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Commentationes Mathematicae Universitatis Carolinae, Vol. 13 (1972), No. 1, 63--78

Persistent URL: http://dml.cz/dmlcz/105396

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13,1 (1972)

A CHARACTERIZATION OF THE EIGENVALUES OF A COMPLETELY CONTINUOUS SELFADJOINT OPERATOR

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1. Introduction. In the present paper we give a characterization of the eigenvalues of a completely continuous selfadjoint operator which acts in a Hilbert space by a variational principle. Our arguments are based on the variant of Ljusternik-Schnirelmann-theory in [4] without the explicit use of the notion category. This procedure makes it possible to dispense with the oddness of the operator and therefore to handle the problem of existence and bifurcation of nontrivial solutions for nonlinear operator equations with not necessary odd operators. This has a great importance in the study of some problems in nonlinear elasticity.

In [2] M.S. Berger has given a similar characterization with an explicit use of the category arguments based on a result about the dimension of the critical set of a functional. Our formulations are, roughly speaking, a "curved" variant with respect to the formulation in [5] and with respect to the well-known variational formulation

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AMS, Primary 47A99 Ref. Ž. 7.974.42
Secondary 58E05

(see e.g. [8]) [1]. But it is not the aim of the present paper to give a treatment of some connections between these various formulations.

The advantage of our results and the variational formulations consist in the immediate applicability to bifurcation theory for nonlinear equations of the type $\lambda Au = Bu$. This will be done in a forthcoming paper about a generalization of the bifurcation procedure in [7].

__2. Preliminaries__. In this section, we recall some wellknown facts about the spectral analysis of completely continuous selfadjoint operators in a Hilbert space and give some inequalities for the use in the next sections (see [4]).

Let H be a real Hilbert space with the scalar product $(\, , \,)$ and the norm $\| \; \|$. In the whole paper we suppose that L __is a completely continuous selfadjoint positive operator__ (i.e. $(Lu, u) > 0$ for $u \neq \theta$) __which acts in__ H. It holds ([1]): There exists a finite or infinite sequence of orthogonal in pairs and normalized eigenvectors

$$e_1, e_2, \dots, e_n, \dots$$

which belong to the eigenvalues

$$\lambda_1 \geqq \lambda_2 \geqq \dots \geqq \lambda_n \geqq \dots \quad .$$

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1) Particularly for Courant-Weinstein-characterisation see: Dunford, N. and Schwartz, J.T.: Linear operators, part II. New York, London 1963.

For each $u \in H$ the expansion holds

(1) $$Lu = \sum_{j=1}^{\infty} \lambda_j (u, e_j) e_j \quad .$$

We denote by

$$\lambda_1^* > \lambda_2^* > \dots > \lambda_m^* > \dots$$

the sequence of all distinct eigenvalues of L (each λ_m^* has a certain finite multiplicity p_m, e.g. $\lambda_1^* = \lambda_1 \equiv \dots \dots \equiv \lambda_{p_1}$) and by E_m $(m = 1, 2, \dots)$ the eigenspaces of λ_m^*. Further we denote by H_m $(m = 1, 2, \dots)$ the closed linear hull of all eigenvectors which belong to the eigenvalues $\lambda_1^*, \dots, \lambda_m^*$. Clearly $H_1 = E_1$.

Let P_m be the projections onto H_m and $P_m^\perp$ the projection onto $H \ominus H_m$, respectively. From (1) follows

$$(LP_1^\perp u, u) = \sum_{j=p_1+1}^{\infty} \lambda_j (P_1^\perp u, e_j)(u, e_j)$$

$$= \sum_{j=p_1+1}^{\infty} \lambda_j (P_1^\perp u, e_j)^2 \quad ,$$

therefore for each $u \in H$

(2) $$(LP_1^\perp u, u) \leq \lambda_2^* \| P_1^\perp u \|^2 \quad .$$

Let be $u \in H \ominus H_m$. Then $(u, e_j) = 0$ for all eigenvectors e_j which belong to the eigenvalues $\lambda_1^*, \dots, \lambda_m^*$ and we obtain

