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ON CNE-PARAMETER FAMILIES OF DIFFEOMORPHISMS II: GENERIC BRANCHING IN HIGHER DIMENSIONS

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§ 1

In [1], we have studied the generic nature of the loci of periodic points of a diffeomorphism of a finite dimensional manifold $M$, depending on a parameter with values in a one dimensional manifold $P$, in $P \times M$. A part of the results (those concerning the branching of periodic points), we have proved for two dimensional $M$ only. It is the purpose of this paper to extend these results for $M$ of arbitrary finite dimension.

Since this paper is a direct continuation of [1], we shall frequently refer to [1] for results of technical character as well as techniques of proof. Nevertheless, for the sake of the reader's convenience, we re-introduce those concepts and results of [1] which are necessary for the understanding of this paper, in the rest of this section. The main results of this paper and their proofs are given in § 3. § 2 has an auxiliary character; it establishes certain generic properties of maps of an interval into the
set of matrices.

Denote $\mathcal{F}$ the space of $C^\infty$ mappings $(1 < \kappa \leq \infty) \times P : P \times M \to M$, where $P, M$ are $C^\infty$ second countable manifolds of dimension $1$, $n < \infty$ respectively, such that for every $\mu \in P$ the map $\mathcal{F}_\mu : M \to M$, given by $\mathcal{F}_\mu(m) = \mathcal{F}(\mu, m)$ is a diffeomorphism, endowed with the $C^\infty$ Whitney topology.

Let us note that, although this topology is not metrizable, it has the property that a residual set in $\mathcal{F}$ (i.e. a countable intersection of open dense sets) is dense in $\mathcal{F}$ (this can be proved similarly as the analogous statement for vector fields is proved in [2], using the openness of $\mathcal{F}$ in the set of all $C^\infty$ mappings $P \times M \to M$).

Denote by $Z_{\mathcal{F}} = Z_{\mathcal{F}}(\mathcal{F})$ the set of $\mathcal{F}$-periodic points of $\mathcal{F}$, i.e. $Z_{\mathcal{F}}(\mathcal{F}) = \{(\mu, m) | \mathcal{F}_\mu^j(m) = m,\mathcal{F}_\mu^j(m) \neq m \text{ for } 0 < j < \infty\}$. In [1, Theorem 1] a residual subset $\mathcal{F}_1$ of $\mathcal{F}$ was defined and it was shown that for every $\mathcal{F} \in \mathcal{F}_1$, $Z_{\mathcal{F}}$ are one dimensional submanifolds of $P \times M$ ($Z_1$ being closed) and, if an eigenvalue of $\mathcal{F}_{\mu}^j(\mu)$ at some point $(\mu, m) \in Z_{\mathcal{F}}$ is 1 (we denote the set of such points by $X_{\mathcal{F}}$), then it meets the unit circle $S$ in the complex plane transversally at $(\mu, m)$ (in the sense of Remark 3) and the remaining eigenvalues of $\mathcal{F}_{\mu}^j(\mu)$ do not lie on $S$. Also, it was shown that the subset $\mathcal{F}_1$ of maps from $\mathcal{F}$, having the.

\textbf{x) In [1] we have assumed $1 < \kappa < \infty$, but Theorems 1 - 4 of [1] are trivially true for the $C^\infty$ case.}
above properties for $1 \leq n \leq 4$, is open dense in $F$.

§ 2

Denote by $\mathcal{U}$ the set of all $n \times n$ matrices with the differential structure induced by its natural identification with $\mathbb{R}^{n^2}$. Further, denote by $\mathcal{U}_1$ the set of matrices having an eigenvalue of multiplicity $\geq 2$ on $S$, $\mathcal{G}_{2\ell}$ the set of matrices having an $\ell$-th root of unity different from $\pm 1$ as eigenvalue, $\mathcal{U}_2 = \bigcup_{k=3}^{\infty} \mathcal{G}_{2k}$. 

Let $I$ be a closed interval on $\mathbb{R}$. Denote by $\phi$ the space of all $C^\infty$ mappings $I \rightarrow \mathcal{U}$ endowed with the $C^\infty$ uniform topology.

Proposition 1. Let $J \subset I$ be a closed interval, $J \subset \text{int} I$. Then, for every $\ell = 3, 4, \ldots$ the set $\Psi_{\ell}(J)$ of all $F \in \phi$ such that $F(J) \cap (\mathcal{U}_1 \cup \mathcal{U}_2) = \emptyset$ is open dense in $\phi$.

Corollary 1. Given $J$ as in Proposition 1, the set $\Psi(J)$ of all $F \in \phi$ such that $F(J) \cap (\mathcal{U}_1 \cap \mathcal{U}_2) = \emptyset$ is residual in $\phi$.

For the proof of Proposition 1 we shall need to prove several lemmas.

Consider the sets $\mathcal{H}_1 = \{(A, \lambda_1, \lambda_2) \in \mathcal{U} \times R^2 | P(A, \lambda_1, \lambda_2) = P_2(A, \lambda_1, \lambda_2) = P_2'(A, \lambda_1, \lambda_2) = P_2'(A, \lambda_1, \lambda_2) = 0, \lambda_1^2 + \lambda_2^2 = 1\}$ and $\mathcal{H}_2(A_{10}, A_{20}) = \{(A, \lambda_1, \lambda_2) \in \mathcal{U} \times R^2 | P(A, \lambda_1, \lambda_2) = P_2(A, \lambda_1, \lambda_2) = 0,\lambda_1 = A_{10}, \lambda_2 = A_{20}\}$, where $P(A) = P_1(\text{Re} \lambda, \text{Im} \lambda) + iP_2(\text{Re} \lambda, \text{Im} \lambda)$ is the characteristic polynomial of
Being defined by polynomial equalities, \( \mathcal{U}_1 \) and \( \mathcal{U}_2(\lambda_{10},\lambda_{20}) \) are real algebraic varieties and the sets \( \mathcal{U}_1, \mathcal{U}_2 \) are the projections of \( \mathcal{U}_1 \) and \( \mathcal{U}_2(\lambda_{10},\lambda_{20}) \) into \( \mathcal{U} \), respectively, where the union is taken over all \( \lambda_{10}, \lambda_{20} \) such that

\[
(\lambda_{10} + i\lambda_{20})^2 = 1 \quad \text{and} \quad \lambda_{20} \neq 0.
\]

