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HAUSDORFF MEASURES OF THE SET OF CRITICAL VALUES OF FUNCTIONS
OF THE CLASS $C^{k,\lambda}$

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This paper deals with the problem of critical values of real functions. The following assertion is known for functions of one variable (see [1]): If f is a function of the class $C^{k,\lambda}$, then $\mu_b(f(Z)) = 0$, where $b = \frac{1}{k+\lambda}$, μ_b is a b -Hausdorff measure and Z denotes the set of all critical points of the function f . In this paper there is proved an analogous assertion for functions defined on some open set in E_n . Theorem 4.2 and Remark 4.1 give a full answer to the question how big the set of critical values can be in dependence of the smoothness of our function f . This result is proved for $\lambda = 0$ (i.e. for $f \in C^k$) in [2],[3],[4].

I am indebted to Professor J. Nečas for his valuable advices.

1. Notations and terminology. We shall denote by Ω a fixed open set in the n -dimensional Euclidean space E_n . Let k be a positive integer number, $\lambda \in \langle 0,1 \rangle$, let f be a function defined on Ω . Then we write $f \in C^{k,\lambda}(\Omega)$ if

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f has on Ω continuous derivatives of all orders not exceeding k and if derivatives of the order k are λ -Hölderian. We shall denote the set of critical points of the given function by $Z = \{x \in \Omega; \frac{\partial f}{\partial x_i}(x) = 0, i = 1, \dots, n\}$. If $\beta = (\beta_1, \beta_2, \dots, \beta_m)$ is a multiindex then we write $|\beta| = \beta_1 + \dots + \beta_m$ and $D^\beta f = \frac{\partial^{|\beta|} f}{\partial x_1^{\beta_1} \partial x_2^{\beta_2} \dots \partial x_m^{\beta_m}}$. Suppose ψ

is a mapping defined on a domain D in E_d , the range of which lies in E_n . We denote by ψ_1, \dots, ψ_m the components of this mapping and write $\psi \in C^{k, \lambda}(D)$ if $\psi_i \in C^{k, \lambda}(D)$.

The composition of the function f and of the mapping ψ is denoted by $f * \psi$, the derivative of this composition is denoted by $D^\beta (f * \psi)$; the symbol $D^\beta f * \psi$ denotes the composition of the function $D^\beta f$ and of ψ .

If $x = (x_1, \dots, x_n) \in E_n$, then we put

$$\|x\| = \left(\sum_{i=1}^n x_i^2 \right)^{\frac{1}{2}}. \quad \text{By } D(x) \text{ we denote an open ball}$$

with the center in the point x . If $x^0 \in E_n$, then by $\overline{xx^0}$ we denote an open segment with the extreme points x, x^0 .

2. General remarks

Remark 2.1. Let $F_1, \dots, F_b \in C^{k, \lambda}(\Omega)$ be functions, $x^0 \in \Omega$. Suppose, for each $i = 1, \dots, b$, there exists j_i such that $\frac{\partial F_i}{\partial x_{j_i}}(x^0) \neq 0, F_i(x^0) = 0$. Denote

$N = \{x \in \Omega; F_i(x) = 0 \text{ for each } i = 1, \dots, b\}$. Then

there exists a number $d < n$, the balls $D(x^0) \subset \Omega$,

$D(x^0) \subset E_d$ and a mapping $\Phi \in C^{k, \lambda}(D(x^0))$ such

that $\Phi(y^0) = x^0$, $N \cap D(x^0) \subset \Phi(D(y^0)) \subset \Omega$ and such that either $d = 1$ or

$$(1) \quad \frac{\partial}{\partial y_j} (F_i * \Phi)(y^0) = 0 \quad \text{for each } i = 1, \dots, b ; \\ j = 1, \dots, d .$$

Proof. We can choose a submatrix I of the matrix

$$M = \left(\frac{\partial F_i}{\partial x_j}(x^0) \right)_{\substack{j = 1, \dots, m \\ i = 1, \dots, b}} \quad \text{with the following properties: } \det I \neq 0 \quad \text{and } \text{rank } I = \max(\text{rank } S),$$

where maximum is taken over all submatrices S of M such that $\text{rank } S < m$.