(3) $$(Lu, u) \leq \lambda_{m+1}^* \| u \|^2 , \quad u \in H \ominus H_m \quad .$$

From this and (1) it follows that

(4) $$(Lu, u) = (LP_m u, P_m u) + (LP_m^\perp u, P_m^\perp u) \leq$$

$$\leq \lambda_1^* \| P_m u \|^2 + \lambda_{m+1}^* \| P_m^\perp u \|^2$$

$$= \lambda_{m+1}^* \| u \|^2 + (\lambda_1^* - \lambda_{m+1}^*) \| P_m u \|^2$$

for arbitrary $u \in H$. By a similar consideration we obtain

(5) $$(Lu, u) \geq \lambda_m^* \| P_m u \|^2 , \qquad u \in H .$$

3. The first eigenvalue. For arbitrary but fixed real $R > 0$ we define

$$S_R = \{ u \mid u \in H, \ \frac{1}{2} \| u \|^2 = R \} ,$$

$$B_R = \{ u \mid u \in H, \ \frac{1}{2} \| u \|^2 \leq R \} .$$

The functional

$$\phi(u) = \frac{1}{2} (Lu, u) , \qquad u \in H$$

has L as its gradient (in the Fréchet sense) on H .

The following result is wellknown, however the proof is given in a nontraditional way by the aid of an argument from [3].

Theorem 1. The variational problem

maximize $\phi(u)$ over B_R

has for each fixed $R > 0$ a solution $u_1 = u_1(R) \in S_R$ such that $\phi(u_1(R)) = \max_{B_R} \phi(u)$. There exists a $\lambda_1 = \lambda_1(R) \in R^1$ with $\lambda_1 u_1 = L u_1$. Furthermore

(i) $\phi(u_1(R)) = \lambda_1(R) R$ for each $R > 0$.

(ii) For each $R > 0$ holds $\lambda_1(R) = \lambda_1^*$.

<u>Proof of Theorem 1</u>. The functional ϕ is weakly continuous on H and after a wellknown theorem of functional analysis there exists an element $u_1 = u_1(R) \in B_R$ such that $\phi(u_1(R)) = \max_{B_R} \phi(u)$. Suppose $\frac{1}{2}\|u_1(R)\|^2 < R$. There would exist a real number $t > 1$ with $\frac{1}{2}\|u_1(R)\|^2 < \frac{1}{2}\|tu_1(R)\|^2 \leqq R$. On account of the maximality of $u_1(R)$ it follows

$$\frac{t^2}{2}(Lu_1(R), u_1(R)) = \phi(tu_1(R)) \leqq \phi(u_1(R)) = \frac{1}{2}(Lu_1(R), u_1(R))$$

a contradiction, since $u_1(R) \neq \theta$. Therefore $u_1(R) \in S_R$ and a trivial consideration shows

$$\phi(u_1(R)) = \max_{B_R} \phi(u) = \max_{S_R} \phi(u) .$$

After a slight modification of the proof of Theorem 4 in [3] we obtain the existence of a $\lambda_1 = \lambda_1(R) \in R^1$ with $\lambda_1 u_1 = Lu_1$. From the last statement it follows simply

$$\phi(u_1(R)) = \frac{1}{2}(Lu_1(R), u_1(R)) = \frac{1}{2}\lambda_1(R)\|u_1(R)\|^2 = \lambda_1(R)R$$

which proves (i).