By [3, splitting (b) of §11], \( \mathcal{U}_1 \) and \( \mathcal{U}_2 \) can be written as a finite disjoint union of submanifolds of strictly decreasing dimensions, \( \mathcal{U}_1 = \bigcup_{j=1}^\infty \mathcal{M}_j \), \( \mathcal{U}_2(\lambda_{10},\lambda_{20}) = \bigcup_{j=1}^\infty \mathcal{N}_j \) such that \( \bigcup_{j=1}^\infty \mathcal{M}_j \) and \( \bigcup_{j=1}^\infty \mathcal{N}_j \) is closed for all \( 0 < \varphi \leq \nu, \, 0 < \delta \leq \rho \).

**Lemma 1.** codim \( \mathcal{M}_j \) \( \geq 4 \) for all \( j \).

For the proof of this lemma we need some more lemmas.

**Lemma 2.** For any \( A \in \mathcal{U} \), the set of all matrices similar to \( A \) is an immersed submanifold of \( \mathcal{U} \) of codimension \( \geq m \).

**Proof.** Consider the group \( GL(m) \), whose action \( \psi \) on \( \mathcal{U} \) is given by \( \psi(TA) = T^{-1}AT \) for \( T \in GL(m), \, A \in \mathcal{U} \). The set of matrices similar to \( A \) under this group action and, according to [4,2.2, Proposition 2], is an immersed submanifold of \( \mathcal{U} \) of codimension equal to the dimension of the closed Lie subgroup \( H = \{ T \in GL(m) \mid \psi(T,A) = A \} \). It is easy to show that \( H \) is identical with the subset of \( GL(m) \) of matrices that commute with \( A \). It follows from [5,VIII, §2, Theorem 2] that \( H \) has the dimension \( \geq m \), q.e.d.
Corollary 2. Denote by \( \mu \) the map \( \mathcal{U} \rightarrow \mathbb{R}^m \) assigning to every matrix from \( \mathcal{U} \) the \( m \)-tuple of coefficients of its characteristic polynomial and \( \tilde{\mu} : \mathcal{U} \rightarrow \mathbb{R}^{m+2} \) as \( \tilde{\mu} = \mu \times \text{id} \). Then, for any point \( x \in \mathbb{R}^{m+2} \), \( \mu^{-1}(x) \) is a finite disjoint union of immersed submanifolds of \( \mathcal{U} \) of codimension \( \geq m \).

Denote by \( V \subset \mathbb{R}^{m+2} \) the set of points \((x_1, \ldots, x_m, \lambda_1, \lambda_2)\) such that \( \lambda = \lambda_1 + i \lambda_2 \in \mathbb{C} \) and is a root of the polynomial \( P(\lambda) = \lambda^n + \alpha_1 \lambda^{n-1} + \ldots + \alpha_n \) of multiplicity \( \geq 2 \). Obviously, \( \tilde{\mu}(\mathcal{U}_1) = V \).

Lemma 3. The map \( \tilde{\mu} \mid_{\mathcal{U}_1} : \mathcal{U}_1 \rightarrow V \) is open (in the topologies on \( \mathcal{U}_1, V \) induced by their imbedding into \( \mathcal{U}, \mathbb{R}^{m+2} \) respectively).

Proof. Obviously, it suffices to prove that \( \mu \mid_{\mathcal{U}_1} : \mathcal{U}_1 \rightarrow \mathring{V} \), where \( \mathring{V} \) is the projection \( (\mathbb{R}^m \times \mathbb{R}^2 \rightarrow \mathbb{R}^m) \) of \( V \) into \( \mathbb{R}^m \), is open. That is, we have to prove that given a neighbourhood \( U \) of \( A \in \mathcal{U}_1 \), for any \( P \in \mathring{V} \) sufficiently close to \( \mu(A) \), there is a \( B \in U \) such that \( \mu(B) = P \).

This statement is obvious if \( A \) has the real canonical form; its extension for \( A \) not in canonical form follows from \( \mu(A) = \mu(T^{-1}AT) \) for \( T \in GL(m) \).

Proof of Lemma 1. \( V \) is an algebraic variety in \( \mathbb{R}^{m+2} \), defined by the polynomial identities \( P_1(\lambda_1, \lambda_2) = P_2(\lambda_1, \lambda_2) = P_1'(\lambda_1, \lambda_2) = P_2'(\lambda_1, \lambda_2) = \lambda_1^2 + \lambda_2^2 - 1 = 0 \), where \( P_1(\lambda_1, \lambda_2) = \text{Re} \ P(\lambda_1 + i \lambda_2) \) etc. Therefore, it can be written as a finite disjoint union of submanifolds.
folds of $\mathbb{R}^{m+2}$ of decreasing dimension, $V = \bigotimes_{i=1}^{d} V_i$.

We prove $\dim V_1 \leq m - 2$. To do this, we note that $\text{codim} V_1 \geq \text{rank}_x V$ for any $x \in V_1$ (cf. [3]), where $\text{rank}_x V$ is the dimension of the linear space spanned by the differentials at $x$ of the polynomials of the ideal associated with $V$. Since $V_1$ is open in $V$, it suffices to prove that the set of those $x$ for which $\text{rank}_x V \geq 4$ is dense in $V$.

