Commentationes Mathematicae Universitatis Carolinae

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Commentationes Mathematicae Universitatis Carolinae, Vol. 13 (1972), No. 1, 149--161

Persistent URL: http://dml.cz/dmlcz/105402

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13,1 (1972)

THE EXISTENCE OF POLAR NON-DEGENERATE FUNCTIONS ON A TOPOLOGICAL MANIFOLD

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1. In this note we prove

Theorem 1.1. If F is a non-degenerate function on a connected n -dimensional topological manifold M with m critical points, $u+1$ critical points of index zero and $v+1$ critical points of index n, there exists a polar non-degenerate function $F^{\#}$ with $m-2u-2v$ critical points such that $F^{\#}$ is identical with F in some neighborhood of each of $F^{\#}$'s critical points. If $n > 2$, the function $F^{\#}$, as defined, will have u fewer critical points of index one than F and v fewer of index $n-1$.

Morse [2] and Smale [4] have proved the differentiable version of this theorem which is of great importance in the study of C^{∞} manifolds. In 1959 [3], Morse states the topological theorem and claims an unpublished direct topological proof. We are unable to prove the theorem by direct topological methods but have reduced it to the differentiable case and then applied the differenti-

AMS, Primary: 57A99, 57D70, 58E05 Ref.Ž. 3.964, 3.983

able theorem. (The theorem is of considerable interest owing to the discovery of a non-triangulable topological manifold.)

Briefly (see 2. Definitions), we first isolate a critical point C of index one at level c such that $f^{-1}([0, c])$ is connected but $f^{-1}([0, c))$ is not. We next construct a homeomorphism H mapping a connected, closed, differentiable manifold $\mathcal{U}$, a subset of R^m, into a subset of M containing C and two critical points A_1 and A_2 of index zero so that $f \circ H$ is a C^∞ non-degenerate function with critical points a_1 and a_2 ($A_i = H(a_i)$, $i = 1, 2$) of index zero and critical point 0 ($C = H(0)$) of index one. By the differentiable case (Morse [2]) there exists $F^{\#}$, agreeing with $f \circ H$ except in a neighborhood of an arc joining a_1 to 0 to a_2, with only one critical point of index zero and no other critical points. Defining $f^{\#}(x) = F^{\#} \circ H^{-1}(x)$ for x in $H(\mathcal{U})$ and $f^{\#}(x) = f(x)$ otherwise, one has reduced by one the number of critical points of index zero and of index one.

Theorem 1.1 then follows easily.

2. Definitions. Let M be a compact, connected, m-dimensional topological manifold and f be a continuous real-valued function on M. The following definitions are adapted from [1] and [3]. For $x = (x_1, \ldots, x_m)$ in R^m, let $r = (x_1, \ldots, x_k)$, $s = (x_{k+1}, \ldots, x_m)$,

$z = (r, s)$ for $0 \leq k \leq m$, $|r| = (x_1^2 + \dots + x_k^2)^{1/2}$ with similar definitions of $|s|$ and $|z|$. For $k = 0, \dots, m$ and $e > 0$, let $N_{e,k} = \{z \text{ in } R^m : |r| < e \text{ and } |s| < e\}$. In particular, for $k = 1$, let $N_{e,1} = N_e$.

A point x in M is a topological regular point of f if there exists $e > 0$ and an associated homeomorphism h mapping N_e into M such that $h(N_e)$ is an open neighborhood of x and $f \circ h(z) = f(x) + z_1$ if $z = (z_1, s)$ is in N_e. A point x in M is a topological critical point of f if it is not a topological regular point, and $f(x)$ is said to be a critical value for f. A point x in M is a topological critical point of index k, $0 \leq k \leq m$, if there exists $e > 0$ and an associated homeomorphism h mapping $N_{e,k}$ into M such that $h(N_{e,k})$ is an open neighborhood of x and $f \circ h(z) = f(x) - |r|^2 + |s|^2$ for z in $N_{e,k}$. Clearly topological critical points of index k are topological critical points. In both of the above cases h is called an f-coordinate function and $h(N_{e,k})$ an f-neighborhood. If every critical point of f is a topological critical point of index k for some k, $0 \leq k \leq m$, f is said to be a topological non-degenerate function. Entirely similar definitions apply in the differentiable case.