Let $u \in E_1 \cap S_R$ be arbitrary. We obtain

$$(6) \quad \lambda_1^* R = \frac{\lambda_1^*}{2}\|u\|^2 = \phi(u) \leqq \max_{S_R} \phi(u) = \lambda_1(R)R .$$

On the other hand, let $u \in H$ be arbitrary. With $u = P_1 u + P_1^{\perp} u$ it follows from (2) that

$$\phi(u) = \frac{\lambda_1^*}{2}(P_1 u, u) + \frac{1}{2}(LP_1^{\perp} u, u) \leqq$$

$$\leq \frac{\lambda_1^*}{2} \|P_1 u\|^2 + \frac{\lambda_2^*}{2} \|P_1^{\perp} u\|^2 \leq \frac{\lambda_1^*}{2} \|u\|^2 .$$

Therefore, for arbitrary $u \in S_R$

$$\lambda_1(R)R = \max_{S_R} \phi(u) = \max_{S_R} \phi(u) \leq \lambda_1^* R$$

and together with (6) finally $\lambda_1(R) = \lambda_1^*$, which completes the proof.

4. Lemmas. In this section and the next we prepare the characterization of the eigenvalues λ_n with $n \geq 2$. We begin with a result about ordinary differential equations in Hilbert spaces.

Let $h \in H$ with $\|h\| = 1$ be chosen arbitrary, but fixed. Now let us consider the following initial value problem

$$(7) \qquad \frac{du(t)}{dt} = h - \frac{(u(t), h)}{2R} u(t), \quad u_0 \in S_R .$$

From [7], [9] we obtain

Lemma 1. There exists a real number $t_0 > 0$ which depends neither from u_0 nor from h such that

(i) in the interval $0 \leq t \leq t_0$ there exists a unique solution $u(t)$ of (7);

(ii) there exists a constant $c = c(R)$ which depends only on R but neither on u_0 nor on h with

$\|u(t) - u_0\| \leq c(R)t$ for all $0 \leq t \leq t_0$;

(iii) $u(t) \in S_R$ holds for all $0 \leq t \leq t_0$.

The crucial importance of this lemma is the uniformity of t_0 with respect to u_0 and h . We call the solution of (7) a trajectory on the sphere S_R .

For arbitrary $u \in S_R$ and $v \in H$, we introduce the following operators

$$S(v,u) = Lv - \frac{(Lv,u)}{2R}u, \quad Qu = Lu - \frac{(Lu,u)}{2R}u \ .$$

Lemma 2. There exists a constant c_1 (independently of R) such that for all $u, u_0 \in S_R$ and $v \in B_R$ there holds

$$(8) \quad \| S(v,u) - Qu_0 \| \leqq c_1 (\| u - u_0 \| + \| v - u_0 \|) \ .$$

Proof of Lemma 2. First of all we obtain

$$\| (Lv,u)u - (Lu_0,u_0) \| \leqq 2R \| L \| (2 \| u - u_0 \| + \| v - u_0 \|) \ .$$

Therefore

$$\| S(v,u) - Qu_0 \| \leqq \| Lv - Lu_0 \| + \frac{1}{2R} \| (Lv,u)u - (Lu_0,u_0)u_0 \|$$
$$\leqq c_1 (\| u - u_0 \| + \| v - u_0 \|)$$

with $c_1 = 2 \| L \|$, q.e.d.

For arbitrary $u, v \in H$ it holds that

$$\phi(u) - \phi(v) = \int_0^1 (L(v + s(u-v)), u - v)\, ds \ .$$

Let $u(t)$ be a solution of the initial value problem (7). For a certain mean value $\xi(t) \in (0,1)$ it follows from the last equation

$$\phi(u(t)) - \phi(u_0) = (L(u_0 + \xi(t)(u(t) - u_0)), u(t) - u_0)$$

and with the notation $u_\xi = u_0 + \xi(t)(u(t) - u_0)$ and after change of integration and scalar product (see [6]) therefore

$$(9) \qquad \phi(u(t)) - \phi(u_0) = \int_0^t (S(u_\xi, u(s)), h)\, ds$$

for all $0 \leqq t \leqq t_0$.

<u>5. Lemmas (continued).</u> Let us consider the space H_m with $m \geqq 2$ (see Section 2). Then we define

$$\sigma(H_m) = \{u \mid u \in H,\ \|P_m u\| > 0\} .$$

Clearly $(S_R \cap H_m) \subset \sigma(H_m)$. The following two lemmas are contained in [4] (see [4] for definitions, too).

