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ON DENSE SUBSPACES OF CERTAIN TOPOLOGICAL SPACES

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In this paper, the following results are obtained:

- (i) Each stratifiable space in which each point has a σ -closure preserving local base has a dense subspace which is an M_1 -space. (ii) There exists a paracompact σ -space (due to R.W. Heath) which has no dense stratifiable subspace. (iii) Each semi-stratifiable space has a dense subspace which is a σ -space.

I. Introduction. Consider the following relationships between certain widely studied abstract spaces (all spaces are to be T_1). (1) Each M_1 -space is a stratifiable space (M_3 -space) [3]. (2) Each stratifiable space is a paracompact σ -space [6]. (3) Each σ -space is a semi-stratifiable space [4]. The converses of statements (2) and (3) are shown to be false in [5] and [1] respectively and the validity of the converse of statement (1) is an open question. However, it follows from the author's results in [9], that in each of the statements (1), (2), and (3) a first countable space of the second type contains a dense subspace

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which is of the first type. It is the purpose of this paper to investigate this relationship for non-first countable spaces.

II. Preliminaries.

Notation 1.1. If M is a subset of the space S , then $CL(M)$ will denote the closure of M in S . If H is a set collection, then H^* will denote the union of the members of H .

Definition 1.2. A collection \mathcal{G} of subsets of the space S is said to be closure preserving provided that for each subcollection H of \mathcal{G} , $CL(H^*) = \{CL(h) \mid h \in H\}^*$. A collection \mathcal{G} of subsets of the space S is said to be σ -closure preserving if it is the union of countably many closure preserving collections.

Definition 1.3. A collection \mathcal{G} of subsets of the space S is said to be a network for S provided that if $\mu \in S$ and D is an open set containing μ , then there exists an element g of \mathcal{G} such that $\mu \in g$ and $g \subset D$.

Definition 1.4. [3] An M_1 -space is a regular space having a σ -closure preserving base.

Definition 1.5. [3] An M_2 -space is a regular space S having a σ -closure preserving quasi-base.

Definition 1.6. [2] A space X is a stratifiable space (M_3 -space) if to each open $U \subset X$, one can assign a sequence U_1, U_2, \dots of open subsets of X such that

- (a) $CL(U_m) \subset U$ for each m ,
- (b) $\bigcup U_m = U$,

(c) $U_m \subset V_m$, whenever $U \subset V$.

Definition 1.7. [7] A σ -space is a space X having a σ -locally finite network.

Definition 1.8. (Due to E.A. Michael.) A space X is semi-stratifiable if to each open $U \subset X$, one can assign a sequence U_1, U_2, \dots of open subsets of X which satisfy (b) and (c) of Definition 1.6.

Lemma 1.9. [4] A necessary and sufficient condition for a space X to be semi-stratifiable is that for each $x \in X$, there exists a sequence $g_1(x), g_2(x), \dots$ of open subsets of X such that (i) $\bigcap g_i(x) = x$ and (ii) if $y \in X$ and x_1, x_2, \dots is a sequence of points in X such that for each i , $y \in g_i(x_i)$, then x_1, x_2, \dots converges to y .

III. Theorems. The author has not been able to decide whether each non-first countable stratifiable space has a dense subspace which is an M_1 -space. However, the following theorem is a partial answer. By the statement that \mathcal{G} is a local base for the point μ of the space S is meant that \mathcal{G} is a collection of open subsets of S such that if μ is contained in the open set D , then there exists an element g of \mathcal{G} such that $\mu \in g$ and $g \subset D$.

Theorem 3.1. Each stratifiable space in which each point has a σ -closure preserving local base has a dense subspace which is an M_1 -space.

Proof. Let S be a stratifiable space and for each point μ of S let $B(\mu)$ denote a σ -closure preserving local

base for \mathcal{B} .

Each stratifiable space is a \mathcal{G} -space [6]. Thus, let $H = \cup H_i$ denote a network for S where for each i , H_i is locally finite. For each i , let K_i denote a point set containing one point from each element of H_i and note that K_i is discrete in S . Since each stratifiable space is paracompact [3] and hence collectionwise normal, for each i , there exists a discrete collection \mathcal{G}_i of open sets in S covering K_i such that each element of \mathcal{G}_i contains only one point of K_i . For each i and each point μ of K_i , let $V_i(\mu) = \{ \ell \in \mathcal{B}_\mu \mid \ell \text{ is contained in the element of } \mathcal{G}_i \text{ which contains } \mu \}$. Note that $V_i(\mu) = \bigcup_j V_{i,j}(\mu)$ where for each j , $V_{i,j}(\mu)$ is closure preserving.

Now, let $K = \cup K_i$ and for each i and j , let $V_{i,j} = \{ \ell \cap K \mid \ell \in V_{i,j}(\mu) \text{ and } \mu \in K_i \}$. It follows that K is a dense subset of S and $\bigcup_i \bigcup_j V_{i,j}$ is a \mathcal{G} -closure preserving base for K , regarded as space. Thus, K is an M_1 -space.