A topological (or C^∞) non-degenerate function clearly has isolated critical points, and if the manifold M is compact (our case) they are finite in number. The set $f^{-1}(t)$ is called f-level t, and if $f(x) \leq t$, x is said to lie below f-level t. Similar definitions for strictly below, above, and strictly above f-level t are immediate.

From now on we shall suppose f is a non-negative topological non-degenerate function on a closed, connected topological manifold M as above, and for each f-coordinate function h associated with some x in M, $h(0) = x$. The f-neighborhoods of the critical points will be disjoint and the f-neighborhoods of the regular points will contain no critical points.

It is easy to show that, as in the differentiable case (see [3], Lemma 2.1, p. 253), it is no loss of generality to suppose f has distinct critical values and we shall do so.

3. <u>Isolating a critical point of index one</u>. Suppose A and B are any two critical points of index 0, $f(A) < f(B)$. Let $c = \inf\{c' : A$ and B lie in the same arc-component of $f^{-1}([0, c'])\}$. Since A and B are arc-connected in M, c exists and $c \geq f(B)$. Let $K_1, \ldots, K_n$ be the arc-components of $f^{-1}([0, c))$ and $Cl(K_i)$ be the closure of K_i.

<u>Lemma 3.1</u>. $Cl(K_i) \cap Cl(K_j)$ is empty or a critical point of index one at level c, $i \neq j$.

<u>Proof</u>. (i) If x is in $Cl(K_i) \cap Cl(K_j)$, $f(x) = c$, for if $f(x) > c$, x is not in $Cl(K_i)$

for any i, and if $f(x) < c$, x is in K_k, some k, and some open neighborhood of x is in K_k so x is not in $Cl(K_j)$, $j \neq k$.

(ii) If x is in $Cl(K_i) \cap Cl(K_j)$, x is a critical point of index one, for if x is either regular or a critical point of index greater than one, there exist p and q arc-connected below f-level c in an arc-connected f-neighborhood of x.

(iii) Since f is non-degenerate, x is then a critical point of index one, of which there are only a finite number in M, and $f(x) = c > f(B)$.

Let C be such a critical point of index one at f-level c, h_0 the f-coordinate function associated with C and $h_0(N_d)$ the f-neighborhood such that $h_0(0) = C$.

4. <u>Reduction to the differentiable case</u>. Our method is to inductively construct a homeomorphism H mapping a connected, closed, differentiable manifold U, a subset of R^m, into a subset of M containing C and two critical points, A_1 and A_2, of index zero, so that $f \circ H$ is a C^∞ non-degenerate function with critical points a_1 and a_2 ($A_i = H(a_i)$) of index zero and critical point 0 of index one.

Our inductive process is relatively straightforward. We must make sure it starts and ends smoothly (a, c and d) and that it does not get hung up at a regular level or at a critical point of index greater than zero (b).

a) H in a neighborhood of 0 . Let e be such that $d > max(\sqrt{e} + e/4, \sqrt{e}/2)$, and let

$U_0' = \{x$ in $R^n : 0 \leq x_1 \leq \sqrt{e}$ and $|s| < \sqrt{e}/2\}$,

$U_0 = \{x$ in $U_0' : 0 \leq x_1 \leq \sqrt{e} - r$ and $|s| \leq$

$\leq B(e)\}$,where

$$0 < r < \sqrt{e}/2 \quad \text{and}$$

$$B(e) = (1/2)\min(\sqrt{e}/2 - r, \sqrt{e/8}, \sqrt{8e - e^2}),$$

$$W_0' = \{x \text{ in } U_0' : -e/4 < -x_1^2 + |s|^2\},$$

v be a C^∞ map of R into R , strictly increasing on $[0, 1]$, such that, if $t \leq 0$, $v(t) = 0$, and

if $t \geq 1$, $v(t) = 1$,

$$b(x) = x_1 - \sqrt{e/4 + |s|^2},$$

$d(x) = \sqrt{e/2} - \sqrt{e/4 + |s|^2}$, and g_0 map U_0' into N_d so that

$$g_0(x_1, s) = ([1 - v(b(x)/d(x)]x_1 + $$
$$+ v(b(x)/d(x))\sqrt{x_1 + |s|^2}, s).$$

<u>Lemma 4.1</u>. g_0 is a C^∞ diffeomorphism on U_0 .