<u>Lemma 3</u>. The set $S_R \cap H_m$ is noncontractible in $\sigma(H_m)$.

<u>Lemma 4</u>. Let $\tilde{H}$ be a proper subspace of H_m . If for a set $V \subset H$ it holds that $P_m V \cap \tilde{H} = \emptyset$ then it follows that $V \subset \sigma(H_m)$ and V is contractible in $\sigma(H_m)$.

<u>Proof of Lemma 4</u>. Suppose there would exist $u \in V$ with $P_m u = \theta$. Then $\theta \in P_m V$ and therefore $\theta \in P_m V \cap \tilde{H}$ which contradicts our assumption. The proof of the second part of Lemma 4 corresponds to the proof of Lemma 2.7 of [4], p.331, q.e.d.

Our next lemma provides an explicit condition for the belonging of an element $u \in H$ to $\sigma(H_m)$.

<u>Lemma 5</u>. Let $0 \leqq \sigma < \lambda_m^* - \lambda_{m+1}^*$. If $\phi(u) \geqq$ $\geqq (\lambda_m^* - \sigma)R$ for $u \in S_R$ then

$$\|P_m u\|^2 \geqq 2R\,\frac{\lambda_m^* - \lambda_{m+1}^* - \sigma}{\lambda_1^* - \lambda_{m+1}^*} > 0 .$$

<u>Proof of Lemma 5</u>. Suppose the contrary holds. Then we obtain with aid of (4)

$$\phi(u) = \frac{1}{2}(Lu, u) \leqq \frac{\lambda_{m+1}^*}{2}\|u\|^2 + \frac{\lambda_1^* - \lambda_{m+1}^*}{2}\|P_m u\|^2 <$$

$$< \frac{\lambda^*_{n+1}}{2} \|u\|^2 + R(\lambda^*_1 - \lambda^*_{n+1}) \frac{\lambda^*_n - \lambda^*_{n+1} - \delta}{\lambda^*_1 - \lambda^*_{n+1}}$$

$$= \lambda^*_{n+1} R + (\lambda^*_n - \lambda^*_{n+1} - \delta) R = (\lambda^*_n - \delta) R \quad .$$

This contradiction proves the assertion of the lemma.

<u>Corollary 1.</u> For $0 \leqq \delta < \lambda^*_n - \lambda^*_{n+1}$ it holds that

$$\{ u \mid u \in S_R , \ \phi(u) \geqq (\lambda^*_n - \delta) R \} \subset \sigma(H_n) \ .$$

<u>6. The eigenvalues</u> λ^*_n <u>with</u> $n \geqq 2$. In this section we proceed to the characterization of the eigenvalues λ^*_n with $n \geqq 2$.

<u>Definition</u>. $[V]_{H_n}$ denotes the class of all compact subsets $V \subset S_R$ which lie in $\sigma(H_n)$ and which are non-contractible in $\sigma(H_n)$.

Consequently $[V]_{H_n}$ contains with a set V all sets which are the result of V by a continuous deformation, remaining in S_R . Lemma 3 provides $(S_R \cap H_n) \in [V]_{H_n}$.

Now we formulate the following variational problem

$$(*) \qquad c_n = c_n(R) = \sup_{[V]_{H_n}} \inf_V \phi(u) \ .$$

Obviously it holds

$$(10) \qquad c_n(R) \geqq \min_{S_R \cap H_n} \phi(u) = \lambda^*_n R \ .$$

It is the aim of our following considerations to show that in the last inequality the equal sign must hold always , which provides the desired characterization by the variational problem ($*$).

For $\varepsilon > 0$ we introduce

$$W_\varepsilon = \{ u \mid u \in S_R , \ |\phi(u) - c_m(R)| \leq \varepsilon \} .$$

We prove now

<u>Lemma 6</u>. For each $\varepsilon > 0$ there exists a $u \in W_\varepsilon$ with $\| Qu \| < \varepsilon$.

