do not know what normed linear spaces have this property.

Corollary 4. Let X and Y be two separable and reflexive Banach spaces. Suppose that for any sequence $(A_n) \subset \mathcal{L}(X, Y)$ such that $A_n \longrightarrow A$ it is $A_n \xrightarrow{c} A$. Then X and Y are finite dimensional spaces.

Proof. By virtue of Theorem 3, $\mathcal{X}$ is a finite dimensional space. Using Corollary 1, Corollary 2 and Theorem 3, we obtain that $\mathcal{Y}$ is a finite dimensional space.

3. The convergence of projections

Theorem 4. Let $\mathcal{X}$ be a reflexive space and let (P_n) be a sequence of commuting projections on $\mathcal{X}$, i.e. $(P_n) \subset \mathcal{L}(\mathcal{X},\mathcal{X})$, $P_n^2 = P_n$, $P_{n+1} P_n = P_n P_{n+1}$ and let $P_{n+1} P_n = P_n$, i.e. $P_n(\mathcal{X}) \subset P_{n+1}(\mathcal{X})$. Then the following conditions are equivalent:

(i) $P_n \longrightarrow I$.

(ii) $P_n^* \longrightarrow I^*$.

Proof. We denote $P_n(\mathcal{X})$ by $\mathcal{X}_n$ and $P_n^*(\mathcal{X}^*)$ by $\mathcal{Y}_n^*$. By the commutativity of (P_n) and $\mathcal{X}_n \subset \mathcal{X}_{n+1}$, we have $\mathcal{Y}_n^* \subset \mathcal{Y}_{n+1}^*$. Further $\mathcal{Y}^* = \overline{\bigcup_{n=1}^{+\infty} \mathcal{Y}_n^*}$ is a closed convex subset of $\mathcal{X}^*$. Now, we can use the Mazur theorem (see e.g. Day [2]) to get that $\mathcal{Y}^*$ is also weakly closed. If (i) holds then $P_n^* \longrightarrow I^*$ which shows that $\mathcal{Y}^*$ is a weakly dense subset of $\mathcal{X}^*$. Therefore $\mathcal{Y}^* = \mathcal{X}^*$. From the assumption (i) it follows that $(\| P_n \|)$ is a bounded sequence and thus $(\| P_n^* \|)$ is also bounded. By the Banach-Steinhaus theorem, it remains to prove that $P_n^* x^* \longrightarrow x^*$ for any $x^* \in \bigcup_{n=1}^{+\infty} \mathcal{Y}_n^*$. But if $x^* \in \mathcal{Y}_{n_0}^*$ then $P_n^* x^* = x^*$ for all $n \geq n_0$.

Corollary 5. Under the assumptions of Theorem 4 the following conditions are equivalent:

(i) $P_n \longrightarrow I$.

(ii) $P_m^* \longrightarrow I^*$.

(iii) $P_m \xrightarrow{c} I$.

(iv) $P_m^* \xrightarrow{c} I^*$.

<u>Remark 5.</u> The case of noncommuting projections will be obtained from Corollary 3.

R e f e r e n c e s

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