For $x \in V$, $x = (\alpha_1, \ldots, \alpha_m, \lambda_1, \lambda_2)$ we have

$$d P_1 = (\ldots, \lambda_1, 1, 0, 0),$$

for $x \in V$, $x = (\alpha_1, \ldots, \alpha_m, \lambda_1, \lambda_2)$ we have

$$d P_1' = (\ldots, 1, 0, \frac{\partial P_1}{\partial \lambda_1}, \frac{\partial P_1}{\partial \lambda_2}),$$

$$d P_2' = (\ldots, 0, 0, \frac{\partial P_2}{\partial \lambda_1}, \frac{\partial P_2}{\partial \lambda_2}),$$

$$d(\lambda_1^2 + \lambda_2^2 - 1) = (\ldots, 0, 0, 2 \lambda_1, 2 \lambda_2),$$

and, since

$$\begin{vmatrix}
\lambda_1, 1, 0, 0 \\
1, 0, \frac{\partial P_1}{\partial \lambda_1}, \frac{\partial P_1}{\partial \lambda_2} \\
0, 0, \frac{\partial P_1}{\partial \lambda_1}, \frac{\partial P_1}{\partial \lambda_2} \\
0, 0, 2 \lambda_1, 2 \lambda_2
\end{vmatrix} = 2 \left[ \lambda_2 \frac{\partial P_1}{\partial \lambda_1} - \lambda_1 \frac{\partial P_1}{\partial \lambda_2} \right] = 2 \left[ \lambda_2 \frac{\partial P_1}{\partial \lambda_1} + \lambda_1 \frac{\partial P_1}{\partial \lambda_2} \right] = 2 \text{Re} \left( \lambda^{-1} P''(\lambda) \right).$$

Thus, it suffices to prove that for a dense subset of $V$, $\text{Re} \left( \lambda^{-1} P''(\lambda) \right) \neq 0$.

It is obvious that the set of those $x \in V$ for which $P''(\lambda) \neq 0$ is dense in $V$. If $\lambda$ is real and $\lambda \in S$, the set is dense. Therefore, $\text{Re} \left( \lambda^{-1} P''(\lambda) \right) \neq 0$.

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Assume that \( \lambda \) is not real, \( \lambda \in \mathbb{S} \) and \( P''(\lambda) \neq 0 \).
Then \( \lambda^{-1}P''(\lambda) = \overline{\lambda}P''(\lambda) = \overline{\lambda}(\lambda - \overline{\lambda})^2 \text{Re}(\lambda) \),
where \( \text{Re}(\mu) \) is real for \( \mu \) real. For \( \varepsilon \) real denote

\[
P_\varepsilon(\mu) = (\mu - \lambda)^2(\mu - \overline{\lambda})^2[\text{Re}(\mu) + \varepsilon] = \mu^m + \alpha_{1\varepsilon} \mu^{m-1} + \ldots + \alpha_m \varepsilon.
\]

\( P_\varepsilon(\mu) \) is real for \( \mu \) real and \( (\alpha_{1\varepsilon}, \ldots, \alpha_m, \lambda_1, \lambda_2) \in V \).

We have \( \text{Re}(\overline{\lambda}P''(\lambda)) - \text{Re}(\overline{\lambda}P''(\lambda)) = \varepsilon \text{Re}[\overline{\lambda}(\lambda - \overline{\lambda})^2] = -4\varepsilon \lambda_1 \lambda_2 \). Since both \( \lambda_1 \neq 0 \) and \( \lambda_2 \neq 0 \), there is an \( \varepsilon > 0 \) arbitrarily small such that \( \text{Re}(\overline{\lambda}P''(\lambda)) \neq 0 \). This proves the density in \( V \) of the set of points \( \lambda \) for which \( \text{Re}(\lambda^{-1}P''(\lambda)) \neq 0 \).

Let \( i \) be such that \( \overline{\mathcal{P}}(M_i) \cap V_j = \emptyset \) for \( j < i \). Since \( \bigcup_{j=1}^i V_j \) is open, \( \mathcal{M} = \overline{\mathcal{P}}^{-1}(V_i) = \overline{\mathcal{P}}^{-1}(\bigcup_{j=1}^i V_j) \) is open in \( M_i \) and, by Lemma 3, \( \nu(M_0) \) is open in \( V_i \). From this and the Sard's theorem ([6, Theorem 15.1]) it follows that there is a point \( \overline{\lambda} \in M_0 \) at which \( \overline{\mathcal{P}} \) is regular. Thus, locally \( \overline{\mathcal{P}}^{-1}(\overline{\mathcal{P}}(\overline{\lambda})) \) is an imbedded submanifold of the dimension \( \dim M_i - \dim V_i \geq \dim M_1 - m + 2 \). On the other hand, from Corollary 2 it follows \( \dim \overline{\mathcal{P}}^{-1}(\overline{\mathcal{P}}(\overline{\lambda})) \leq n^2 - n \). Consequently, \( \dim M_i \leq n^2 - 2 \), q.e.d.