<u>Proof of Lemma 6</u>. Suppose the contrary holds, i.e. it holds that $\| Qu \| \geq \varepsilon_0$ for a certain $\varepsilon_0 > 0$ and all $u \in W_{\varepsilon_0}$.

Let $u_0 \in W_{\varepsilon_0}$ be arbitrarily chosen. There exists a $h = h_{u_0}$ with $\| h \| = 1$ and $(Qu_0, h) \geq \frac{1}{2} \| Qu_0 \|$. Further, let $u(t)$ be the unique solution of the initial value problem (7) in the interval $0 \leq t \leq t_0$ with the initial value u_0 and $h = h_{u_0}$ (see Lemma 1). Now we take

$$t_1 = \min \{ t_0 , \frac{\varepsilon_0}{8c_1 c(R)} \} .$$

From Lemma 1 (ii) it follows

$$\| u(t) - u_0 \| \leq \frac{\varepsilon_0}{8c_1} \quad \text{for all } 0 \leq t \leq t_1$$

and from this

$$\| u_\xi - u_0 \| \leq \frac{\varepsilon_0}{8c_1} \quad \text{for all } 0 \leq t \leq t_1 .$$

Moreover, it holds that $\| u_\xi \| \leq (2R)^{\frac{1}{2}}$ and (8) provides for all $0 \leq s \leq t_1$ the estimation

$$\| S(u_\xi , u(s)) - Qu_0 \| \leq \frac{\varepsilon_0}{4} .$$

We obtain

$$(S(u_\xi, u(s)), h) \geq \frac{1}{2}\| Q u_0 \| - |(S(u_\xi, u(s)) - Q\dot{u}_0, h)|$$

$$\geq \frac{\varepsilon_0}{2} - \frac{\varepsilon_0}{4} = \frac{\varepsilon_0}{4}, \quad 0 \leq s \leq t_1 .$$

From (9) it follows therefore

$$(11) \quad \phi(u(t_1)) - \phi(u_0) = \int_0^{t_1} (S(u, u(s)), h)\, ds \geq \frac{1}{4}\, \varepsilon_0 t_1 .$$

This estimation holds <u>uniformly</u> for all $u_0 \in W_{\varepsilon_0}$.

We take $\varepsilon_1 = min(\frac{1}{8}\, \varepsilon_0 t_1, \varepsilon_0)$. For this ε_1 there exists a $V_{\varepsilon_1} \in [V]_{H_m}$ such that $\min_{V_{\varepsilon_1}} \phi(u) \geq$ $\geq c_m(R) - \varepsilon_1$.

Now we consider the decomposition

$$V_{\varepsilon_1}^{(1)} = \{u \mid u \in V_{\varepsilon_1},\ c_m(R) + \varepsilon_1 > \phi(u) \geq c_m(R) - \varepsilon_1\} ,$$

$$V_{\varepsilon_1}^{(2)} = \{u \mid u \in V_{\varepsilon_1},\ \phi(u) \geq c_m(R) + \varepsilon_1\} .$$

Obviously $V_{\varepsilon_1} = V_{\varepsilon_1}^{(1)} \cup V_{\varepsilon_1}^{(2)}$. Let $u \in V_{\varepsilon_1}^{(1)}$, i.e. $|\phi(u) - c_m(R)| \leq \varepsilon_1$. Since $\varepsilon_1 \leq \varepsilon_0$, it follows that $u \in W_{\varepsilon_0}$. Then we obtain from (11) (take $u = u_0 = u(0)$)

$$(12) \quad \begin{aligned} \phi(u(t_1)) &\geq \phi(u) + \frac{1}{4}\, \varepsilon_0 t_1 \geq \phi(u) + 2\varepsilon_1 \\ &\geq c_m(R) + \varepsilon_1 . \end{aligned}$$

Now we displace the element $u \in V_{\varepsilon_1}^{(1)}$ along the trajectory $u(t)$ with an initial point $u = u_0 = u(0)$ and $h = h_{u_0}$ up to a point $u(t)$ with $\phi(u(t)) \geq c_m(R) +$ $+ \varepsilon_1$. The inequality (12) shows that this is always possible.