<u>Proof</u>. (i) g_0 is one to one on U_0' . It suffices to prove $g_0(x) \neq g_0(x')$ where $x = (x_1, \ldots, x_m)$, $x' = (x_1', \ldots, x_m')$, x, x' in U_0' , $x_i = x_i'$ except $x_1 < x_1'$, and $v(b(x)/d(x)) < 1$. But if $g_0(x) = g_0(x')$, then $0 < x_1' - x_1 =$

$= v(b(x)/d(x))(\sqrt{x_1 + |s|^2} - x_1) -$

$- v(b(x')/d(x'))\,(\sqrt{x_1' + |s|^2} - x_1') <$
$< v(b(x')/d(x'))(x_1' - x_1) \leq x_1' - x_1$, a contradiction.

(ii) The Jacobian of g_0 is easily seen to be non-zero in U_0' .

Define H mapping U_0' into M by $H(x) = h_0 \circ g_0(x)$, so 0 is a critical point of index one for $f \circ H$ with g_0 and W_0' for its $f \circ H$ -coordinate function and neighborhood, and $f \circ H$ is regular on the rest of U_0' . Note, then, that $f \circ H$ is C^∞ and non-degenerate on U_0 .

b) A sequence of regular points in M accumulating at a critical point, A_1 , of index zero, and H defined around the corresponding sequence in R^n . We shall first discuss the general choice of points p_i in M , and then show how to avoid the two problems of accumulation at a regular level or at a critical point of index greater than zero.

(i) Let $p_1 = h_0(\sqrt{e/2}, 0, \dots, 0)$, a regular point in M with f -coordinate function h_1 and f -neighborhood $h_1(N_{e'})$. Choose $p_2 = h_1(-e'/2, 0, \dots, 0)$ and define a C^∞ homeomorphism g_1 mapping $N_{e'}$ into R^n by $g_1(x_1, s) = (-x_1 + f(C) - f(p_1), s)$.
Let $U_1 = U_0 \cup \{x \text{ in } g_1(N_{e'}) : f(C) - f(p_1) \leq x_1 \leq f(C) - f(p_2)$ and $|s| \leq B(e')\}$ and extend H over U_1 by $H(x) = h_1 \circ g_1^{-1}(x)$.

From now on, having chosen e , the succeding choise, e', will be referred to simply as e - in the end it will be possible to choose such a common e for the entire argument. Therefore $B(e)$, $B(e')$, etc. will be called simply B.

Having chosen $p_{m-1} = h_{m-2}(-e/2, 0, \dots, 0)$ for $m \geq 3$, and extended H over

$\mathcal{U}_{m-2} = \mathcal{U}_{m-3} \cup \{x \text{ in } g_{m-2}(N_e) : f(C) - f(p_{m-2}) \leq$

$\leq x_1 \leq f(C) - f(p_{m-1})$ and $|s| \leq B\}$ by

$H(x) = h_{m-2} \circ g_{m-2}^{-1}(x) = h_{m-2}(-x_1 + f(C) - f(p_{m-2}), s)$

where h_{m-2} and $h_{m-2}(N_e)$ are the f-coordinate function and neighborhood for p_{m-2}, let

$p_m = h_{m-1}(-e/2, 0, \dots, 0)$ and extend H over

$\mathcal{U}_{m-1} = \mathcal{U}_{m-2} \cup \{x \text{ in } g_{m-1}(N_e) : f(C) - f(p_{m-1}) \leq x_1 \leq f(C) -$

$- f(p_{m-1}) + e/2$ and $|s| \leq B\}$ by

$H(x) = h_{m-1} \circ g_{m-1}^{-1}(x)$.

Clearly $f \circ H$ is C^{∞} and non-degenerate on $\mathcal{U}_i$ for any i.