**Lemma 4.** If \( \lambda_20 \neq 0 \), then \( \text{codim} \ N_1 \geq 4 \).

The proof of this lemma is similar to that of Lemma 1, with \( V \) replaced by the set \( W \subset \mathbb{R}^{m+2} \) of points

\( (\alpha_1, \ldots, \alpha_m, \lambda_0, \lambda_2) \) for which \( \lambda_0 = \lambda_0 + i \lambda_2 \)
is a root of \( P(\lambda) = \lambda^m + \alpha_1 \lambda^{m-1} + \ldots + \alpha_m \).
This is again an algebraic variety defined by the equations
\[ \lambda_1 - \lambda_{10} = \lambda_2 - \lambda_{20} = 0, P_1(\lambda_1, \lambda_2) = P_2(\lambda_1, \lambda_2) = 0. \]
The differentials of the polynomials at the points of \( W \) are
\[
\begin{align*}
dP_1 &= (\ldots, \lambda_{10}, 1, \frac{\partial P_1}{\partial \lambda_1}, \frac{\partial P_1}{\partial \lambda_2}), \\
dP_2 &= (\ldots, \lambda_{20}, 0, \frac{\partial P_2}{\partial \lambda_1}, \frac{\partial P_2}{\partial \lambda_2}), \\
d(\lambda_1 - \lambda_{10}) &= (\ldots, 0, 0, 1, 0), \\
d(\lambda_2 - \lambda_{20}) &= (\ldots, 0, 0, 0, 1).
\end{align*}
\]
Obviously, they are independent if \( \lambda_{20} \neq 0 \). The rest of the proof is analogous to the proof of Lemma 1.

**Proof of Proposition 1.** Openness follows from the fact that both \( \mathcal{U}_1 \) and \( \mathcal{U}_2 \) are closed.

For the proof of density we consider the sets
\[ \mathcal{U}_1, \mathcal{U}_2(\lambda_{10}, \lambda_{20}) \] with \( \lambda_{20} \neq 0 \) and the space \( \tilde{\mathcal{F}} \) of maps \( F : \text{int } I \times \mathbb{R}^2 \to \mathcal{U} \), defined by
\[
\tilde{F} = F|_{\text{int } I \times \text{id}}, \quad F \in \tilde{\mathcal{F}},
\]
edowed with the \( C^\infty \) uniform topology. Further, we denote by \( \tilde{\mathcal{V}}_i = \{ \tilde{F} \mid \tilde{F}(I) \cap \bigcap_{\delta \in \Lambda_{i-1}} M_\delta = \emptyset \} \) for \( 1 \leq i \leq n \), \( \tilde{\mathcal{V}}_{n+i} = \{ \tilde{F} \mid \tilde{F}(I) \cap \bigcap_{\delta \in \Lambda_{i-1}} M_\delta = \emptyset \} \) for \( 1 \leq i \leq b \). Since \( \tilde{\mathcal{V}}_2 \) is the intersection of the projections of \( \tilde{\mathcal{V}}_{n+i} \) taken over all nonreal \( \ell \)-th roots of unity, it suffices to prove that \( \tilde{\mathcal{V}}_{n+i} \) is dense in \( \tilde{\mathcal{F}} \). We prove this by induction showing that every \( \tilde{F} \in \tilde{\mathcal{V}}_i \) can be approximated arbitrarily closely by an \( \tilde{F}' \in \tilde{\mathcal{V}}_{i+1} \). Without loss
of generality we assume $1 < i < \pi$.

The map $\varphi : \Phi \rightarrow \tilde{\Phi}$ given by $\varphi (F) = \tilde{F}$ is a $C^\gamma$-representation (here and further in this proof we use the terminology of [6]) and the evaluation map meets $M_{\mathbb{K} - i}$ transversally. Due to the dimension estimates of Lemma 1 and Lemma 4, the existence of the approximation of $F$ not intersecting $M_{\mathbb{K} - i}$ follows from the transversality theorem [6, Theorem 19.1] and the openness of $\tilde{\mathcal{F}}_i$, q.e.d.

Denote $\mathcal{U}_3$ the subset of $\mathcal{U}$ consisting of matrices having an eigenvalue on $S$. Again, we associate with $\mathcal{U}_3^i$ the algebraic variety $\tilde{\mathcal{U}}_3$ in $\mathcal{H}$, defined by

$$\tilde{\mathcal{U}}_3 = \{(A, \lambda_1, \lambda_2) \mid P_1(\lambda_1, \lambda_2) = P_2(\lambda_1, \lambda_2) = \lambda_1^2 + \lambda_2^2 - 1 = 0 \}$$

whose projection is $\mathcal{U}_3^i$. Thus, $\tilde{\mathcal{U}}_3 = \bigcup_{i=1}^n \mathcal{U}_3^i$, where $\mathcal{U}_3^i$ are mutually disjoint manifolds of decreasing dimension and $\bigcup_{i=1}^n \mathcal{U}_3^i$ is closed in $\tilde{\mathcal{U}}_3$ for every $i$.

**Lemma 5.** $\text{codim } \mathcal{X}_1^i = 3$.

**Proof.** The proof of the inequality $\dim \mathcal{X}_1^i \geq 3$ is analogous to that of Lemma 1. We only note that the differentials of the defining polynomials $P_1, P_2, \lambda_1^2 + \lambda_2^2 - 1$ of $\tilde{\mathcal{U}}_3$ in $\mathbb{R}^{n+2}$ (defined as in Corollary 2) are independent if $\Re (\lambda P'(\lambda)) \neq 0$; it can be shown similarly as in the proof of Lemma 1 that this is true for a dense subset of $\tilde{\mathcal{U}}_3$.

To prove the opposite inequality assume $I = [0, 2]$ and consider the map $F(t) = \text{diag}\{t, 0, \ldots, 0\}$. If
\text{codim } \mathcal{K}_1 < 3 \quad \text{then it would follow from the transversality argument used in the proof of Proposition 1 that there should exist a small } C^\infty \text{ perturbation } \hat{F} \text{ of } F \text{ no value of which would have an eigenvalue on } S . \text{ This, however, is obviously impossible.}

\textbf{Proposition 2.} Let } J \subset I \text{ be a closed interval, } J \subset \text{int } I \text{ . Then, for every } \ell > 2 \text{ the subset } 
\mathcal{Y}_\ell (J) \subset \mathcal{Y}_\ell (J) \text{ of all } F \text{ such that } F \text{ meets } \mathcal{K}_3 \text{ transversally (i.e. } F \text{ meets transversally } \mathcal{K}_3 \text{ and does not meet } \mathcal{K}_i \text{ for } i > 1 \text{ at all) is open dense in } 
\mathcal{Y}_\ell (J) , \text{ and, thus, in } \Phi . 

