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A NOTE ON THE RIEMANN CURVATURE TENSOR

Oldfich KOWALSKI, Praha

In Paper [2] the problem was discussed whether, and how, a Riemann metric can be derived from a "generalized" curvature tensor, under a natural assumption of regularity. The purpose of this Note is to extend our results to a wider class of curvature tensors.

We shall start with some preparatory lemmas.

**Lemma 1.** Let $V$ be a real vector space with a positive scalar product $g$. Let $G \subset O(V)$ be a connected Lie group of orthogonal transformations of $V$ and $\mathfrak{g} \subset \mathfrak{so}(V)$ its Lie algebra. Then for any symmetric bilinear form $\mathcal{h}$ on $V$ the following is true:

$$\mathcal{h}(AX, Y) + \mathcal{h}(X, AY) = 0.$$  

**Proof.** See [1], Chapter I.

**Lemma 2.** (See [1], Appendix 5.) Let $G$ be a subgroup of $O(n)$ which acts irreducibly on the $n$-dimensional coordinate space $\mathbb{R}^n$. Then any symmetric bilinear form

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on $\mathbb{R}^n$ which is invariant by $G$ is a multiple of the standard scalar product

$$(x, y) = \sum_{i=1}^{n} x^i y^i$$

Let $\mathcal{L}$ be a set of linear endomorphisms of a vector space $V$. Put

$$(2) \quad \Theta(\mathcal{L}) = \{ h \in S^2(V) \mid h(AX, Y) + h(X, AY) = 0; X, Y \in V, A \in \mathcal{L} \}$$

where $S^2(V)$ denotes the space of all symmetric bilinear forms on $V$.

We say that $\mathcal{L}$ generates a Lie algebra $\mathfrak{g} \subset \mathfrak{gl}(V)$ if $\mathfrak{g}$ is the least Lie subalgebra of $\mathfrak{gl}(V)$ containing $\mathcal{L}$. Finally, $G(\mathcal{L})$ will denote the connected subgroup of $GL(V)$ whose Lie algebra is generated by $\mathcal{L}$.

**Proposition 1.** Let $V$ be a vector space with a (positive) scalar product $\langle \cdot, \cdot \rangle$ and $G \subset O(V)$ an irreducible Lie group of orthogonal transformations of $V$. Let $\mathcal{L} \subset \mathfrak{so}(V)$ be a set of linear endomorphisms generating the Lie algebra $\mathfrak{g}$ of $G$. Then

(i) $\dim \Theta(\mathcal{L}) = 1$, i.e., $\Theta(\mathcal{L}) = \langle \mathfrak{g} \rangle$.

(ii) If $X \in V$ and $AX = 0$ for any $A \in \mathcal{L}$, then $X = 0$.

**Proof.** ad (i). If $\mathcal{L} = \mathfrak{g}$, the assertion is nothing else than an infinitesimal version of Lemma 2 (cf. Lemma 1).

In a general case we have $\Theta(\mathfrak{g}) \subset \Theta(\mathcal{L})$. Put $\mathcal{L}' = \{ A \in \mathfrak{g} \mid \Theta(\mathcal{L}) \subset \Theta(\{A\}) \}$. Because $\Theta(\mathcal{L}') = \cap \Theta(\{A\})$ ($A \in \mathcal{L}$), we get $\Theta(\mathcal{L}') = \Theta(\mathcal{L})$.

It suffices to show that $\mathcal{L}' = \mathfrak{g}$. Clearly, if
A \cdot B \in \mathcal{L} \quad \text{then} \quad \alpha A + \beta B \in \mathcal{L}' \quad \text{Now, for any} \quad X \in \mathcal{V},

\nu \in \Theta(\mathcal{L}), \quad A, B \in \mathcal{V}, \quad \nu ([A, B] X, X) = \nu (ABX, X) - \nu (BAX, X) = -\nu (BX, AX) + \nu (AX, BX) = 0 \quad \text{and hence} \quad [A, B] \in \mathcal{L}'

\text{ad (ii). Let first} \quad \mathcal{L} = \mathcal{g} \quad \text{Then if a non-zero} \quad X \in \mathcal{V} \quad \text{exists with} \quad AX = 0 \quad \text{for any} \quad A \in \mathcal{g}, \quad \text{the corresponding group} \quad G \quad \text{pointwise preserves the vector subspace} \quad (X) \subset \mathcal{V} \quad \text{and hence} \quad G \quad \text{is not irreducible} \quad \text{a contradiction.}

\text{Now, let} \quad \mathcal{L} \subset \mathcal{g} \quad \text{be general, and let} \quad X \in \mathcal{V} \quad \text{be such that} \quad AX = 0 \quad \text{for any} \quad A \in \mathcal{L} \quad \text{Then the same is true for any} \quad B \in \mathcal{g} \quad \text{This completes the proof.}

\text{Let} \quad B \quad \text{be a tensor of type} \quad (1, 3) \quad \text{on a vector space} \quad \mathcal{V}, \quad \text{i.e., a bilinear map of} \quad \mathcal{V} \times \mathcal{V} \quad \text{into} \quad \mathfrak{gl}(\mathcal{V}). \quad \text{Then}

\mathcal{B} = \{ B(X, Y) \mid X, Y \in \mathcal{V} \} \quad \text{is a subset of} \quad \mathfrak{gl}(\mathcal{V}) \quad \text{and we shall put}

G(B) \overset{\text{def}}{=} G(\mathcal{B}), \quad \Theta(B) \overset{\text{def}}{=} \Theta(\mathcal{B})