(ii) Suppose the p_m accumulate at a regular level k. Let p be an accumulation point - $f(p) = k < f(p_i)$ for every i. Let h and $h(N_e)$ be the f-coordinate function and neighborhood for p and choose p_m in $h(N_e)$. If $h^{-1}(p_m) = (y_1, \dots, y_n)$, suppose $y_1 = e/2$. Let $p_{m+1} = h(-e/2, y_2, \dots, y_n)$, g be the C^{∞} homeomorphism mapping R^n into R^n such that $g(x_1, \dots, x_n) = (e/2 + f(C) - f(p_m) - x_1,$ $x_2 + y_2, \dots, x_n + y_n)$ and

$\mathcal{U}_m = \mathcal{U}_{m-1} \cup \{x \text{ in } g^{-1}(N_e) : f(C) - f(p_m) \leq x_1 \leq f(C) - f(p_{m+1})$ and $|s| \leq B\}$, and extend H over $\mathcal{U}_m$ by $H(x) = h \circ g(x)$. Note $f(p_{m+1}) < r$.

(iii) Suppose the p_m accumulate at a critical point p of f of index $k > 0$ with f-coordinate function h and neighborhood $h(N_{e,k})$, so that $f(p) = r$.

Select p_m in $h(N_{e,k})$. Suppose $h^{-1}(p_m) = (0, \dots, 0, x_{k+1}, \dots, x_n)$. p_m is regular and has f-coordinate function h_m and neighborhood $h_m(N_e)$. Let $p_{m+1} = h_m(-e/2, s_0)$ such that $h^{-1}(p_{m+1})$ is not on the $x_{k+1}, \dots, x_n$ axis. Without loss of generality suppose $f(p_{m+1}) - r < e/4$. Define g_m^{-1} mapping $g_m(N_e)$ into N_e by $g_m^{-1}(x) = g_m^{-1}(x_1, s) = (-x_1 + f(C) - f(p_m), s + s_0((x_1 - f(C) + f(p_m))/(f(p_m) - f(p_{m+1}))))$ for x in $g_m(N_e)$, and let $\mathcal{U}_m = \mathcal{U}_{m-1} \cup \{x \text{ in } g_m(N_e) : f(C) - f(p_m) \leq x_1 \leq f(C) - f(p_m) + e/2$ and $|s| \leq B\}$. Extend H to $\mathcal{U}_m$ by $H(x) = h_m \circ g_m^{-1}(x)$. Clearly $f \circ H$ is C^∞ and non-degenerate on $\mathcal{U}_m$.

If $h^{-1}(p_m) = (x_1, \dots, x_n)$ and $x_i \neq 0$ for some $i \leq k$, identify p_m and p_{m+1}.

Let $g = f \circ h$, t be an element of R, x' be an element of R^n and $R(x'; 0) = x'$ and consider the differential equation $\dfrac{dR}{dt} = \dfrac{\nabla g}{|\nabla g|^2}$. Since

$$\frac{d(g \circ R)}{dt} = 1, \quad g \circ R(x'; t) = t + c \;,$$

or since $g \circ R(x'; 0) = g(x') = c$, $g \circ R(x'; t) = t + g(x')$.

For $x = (x_1, \ldots, x_n)$ define

$H(x) = h \circ R(h^{-1} \circ H(f(C) - f(p_{m+1}), x_2, \ldots, x_n);$

$f(C) - f(p_{m+1}) - x_1)$ on $U_{m+1} - U_m$ where

$U_{m+1} = U_m \cup \{x = (x_1, s) : f(C) - f(p_{m+1}) \leq x_1 \leq f(C) -$

$- f(p_{m+1}) + e/2$ and $|s| < B\}$ and let $p_{m+2} =$

$= H(f(C) - f(p_{m+1}) + e/2, 0, \ldots, 0)$ so that

$f \circ H(p_{m+2}) = f(p_{m+1}) - e/2 < r$.

Clearly $f \circ H$ is C^∞ and non-degenerate on U_{m+1} .