\text{The proof is analogous to that of Proposition 1.}

\textbf{Corollary 3.} Given } J \text{ as in Proposition 2, the set } 
\mathcal{Y}_0 (J) \text{ of maps } F \in \Phi \text{ such that } F(J) \cap (\mathcal{K}_1 \cup \mathcal{K}_2 ) = 
= 0 \text{ and } F \text{ meets } \mathcal{K}_3 \text{ transversally over } J \text{ is residual in } \Phi . 

\textbf{Lemma 6.} Let } F \in \Phi \text{ and let } \lambda_0 \text{ be a simple eigenvalue of } F(t_0) , \text{ where } t_0 \in I . \text{ Then there is a neighbourhood } N \text{ of } t_0 \text{ in } I \text{ and a unique function } \lambda : N \to C \text{ such that } \lambda(t_0) = \lambda_0 \text{ and } \lambda(t) \text{ is an eigenvalue of } 
F(t) \text{ for } t \in N . \text{ Further, there is a nonsingular } C^\infty \text{ matrix } C(t) \text{ on } N \text{ such that } C^{-1} FC = B , \text{ where the first column of } B(t) \text{ is the transpose of } 
(\lambda(t), 0, \ldots, 0) . 

\textbf{Proof.} \text{Without loss of generality we may assume that } 
F(t_0) \text{ is in the Jordan canonical form with } \lambda_0 \text{ in the first column. Choose } C(t_0) = E \text{ (the unity matrix) and } 
C(t) = (c_1(t), \ldots, c_m(t)) , \lambda(t) \text{ as the solution of }
the set of equations $F(t)c_i(t) = \lambda(t)c_i(t)$, 
$c_i(t) = c_i(t_0), i > 1, |c_1(t)| = 1$ (|.| being the Euclidean norm). It is easy to check that the Jacobian of this set of equations at $t_0$ is not zero. The implicit function theorem completes the proof.

**Remark 1.** Under the assumptions of Lemma 6, for $\lambda_0$ not real, starting from the real canonical form of $F(t_0)$, one can similarly prove that there is a $C^\omega$ real matrix $C(t)$ in some neighbourhood of $t_0$ in $I$ that brings $F(t)$ into the form 

$$
\begin{pmatrix}
B_1(t), & B_2(t) \\
0, & B_3(t)
\end{pmatrix},
$$

where $B_1(t) = (\Re \lambda(t), \Im \lambda(t))$, $B_3(t) = (-\Im \lambda(t), \Re \lambda(t))$.

**Corollary 4.** Let $F \in \Phi$, $t_0 \in I$ and let $\lambda_{i_0}, ..., \lambda_{i_N}$ be simple eigenvalues of $F(t_0)$. Then, there is a neighbourhood $N$ of $t_0$ in $I$ and unique $C^\omega$ functions $\lambda_i : N \to \mathbb{C}$ such that $\lambda_i(t_0) = \lambda_{i_0}$ and $\lambda_i(t)$ are eigenvalues of $F(t)$ for $t \in N$. Further, there is a $C^\omega$ matrix $C(t)$ on $N$ such that $C^{-1}AC = B$, where $B$ has the form $\begin{pmatrix} \hat{B}_1, & \hat{B}_2 \\
0, & \hat{B}_3\end{pmatrix}$ and $\hat{B}_1$ is triangular with $\lambda_1, ..., \lambda_{i_N}$ on the diagonal. Also, there is a real $C^\omega$ matrix $\hat{C}(t)$ on $N$ that brings $F(t)$ into the form $\begin{pmatrix} \hat{B}_1(t), & \hat{B}_2(t) \\
0, & \hat{B}_3(t)\end{pmatrix}$, where $\hat{B}_i(t)$ is block diagonal with blocks as in Remark 1.

**Proposition 3.** Let $F \in \mathcal{W}_2^r(J)$ for some $r > 2$. Then, the eigenvalues of $F$ meet $S$ transversally.
By this proposition we mean that the functions $X$, defined in Lemma 6 for $\lambda_0 \in S$ (note that such $\lambda_0$ are simple) meet $S$ transversally.

**Proof.** Let $\lambda(t_0) \in S$ be an eigenvalue of $F(t_0)$. By Lemma 6, there is a nonsingular $C^r$ matrix $C(t)$ such that $C^{-1}(t)F(t)C(t) = B(t)$, where $B(t)$ has the form specified in Lemma 6. Denote $B(t, \mu)$ the matrix obtained from $B(t)$ by replacing in the first column $\lambda(t)$ by $\mu$. Denote by $\mu(t)$ the orthogonal projection of $\lambda(t)$ on $S$, $\phi$ the Euclidean distance. Since $C(t)B(t, \mu(t))C^{-1}(t) \in U_3$ and $U_1$ is open in $\bar{U}_3, (C(t)B(t, \mu(t))C^{-1}(t), \mu_1(t), \mu_2(t)) \in U_1$, for $t$ sufficiently close to $t_0$, where $\mu = \mu_1 + i \mu_2$. We have $|\lambda(t)| - 1 = |\lambda(t) - \mu(t)| = \phi(B(t), B(t, \mu(t))) \geq |C(t)|^{-1}$, $|C(t)|^{-1}\phi(F(t), C(t)B(t, \mu(t))C^{-1}(t)) \geq \lambda_1 \phi(F(t), U_1)$, where $\lambda_1 > 0$ is a suitable constant. If $F$ meets $U_1$ transversally, then obviously $\phi(F(t), U_1) \geq \lambda_2 |t - t_0|$ for some $\lambda_2 > 0$. Consequently, $\frac{d|\lambda(t)|}{dt}|_{t=t_0} \neq 0$, q.e.d.