Therefore it is possible to displace the whole set $V_{\varepsilon_1}^{(1)}$ continuously. The length of the displacement of a $u \in V_{\varepsilon_1}^{(1)}$ tends to zero, if $(c_m(R) + \varepsilon_1 - \phi(u))' \longrightarrow 0+$. The points in $V_{\varepsilon_1}^{(2)}$ remain fixed.

We denote the set which is displaced in such a way by $\tilde{V}_{\varepsilon_1}$. Obviously $\tilde{V}_{\varepsilon_1} \subset S_R$ (see Lemma 1 (iii)) and $\tilde{V}_{\varepsilon_1}$ is the result of a continuous deformation of V_{ε_1} . Since $\phi(\tilde{u}) \geqq c_m(R) + \varepsilon_1 \geqq \lambda_m^* R + \varepsilon_1$ for all $\tilde{u} \in \tilde{V}_{\varepsilon_1}$ it follows from Corollary 1 that $\tilde{V}_{\varepsilon_1} \subset \sigma(H_m)$. After the supposition V_{ε_1} is noncontractible in $\sigma(H_m)$ and since $\tilde{V}_{\varepsilon_1}$ is the result of a continuous deformation, it follows that $\tilde{V}_{\varepsilon_1}$ is noncontractible in $\sigma(H_m)$. Therefore $\tilde{V}_{\varepsilon_1} \in [V]_{H_m}$ and

$$c_m(R) = \sup_{[V]_{H_m}} \inf_{V} \phi(u) \geqq \inf_{\tilde{V}_{\varepsilon_1}} \phi(u) \geqq c_m(R) + \varepsilon_1 .$$

This contradiction completes the proof.

The next theorem presents our main result.

__Theorem 2__. For each natural integer $m \geqq 2$ it holds:

(i) The variational problem

$$(*) \qquad c_m = c_m(R) = \sup_{[V]_{H_m}} \inf_{V} \phi(u)$$

has a solution $u_m(R) \in S_R$ for each $R > 0$ such that

$$\phi(u_m(R)) = c_m(R) = c_m .$$

(ii) There exists a real number $\lambda_m(R)$ with $\lambda_m(R) u_m(R) = L u_m(R)$.

(iii) It holds $\lambda_n(R) = \lambda_n^*$ for all $R > 0$.

Corollary 2. For each $R > 0$ there holds $c_n(R) = \lambda_n^* R$.

Proof of Corollary 2. From Theosem 2 (ii) it follows that

$$\phi(u_n(R)) = \frac{1}{2}(Lu_n(R), u_n(R)) = \frac{1}{2}\lambda_n(R)\|u_n(R)\|^2$$

and since $u_n(R) \in S_R$ we obtain with (i) and (iii)

$$c_n = c_n(R) = \phi(u_n(R)) = \lambda_n^* R \ , \quad \text{q.e.d.}$$

Proof of Theorem 2 (i) and (ii). We choose a sequence $\{\varepsilon_j\}$ of real numbers with $\varepsilon_j \longrightarrow 0$ for $j \longrightarrow \infty$. From Lemma 6 we can conclude the existence of a sequence $\{u_{j,n}\} \in W_{\varepsilon_j}$ with $\|Qu_{j,n}\| \leqq \varepsilon_j$ (in the whole proof we denote $u_{j,n}(R)$ simply by $u_{j,n}$). Without loss of generality we can assume that $u_{j,n} \longrightarrow u_n$ (weak convergence). Since ϕ is weakly continuous, it follows that $\phi(u_n) = c_n$.