\text{Following} \quad [2], \quad \text{a linear map} \quad B : \mathcal{V} \wedge \mathcal{V} \longrightarrow \mathfrak{gl}(\mathcal{V}) \quad \text{is called regular if the endomorphism} \quad B(X \wedge Y) \quad \text{is non-trivial for any} \quad X \wedge Y \neq 0 \quad \text{(we can write also} \quad B(X, Y) \quad \text{instead of} \quad B(X \wedge Y) \quad \text{as} \quad B \quad \text{corresponds to a unique anti-symmetric bilinear map of} \quad \mathcal{V} \times \mathcal{V} \quad \text{into} \quad \mathfrak{gl}(\mathcal{V}). \quad \text{)}

\text{Further, suppose that a scalar product} \quad \varphi \quad \text{on} \quad \mathcal{V} \quad \text{exists satisfying} \quad \varphi(B(U, T) Y, X) = -\varphi(B(U, T) X, Y), \quad \varphi(B(U, T) X, Y) = \varphi(B(X, Y) U, T), \quad \text{for any} \quad U, T, X, Y \in \mathcal{V} \quad \text{Then} \quad B \quad \text{is called a curvature structure with respect to} \quad \varphi \quad \text{Now, we have}
Proposition 2. Let $V$ be a vector space provided with a scalar product $\varphi$ and let $B : \bigwedge V \to gl(V)$ be a regular curvature structure with respect to $\varphi$. Then the group $G(B)$ is an irreducible subgroup of $O(V)$.

Proof. The inclusion $G(B) \subset O(V)$ is obvious because $B \subset \sigma(V)$. We show that $G(B)$ is irreducible. According to [2], Lemma 1, for any two vectors $X \perp Y$ of $V$ there are transformations $B(U_i \wedge T_i)$ such that $\sum B(U_i \wedge T_i) X = Y$.

If the group $G(B)$ were reducible, the corresponding Lie algebra generated by $\{ B(U \wedge T) \mid U, T \in V \}$ would possess a proper invariant subspace $V' \subset V$, a contradiction.

Let $(M, \varphi)$ be a Riemann manifold of class $C^\infty$ having the curvature tensor $R$. Following C. Teleman [4], the space $(M, \varphi)$ is called non-divisible if, at each point $x \in M$, the group $G(R_x)$ is irreducible. It is obvious that each non-divisible Riemann manifold is irreducible (see [1], Ch.III.,IV.).

More generally, we shall call a tensor field $B$ of type $(1,3)$ on $(M, \varphi)$ non-divisible if the group $G(B_x)$ is irreducible for each $x \in M$.

Further, the tensor field $B$ is called a curvature structure with respect to $\varphi$ (or on $(M, \varphi)$) if so is each algebraic tensor $B_x (x \in M)$. For example, the Riemann curvature tensor $R$ of $(M, \varphi)$ and the corresponding Weyl tensor of conformal curvature $C$ are curvature structures.
According to Proposition 2, any regular curvature structure on \((M, g)\) is non-divisible. (Here "regular" means "regular at each point \(x \in M\)."

One can re-write Proposition 1 as follows:

**Proposition 3.** Let \((M, g)\) be a Riemann space (of class \(C^\infty\)) and \(B\) a non-divisible curvature structure on \((M, g)\). Then

(i) \(\dim (B_x) = 1\) for each \(x \in M\), i.e., \(\Theta(B) = \cup \Theta(B_x)(x \in M)\) is a line bundle; and \(g\) is a section of \(\Theta(B)\).

(ii) If \(B(X, Y) Z = 0\) for any vector fields \(X, Y\) on \(M\) then \(Z\) is a null field.

Now, we can see easily that Theorem 2 and all the paragraphs 3 - 7 of \([2]\) remain true if we replace the word "regular" by the word "non-divisible" everywhere. Particularly, we get the following theorems (the reader is referred to the original paper \([2]\) for details).

**Theorem 1.** (C. Teleman, \([4]\).) Let \((M, g)\) be a connected non-divisible Riemann space of dimension \(n \geq 3\), and let \(\phi\) be a curvature tensor-preserving diffeomorphism of \((M, g)\) onto a Riemann space \((M', g')\). Then \(\phi\) is a homothety.

**Corollary.** (See K. Nomizu and K. Yano, \([3]\).) Let \((M, g)\) be a connected, analytic, irreducible, locally symmetric Riemann space of dimension \(n \geq 3\) and let \(\phi\) be a curvature tensor-preserving diffeomorphism of \((M, g)\)
onto a Riemann space \((\mathcal{M}', \varphi')\). Then \(\Phi\) is a homothety.

**Proof** of the corollary: one can see easily that, for any point \(x \in \mathcal{M}\), \(G(R_x)\) is the restricted homogeneous holonomy group of \((\mathcal{M}, \varphi)\) at \(x\). Thus \((\mathcal{M}, \varphi)\) is non-divisible.

**Theorem 2.** (Cf. [2], paragraph 5 for details.) Let \(B\) be a non-divisible tensor field of type \((1, 3)\) on a \(C^\infty\)-manifold \(\mathcal{M}, \dim \mathcal{M} \geq 3\). Then one can decide whether or not \(B\) is locally a Riemann curvature tensor only by algebraic operations and differentiations.

**Theorem 3.** Let \(\mathcal{M}\) be a \(C^\infty\)-manifold, \(\dim \mathcal{M} \geq 3\). A local reconstruction of a non-divisible Riemannian metric \(\varphi\) on \(\mathcal{M}\) from its curvature tensor \(R\) requires only algebraic operations, differentiations and the integration of an exact differential.

**References**


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