**Corollary 5.** The number of such $t \in J$ for which an eigenvalue of $F(t)$ is on $S$, is finite for every $F \in \mathcal{F}^0(J)$.

**Theorem 1.** Let $J \subseteq \text{int} I$ be a closed interval. Then, the set $\Phi(A)(J)$ of those $F \in \Phi$, satisfying

(i) $F(t)$ has no double eigenvalue on $S$,
(ii) $F(t)$ has no non-real $L$-th root of unity as ei-
(iii) the eigenvalues of \( F(t) \) meet \( S \) transversally,
(iv) if an eigenvalue of \( F(t) \) lies on \( S \), then no other eigenvalue of \( F(t) \) lies on \( S \) except its complex conjugate,
for every \( t \in J \), is open dense in \( \Phi \).

**Corollary 6.** The set \( \Phi_1(J) \) of those \( F \in \Phi \) satisfying (i), (iii), (iv) of Theorem 1 and such that for every \( t \in J \), \( F(t) \) has no non-real root of unity as eigenvalue, is residual in \( \Phi \).

**Proof.** Openness is obvious. From Propositions 1 - 3 it follows that the set of maps from \( \Phi \), satisfying (i) - (iii) (i.e. the set \( \mathcal{Y}^0(J) \)), is open dense in \( \Phi \). Therefore, it suffices to prove that every \( F \in \mathcal{Y}^0(J) \) can be arbitrarily closely approximated by an \( \tilde{F} \in \mathcal{Y}^0(J) \) satisfying (iv). In virtue of Corollary 4 it suffices to show that if for some \( t_0 \neq 0 \), (iv) is not satisfied it is possible to perturb \( F \) in an arbitrary small neighbourhood \( N \) of \( t_0 \) by an arbitrary small perturbation, without changing it outside \( N \), in such a way that (i) - (iv) will be true for the perturbation of \( F \) for every \( t \in N \).

Assume that for some \( t_0 \in J \), \( \lambda \) pairs of conjugate eigenvalues \( \lambda_1, -\lambda_2, \ldots, \lambda_k \) lie on \( S \) (the modification of the proof for the case of some eigenvalue being real is straightforward). Let \( \alpha \) be so small that the functions \( \lambda_2, \ldots, k \) defined by \( \lambda_2, t_0 \) as in Lemma 6 exist and do not meet \( S \) except at \( t_0 \) and no other eigenvalue of \( F(t) \) lies on \( S \) on \( K \cap J \), where
\[ K = [t_0 - \alpha, t_0 + \alpha] \], and that there is a \( C^\infty \) matrix \( C \)

such that \( C^{-1}(t)F(t)C(t) = B(t) \) has the form

\[ B = \text{diag}\left\{ \begin{pmatrix} \lambda_{11}, \lambda_{12} \\ -\lambda_{21}, -\lambda_{22} \end{pmatrix}, \ldots, \begin{pmatrix} \lambda_{n1}, \lambda_{n2} \\ -\lambda_{n2}, -\lambda_{n1} \end{pmatrix}, B_1 \right\} \]

where \( \lambda_j = \lambda_{j1} + i \lambda_{j2} \) (cf. Remark 1). Choose an

\[ \varepsilon < \frac{\alpha}{2}, \text{ an real mutually distinct numbers } \varepsilon_j, j = 1, \ldots, n \]

such that \( |\varepsilon_j| < \varepsilon \) and a bump function \( \chi : N \to \mathbb{R} \) such that \( \chi(t) = 0 \) outside \( K \), \( \chi(t) = 1 \) for \( t \in K \), \( = [t_0 - \frac{\alpha}{2}, t_0 + \frac{\alpha}{2}] \), \( \hat{\lambda}_{ij}(t) = \lambda_j(t + \varepsilon_j \chi(t)) \),

\[ \hat{B}(t) = \text{diag}\left\{ \begin{pmatrix} \hat{\lambda}_{11}(t) \hat{\lambda}_{12}(t) \\ -\hat{\lambda}_{21}(t) \hat{\lambda}_{22}(t) \end{pmatrix}, \ldots, \begin{pmatrix} \hat{\lambda}_{n1}(t) \hat{\lambda}_{n2}(t) \\ -\hat{\lambda}_{n2}(t) \hat{\lambda}_{n1}(t) \end{pmatrix}, B_1(t) \right\} \],

\[ F(t) = \begin{cases} F(t) \text{ for } t \notin K \\ C(t) \hat{B}(t)C^{-1}(t) \text{ for } t \in K \end{cases} \]

It is obvious that \( \hat{F} \in \mathcal{F}_\varepsilon \) and, in \( K \cap J \), \( \hat{\lambda}_{ij} \) meets \( S \) exclusively at the point \( t_0 - \varepsilon_j \). If \( \varepsilon_j \) are chosen small enough, \( F \) will be arbitrarily close to \( F \), q.e.d.

§ 3

In [1, 2] it was shown that for \( f \in \mathcal{F}_\varepsilon \), each point of \( \overline{Z} \setminus Z_{K_0} \) (such points have been called branching points) is contained in some set \( Z_l \) with \( l \) being a di-

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visor of $\mathcal{A}$ and that some eigenvalue of $df_{\tau}$ at such point has to be a root of unity different from 1.