Theorem 3.2. There exists a paracompact \mathcal{G} -space which has no dense stratifiable subspace.

Proof. In [7], Heath gave an example of a regular, countable space X which is not stratifiable. Since X is a paracompact \mathcal{G} -space, it suffices to show that X also has no dense stratifiable subspace.

The space X in [7] is based on the existence of a collection \mathcal{F}' of subsets of N , the set of all natural numbers, such that (1) \mathcal{F}' has c members, (2) for any choice of $m + m$ distinct members, $F_1, F_2, \dots, F_m, F_{m+1}, \dots$

\dots, F_{m+m} of \mathcal{F}' , $F_1 \cap F_2 \cap \dots \cap F_m \cap$
 $\cap (N - F_{m+1}) \cap (N - F_{m+2}) \cap \dots \cap (N - F_{m+m}) \neq \emptyset$

and (3) for any two natural numbers x and y , there is a member of \mathcal{F}' that contains exactly one of x and y . The points of X are the points of N and $\mathcal{F} = \mathcal{F}' \cup \{N - F \mid F \in \mathcal{F}'\}$ is a subbasis for the topology of X .

Now, suppose that S is a dense subspace of X . Since for each element F of \mathcal{F}' , both F and $(N - F)$ are open in X , $F \cap S \neq \emptyset$ and $(N - F) \cap S \neq \emptyset$. Thus let $\mathcal{G}' = \{F \cap S \mid F \in \mathcal{F}'\}$. It follows that the collection \mathcal{G}' has properties (1), (2), and (3) above with respect to the subset S of N and that $\mathcal{G} = \mathcal{G}' \cup \{S - G \mid G \in \mathcal{G}'\}$ is a subbasis for S . Replacing N by S , \mathcal{F}' by \mathcal{G}' , and \mathcal{F} by \mathcal{G} , one can use the same argument given by Heath to show that S is also not stratifiable.

The proof given for the following theorem is a modification of the proof given in [9] for the existence of a dense developable subspace in a semi-metric space.

Theorem 3.3. Each semi-stratifiable space S has a dense subspace which is a \mathcal{G} -space.

Proof. It is sufficient to show that S has a dense subspace X which is the union of countably many subsets each of which is discrete in X .

For each point μ of S , let $g_1(\mu), g_2(\mu), \dots$ be a sequence of open sets in S as in Lemma 1.9. Denote by Ω a well-ordering of the points of S . For each j , let X_j

be the subset of S such that: (1) the first element of K_j is the first element of S with respect of Ω . (2) If I is an initial segment of K_j , then the first element μ of $K_j - I$ is the first element of S with respect to Ω such that μ is not a limit point of I and μ is not in $g_j(q)$ for q in I . (3) If K'_j is a subset of S having properties (1) and (2) then either K'_j is K_j or K'_j is an initial segment of K_j .

It follows that $K = \cup K_i$ is dense in S . For suppose that $\mu \in S - CL(K)$. If for each i , $g_i(k_i)$ contains μ for some point k_i in K , then the sequence k_1, k_2, \dots would converge to μ and μ would be in $CL(K)$. Thus for some j , there exists no element k of K_j such that $g_j(k)$ contains μ . But if this were true, μ would be in K_j and hence in K . This contradicts the choice of μ .

Now, let $X_1 = K_1$ and for each $i > 1$, let $X_i = K_i - (CL(\bigcup_{j=1}^{i-1} X_j) \cap K_i)$. It follows that $X = \cup X_i$ is dense in S . Consider X_i for each i . By the construction of K_i , no point of X_i is a limit point of X_i . And by the construction of X_i , no point of $\bigcup_{j=i+1}^{\infty} X_j$ is a limit point of X_i . Thus if X_i has a limit point q in X , q must be in $\bigcup_{j=1}^{i-1} X_j$. But for each μ in X_i , there exists an m such that μ is not in $g_m(q)$ for q in $\bigcup_{j=1}^{i-1} X_j$. If this were not true, the sequence q_1, q_2, \dots where for each m , q_m is in $\bigcup_{j=1}^{i-1} X_j$ and μ is in $g_m(q_m)$, would converge to μ and hence μ would be a

limit point of $\bigcup_{j=1}^{i-1} X_j$. Thus for each n , let $X_{i,n} = \{ \rho \text{ in } X_i \mid \rho \text{ is not in } \mathcal{O}_m(\rho) \text{ for } \rho \text{ in } \bigcup_{j=1}^{i-1} X_j \}$. Note for each n , $X_{i,n}$ has no limit point in X .

Thus $X = \bigcup_i \bigcup_m X_{i,n}$ is a dense subspace of S which is the union of countably many subsets each of which is discrete in X .

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