(iv) Continuing to choose p_m's and extend H so that the p_m do not accumulate at a regular level or at a critical point of index $k > 0$, it is then clear, since M is compact, that the p_m accumulate at a critical point A_1 of index zero.

c) H in a neighborhood of A_1. Let A_1 have f-coordinate function and neighborhood h_m and $h_m(N_{e,0})$ such that, having chosen p_m, $h_m^{-1}(p_m) = (\sqrt{e/2}, 0, \ldots, 0)$ in $N_{e,0}$, $f(C) - f(A_1) > e$ and $H^{-1}(p_m) =$ $= (f(C) - f(A_1) - e/2, 0, \ldots, 0)$. Let

$a_1 = (f(C) - f(A_1), 0, \ldots, 0)$,

$V_1 = \{ x = (x_1, s) : f(C) - f(p_m) \leq x_1 < f(C) -$

$- f(A_1) + \sqrt{e/8 - |s|^2}$ and

$|s|^2 < min(e/8, 8e - e^2) \}$,

$W_1 = \{ x$ in $R^n : |x - a_1|^2 \leq e/8 \}$,

w be a C^∞ function, strictly decreasing on $[0, 1]$, such that $w(t) = 1$ if $t \leq 0$ and $w(t) = 0$ if $t \geq 1$,

$d(x) = \sqrt{e/8 - |s|^2} - f(C) + f(A_1) + e/4$,

$b(x) = x_1 - f(C) + f(A_1) + e/4$, and define g_m mapping V_1 into R^n by $g_m(x) = g_m(x_1, s) =$
$= (x_1 - f(C) + f(A_1), s)$ if $f(C) - f(A_1) - e/8 \leq x_1$,

and $g_m(x_1, s) = ([1 - w(b(x)/d(x))] x_1 +$
$+ w(b(x)/d(x)) [f(C) - f(A_1) + \sqrt{f(C) - f(A_1) - x_1 - |s|^2}]$
$- f(c_1) + f(A_1), s)$ if $x_1 < f(C) - f(A_1) - e/8$.

As in the case of g_0 , g_m is a C^∞ diffeomorphism on V_1 . Extend H onto V_1 by $H(x) = h_m \circ g_m(x)$. Then $f \circ H$ is C^∞ and non-degenerate in particular on $U_{m+1} = U_m \cup \{x$ in $V_1 : x_1 \leq f(C) - f(A_1) +$
$+ (1/2)\sqrt{e/8 - |s|^2}$ and $|s| \leq B\}$ and a_1 is a critical point of $f \circ H$ of index zero with $f \circ H$ -coordinate function g_m^{-1} and neighborhood W_1 .

d) H and $\mathcal{U}$. Returning to N_d (see p.153), define a function similar to g_0 for $x_1 < 0$, and let $q_1 = h_0(-\sqrt{e/2}, 0, \dots, 0)$. Choose a sequence of q_m similar to the p_m and sets W_m similar to the $\mathcal{U}_m$, again bypassing critical points of index greater than zero and accumulation at a regular level, until the q_m accumulate at A_2, a critical point of index zero. Extend H in a neighborhood of the x_1 axis from 0 through negative values of x_1 past $x_1 = f(A_2) - f(C)$.
Let $a_2 = (f(A_2) - f(C), 0, \dots, 0)$.

Then $f \circ H$ is C^∞ and non-degenerate on $\mathcal{U} = \mathcal{U}_{n+1} \cup W_{n+1}$ containing the points of the x_1-axis from a_2 through 0 to a_1 as interior points, where W_{n+1} is the set corresponding to $\mathcal{U}_{n+1}$ as above.

Since M is closed, there are a finite number of p_i and q_i, and so a suitably small e may be chosen and used all through the argument. Then $\mathcal{U}$ is a connected closed differentiable manifold on which $f \circ H$ is a C^∞ non-degenerate function with two critical points of index zero and one of index one.

5. Proof of Theorem 1.1. Let $f \circ H = F$. Then, as in [2], there exists $F^\#$, C^∞ and non-degenerate on $\mathcal{U}$, agreeing with F except in some small neighborhood, N, of the x_1-axis from a_2 to a_1, with only one critical point in $\mathcal{U}$ – a critical point of index zero – in some neighborhood of which it agrees with F. Let $f^\#(x) = F^\# \circ H^{-1}(x)$ for x in $H(\mathcal{U})$ so that, except

in $H(N)$, $f^{\#}(x) = f(x)$. Then $f^{\#}$ is a topological non-degenerate function on M with one less critical point of index zero and one less of index one than f. Continuing in this fashion, there exists a non-degenerate topological function on M with only one critical point of index zero, and since critical points of index m for f are critical points of index zero for $-f$, there exists a polar non-degenerate function on M.

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(Oblatum 23.8.1971)

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