**Theorem 2.** There is a subset $\mathcal{F}_2$ of $\mathcal{F}_1$, residual in $\mathcal{F}$ such that for every $f \in \mathcal{F}_2$, the following is true for every $(\nu, m_0) \in \mathcal{E}_\nu(f)$, $\mathcal{A} \geq 1$:

(i) $df_{\nu}$ has no double eigenvalue on $S$.

(ii) $df_{\nu}$ has no non-real root of 1 as an eigenvalue.

(iii) The eigenvalues of $df_{\nu}$ meet $S$ transversally at $(\nu, m_0)$.

(iv) If an eigenvalue of $df_{\nu}(m_0)$ lies on $S$, then there is no other eigenvalue of $df_{\nu}(m_0)$ on $S$ except of its complex conjugate.

**Corollary 7.** For $f \in \mathcal{F}_2$, $(\nu, m) \in \mathcal{E}_\nu(f)$ can be a branching point only if one of the eigenvalues of $df_{\nu}(m)$ is $-1$, the other being outside $S$.

**Remark 2.** Denote $\mathcal{F}_{2h}$ the subset of $\mathcal{F}_{1h}$ of those mappings, satisfying (i),(iii),(iv) for $1 \leq h \leq h$ and (ii) with "roots" replaced by "$h$-th roots" for $1 \leq h \leq h$. Then, $\mathcal{F}_{2h}$ is open dense in $\mathcal{F}$.

**Remark 3.** (iii) should be understood as follows: If an eigenvalue $\lambda_0$ of $df_{\nu}(m_0)$ is on $S$, then in some neighbourhood $N$ of $(\nu, m_0)$ in $\mathcal{E}_\nu$, there is a unique $C^\infty$ function $\lambda : N \rightarrow \mathbb{C}$ such that $\lambda(\nu, m)$ is
an eigenvalue of \( d_{\mu}^m \lambda(m) \) for \((\mu, m) \in \mathbb{N}\) and

\[ \lambda(\mu_0, m_0) = \lambda_0. \]

This \( \lambda \) meets \( S \) transversally.

**Proof.** It suffices to prove Remark 2, from which Theorem 2 follows. We carry out the proof for \( \mathcal{F} = 1 \), i.e. we prove that \( \mathcal{F}_{21,2} \) is open dense for any \( \mathcal{F} \); the extension for \( \mathcal{F} > 1 \) is similar as in the proof of [1, Theorem 1].

The openness of \( \mathcal{F}_{21} \) is obvious. To prove density, assume \( \mathcal{F} \in \mathcal{F}_{11} \). Then, by [1, Theorem 1], there is an open set \( \mathcal{U} \) containing \( X_1(f) \) such that for every \( (\mu_0, m_0) \in \mathcal{U} \), (i) - (iv) is trivially satisfied.

\[ \mathcal{Z}_1 \setminus \mathcal{U} \]

can be covered locally finitely by a countable family \( (\mathcal{W}_\alpha, (\mu_\alpha \times x_\alpha)), \mathcal{W}_\alpha = \mathcal{U}_\alpha \times V_\alpha \) of coordinate neighbourhoods in such a way that for any \( \mathcal{F}_\alpha \in P \times M \) compact, \( \mathcal{W}_\alpha \cap \mathcal{K} = \emptyset \) for a finite number of \( \alpha \)'s only and \( (\mathcal{W}_\alpha, (\mu_\alpha \times x_\alpha)) \) satisfy (iv) of [1, Theorem 1] (i.e. \( \mathcal{W}_\alpha \cap \mathcal{Z}_1 \) is the graph of a \( C^k \) function \( \mathcal{g}_\alpha : \mathcal{U} \to \mathcal{V} \)).

We show how for any open \( \mathcal{W}_\alpha', \mathcal{W}_\alpha' = \mathcal{W}_\alpha = \mathcal{U}_\alpha \times V_\alpha \), \( \mathcal{F} \) can be approximated by \( \tilde{\mathcal{F}} \) such that \( \tilde{\mathcal{F}} \) coincides with \( \mathcal{F} \) outside \( \mathcal{W}_\alpha \) and satisfies (i) - (iv) of Theorem 2 for every \( (\mu_0, m_0) \in \mathcal{Z}_1 \cap \mathcal{W}_\alpha \). The construction of an approximation of \( \mathcal{F} \) satisfying (i) - (iv) for any \( (\mu_0, m_0) \in \mathcal{Z}_1 \) is then standard. In the rest of the proof we drop the subscript \( \alpha \).

In the coordinates \( (\mu, m) \mapsto (\mu, y), y = x - x_0 \mathcal{g}(\mu), \mathcal{F} \)
can be represented by

\[ y' = \lambda(\mu) y + \gamma(\mu, y) \]

where the primed coordinates are those of the image,
\( Y(\mu, 0) = 0 \), \( dY(\mu, 0) = 0 \).

By Theorem 1, we can approximate \( A : \mu(U) \rightarrow \mathcal{U} \) by a map \( \hat{A} : \mu(U) \rightarrow \mathcal{U} \) such that \( A \) satisfies (i) - (iv) of Theorem 1 on \( U \).

Let \( \psi : (\mu \times x)(W) \rightarrow \mathbb{R} \) be a \( C^\infty \) bump function such that \( \psi = 1 \) on \( (\mu \times x)(\overline{W}') \) and \( \psi = 0 \) outside \( (\mu \times x)(W) \). Denote by \( \hat{f} \) the map which coincides with \( f \) outside \( W \) and is given in \( W \) by the coordinate representation
\[ \psi' = [A(\mu) + \psi(\mu, x)(\hat{A}(\mu) - A(\mu)))] \psi + Y(\mu, \psi). \]

If we choose \( A \) sufficiently close to \( A, \hat{f} \) will be arbitrarily close to \( f \) and will satisfy (i) - (iv) for every \( (\mu_o, m_o) \in W' \).