Suppose now j_0 is sufficiently large such that $0 < \varepsilon_j \leqq \sigma R$ for all $j \geqq j_0$, while σ is chosen as in Lemma 5. We obtain for $j \geqq j_0$

$$\phi(u_{j,n}) \geqq c_n(R) - \varepsilon_j \geqq (\lambda_n^* - \sigma)R$$

and from the inequality (5) and Lemma 5 it follows that

$$\phi(u_{j,n}) = \frac{1}{2}(Lu_{j,n}, u_{j,n}) \geqq \frac{\lambda_n^*}{2}\|P_n u_{j,n}\|^2$$

$$\geqq \lambda_n^* R \, \frac{\lambda_n^* - \lambda_{n+1}^* - \sigma}{\lambda_1^* - \lambda_{n+1}^*} \ , \qquad j \geqq j_0 \ .$$

The tending of $j \longrightarrow \infty$ provides

(13) $$(Lu_n, u_n) > 0 .$$

We define

$$\lambda_{j,n} = \lambda_{j,n}(R) = \frac{1}{2R}(Lu_{j,n}, u_{j,n}) .$$

Again we can assume without loss of generality that $\lambda_{j,n} \to \lambda_n$. A simple argument which makes use of (13) shows that $\lambda_n > 0$. Therefore without loss of generality

$$\frac{1}{\lambda_{j,n}} Lu_{j,n} \longrightarrow \frac{1}{\lambda_n} Lu_n .$$

Now suppose that for $j \geqq j_1$ there holds $\lambda_{j,n} \geqq \frac{1}{2}\lambda_n$. We obtain for $j \geqq j_1$

$$\| u_{j,n} - \frac{1}{\lambda_n} Lu_n \| \leqq$$

$$\leqq \frac{2}{\lambda_n} \| Qu_{j,n} \| + \| \frac{1}{\lambda_{j,n}} Lu_{j,n} - \frac{1}{\lambda_n} Lu_n \| \xrightarrow[j\to\infty]{} 0$$

and from this follows

$$\| u_{j,n} - u_n \|^2 = (u_{j,n}, u_{j,n}) - 2(u_{j,n}, u_n) + \| u_n \|^2 \to$$

$$\to \frac{1}{\lambda_n}(Lu_n, u_n) - \frac{1}{\lambda_n}(Lu_n, u_n) - (u_n, u_n) + \| u_n \|^2 = 0 .$$

We conclude $u_{j,n} \longrightarrow u_n$ (strong convergence) and furthermore

$$u_n \in S_R \quad \text{and} \quad u_n = \frac{1}{\lambda_n} Lu_n , \quad \text{q.e.d.}$$

<u>Proof of Theorem 2 (iii)</u>. E_n is a proper subspace of H_n. By Lemma 4 for each $V \in [V]_{H_n}$ there must exist $\bar{u} \in V$ with $P_n \bar{u} \in E_n$. From (3) it follows that

$$\Phi(\bar{u}) = \frac{1}{2}(LP_n\bar{u}, P_n\bar{u}) + \frac{1}{2}(LP_n^{\perp}\bar{u}, P_n^{\perp}\bar{u})$$

$$= \frac{\lambda^*_n}{2} \| P_n \bar{u} \|^2 + \frac{1}{2}(LP_n^{\perp}\bar{u}, P_n^{\perp}\bar{u}) \leqq$$

$$\leq \frac{\lambda_n^*}{2} \| P_n \bar{u} \|^2 + \frac{\lambda_{n+1}^*}{2} \| P_n^{\perp} \bar{u} \|^2 \leq \lambda_n^* R \quad .$$

Therefore

$$\min_V \phi(u) \leq \phi(\bar{u}) \leq \lambda_n^* R$$

and since $V \in [V]_{H_n}$ was arbitrary,

$$c_n = c_n(R) = \sup_{[V]_{H_n}} \inf_V \phi(u) \leq \lambda_n^* R \quad .$$

In consequence

$$\lambda_n(R) R = \frac{1}{2} \lambda_n(R) \| u_n(R) \|^2 = \frac{1}{2} (L u_n(R), u_n(R))$$

$$= \phi(u_n(R)) = c_n(R) \leq \lambda_n^* R \quad .$$

We combine this with the inequality (10), and the proof is complete.

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(Oblatum 26.5.1971)