Denote by \( Y_{\mu} \) the set of points \( (\mu, m) \in \mathbb{Z}_\mu \) for which one eigenvalue of \( df_{\mu}(m) \) is \(-1\). For \( (\mu, m) \in \mathbb{Z}_\mu \) denote \( \lambda(\mu, m) \) the number of eigenvalues of \( df_{\mu}(m) \) with modulus less than 1.

**Theorem 3.** Assume \( \mu > 2 \). Then, there is a subset \( \mathcal{F}'_2 \) of \( \mathcal{F}_2 \), residual in \( \mathcal{F} \), such that every \( f \in \mathcal{F}'_2 \) has the following properties:

(i) \( Y_{\mu} \) coincides with the set of \( \mu \)-periodic branching points,

(ii) for every \( (\mu_o, m_o) \in Y_{\mu} \), there is a coordinate neighbourhood \( (W, \mu \times x), W = U \times V \) of \( (\mu_o, m_o) \) such that \( \mu(\mu_o) = \mu, x(m_o) = 0, \mathbb{Z}_\mu \cap W = U \times \{0\} \) and

(a) \( \mathbb{Z}_2 \cap W \) consists of two components, separa-
ted by \((n_0, m_0)\); all points \((n, m) \in \mathbb{Z}_{2^k} \cap W\) satisfy 
\(\mu(n) > 0\) and \(\mathbb{Z}_{2^k} \cap W \cup \{n_0, m_0\}\) is a \(C^2\) (but 
not \(C^2\)) submanifold of \(W\).

(b) No eigenvalue of \([(\mathbb{Z}_{2^k} \cup \mathbb{Z}_{2^{2k}}) \cap \mathbb{W}] \setminus \{n_0, m_0\}\) 
is on \(S\); either \(\lambda(n, m) = \lambda(n', m') = \lambda(n'', m'') + 1\) or 
\(\lambda(n, m) = \lambda(n', m') = \lambda(n'', m'') - 1\) for 
any \((n, m) \in \mathbb{Z}_{2^k} \cap W, \mu(n) < 0, (n', m') \in \mathbb{Z}_{2^{2k}} \cap W, 
(n'', m'') \in \mathbb{Z}_{2^k} \cap W, \mu(n'') > 0\),

(c) \(W \setminus (\mathbb{Z}_{2^k} \cup \mathbb{Z}_{2^{2k}})\) contains no invariant set.

Proof. Again, we carry out the proof for \(k = 1\), the 
proof of its extension for \(k > 1\) being as in [1, Theorem 
1].

Let \(f \in \mathcal{F}_{2^k}^2\). Then, if \((n_0, m_0) \in \mathcal{Y}_1\), one eigenvalue of \(df_{n_0, m_0}\) is \(-1\) 
and the remaining ones can be divided into two groups ac-
cording to whether their moduli are < 1 or > 1, the 
number of the former ones being \(\lambda(n_0, m_0)\). Thus,

using [6, Appendix 3] as in [1, Lemma 4], it follows that 
we can choose the coordinates \((\mu, x)\) in such a way that 
\(x = (x_1, \psi, x), \dim x_1 = 1, \dim \psi = \lambda(n_0, m_0)\) and the 
coordinate representation of \(f\) in these coordinates is as 
follows:

\[
x_1 = -x_1 + \alpha(x_1 + \beta x_1^2 + \gamma x_1^3 + \omega(\mu, x_1, \psi, x)),
\]

\[
(3) \quad \psi = A\psi + Y(\mu, x_1, \psi, x),
\]

\[
x = Cx + Z(\mu, x_1, \psi, x),
\]

where \(\omega, Y, Z\) are \(C^k\) and

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\[ \omega, Y, Z \] are \( C^* \) and \( \gamma \), we denote by \( \mathcal{F}_{31} \) the subset of \( \mathcal{F}_1 \) of those maps in the coordinate representation (3) of which \( \beta^2 + \gamma \neq 0 \) for every \( (\mu_0, m_0) \in \mathcal{V}_1 (\mathcal{F}) \). The definition of \( \mathcal{F}_{31} \) does not depend on the choice of particular coordinates and the set \( \mathcal{F}_{31} \) is open dense in \( \mathcal{F} \). The proof of this as well as the proof that the maps of \( \mathcal{F}_{31} \) satisfy (i), (ii) for \( \kappa = 1 \) does not differ from the corresponding part of the proof of [1, Theorem 3], except of the proof of (ii)(c), where, because of the possible presence of the eigenvalues of moduli both \( < 1 \) and \( > 1 \) one has to use the argumentation of the proof of [1, Lemma 4].

As a corollary of [1, Theorem 1] and Theorem 3 we obtain

**Theorem 4.** Assume \( \kappa > 2 \). Then, for every \( \mathcal{F} \in \mathcal{F}_3 \):

(i) for \( \kappa \) odd, \( \mathcal{Z}_{\kappa} \) is a closed submanifold of \( P \times M \),

(ii) for \( \kappa \) even, either \( \mathcal{Z}_{\kappa} \) is closed and \( \mathcal{Y}_{\kappa/2} \) is empty, or \( \mathcal{Z}_{\kappa} \) is a \( C^1 \) (but not \( C^2 \)) submanifold of \( P \times M \) and \( \mathcal{Z}_{\kappa} \) is discrete and coincides with \( \mathcal{Y}_{\kappa/2} \).

**Remark 4.** This theorem corrects the erroneous formulation of its two dimensional version [1, Theorem 4], in which the possibility of \( \mathcal{Z}_{\kappa} \) being closed was omitted.

**References:**

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