We can suppose

$$I = \left(\frac{\partial F_i}{\partial x_j}(x^0) \right)_{\substack{j = 1, \dots, \kappa \\ i = 1, \dots, \kappa}}, \quad \text{where } 0 < \kappa < m, \quad \kappa \leq b .$$

From the implicit function theorem it follows that there exist the balls $D(x^0) \subset \Omega$, $D(y^0) \subset E_d$, where $d = m - \kappa$ and the functions $\varphi_1, \dots, \varphi_\kappa \in C^{k, \lambda}(D(y^0))$ such that

$$(2) \quad F_i(\varphi_1(y), \dots, \varphi_\kappa(y), y_1, \dots, y_{m-\kappa}) = 0$$

for $i = 1, \dots, \kappa$, $y = (y_1, \dots, y_{m-\kappa}) \in D(y^0)$,

$$(3) \quad \text{if } x \in D(x^0), \quad x \in N, \quad \text{then } x_i = \varphi_i(x_{\kappa+1}, \dots, x_m)$$

for $i = 1, \dots, \kappa$.

Define $\Phi(y) = (\varphi_1(y), \dots, \varphi_\kappa(y), y_1, \dots, y_d)$ for

$\psi = (\psi_1, \dots, \psi_d) \in D(\psi^0)$. By (3) we have

$N \cap D(x^0) \subset \Phi(D(\psi^0))$. The condition (1) for $i = 1, \dots, \kappa$ follows from (2). If $d > 1$, then $\text{rank } M = \kappa$

and the vectors $\left(\frac{\partial F_i}{\partial x_1}(x^0), \dots, \frac{\partial F_i}{\partial x_n}(x^0) \right)$ for $i =$

$= \kappa + 1, \dots, b$ are linear combinations of

$$\left(\frac{\partial F_i}{\partial x_1}(x^0), \dots, \frac{\partial F_i}{\partial x_n}(x^0) \right) \text{ for } i = 1, \dots, \kappa.$$

From here the condition (1) follows for $i = \kappa + 1, \dots, b$ too.

Remark 2.2. Let $F \in C^l(\Omega)$ be a function, $x^0 \in \Omega$,

$D^\beta F(x^0) = 0$ for each $0 < |\beta| \leq l-1$. Suppose D is a

ball in E_d , $d \leq n$. Let $\psi \in C^{(1)}(D)$ be a mapping,

$\psi(D) \subset \Omega$, $x^0 \in D$, $\psi(x^0) = x^0$. Denote

$$C_1 = \max_{\substack{i=1, \dots, n \\ j=1, \dots, d}} \left(\sup_{x \in D} \left| \frac{\partial \psi_i}{\partial x_j}(x) \right| \right) < +\infty.$$

Then for each $x \in D$ there exists $x^1 \in \overline{x x^0}$ and $C > 0$ (C depends on C_1 and l only) such that

$$|F(\psi(x)) - F(\psi(x^0))| \leq C \cdot \sum_{|\beta|=l} |D^\beta F(\psi(x^1))| \cdot \|x - x^0\|^l.$$

Proof. There exists $x^1 \in \overline{x x^0}$ such that

$$\begin{aligned} |F(\psi(x)) - F(\psi(x^0))| &= \left| \sum_{j=1}^d \frac{\partial}{\partial x_j} (F * \psi)(x^1) \cdot (x_j - x_j^0) \right| = \\ &= \left| \sum_{j=1}^d \sum_{i=1}^n \frac{\partial F}{\partial x_i}(\psi(x^1)) \cdot \frac{\partial \psi_i}{\partial x_j}(x^1) \cdot (x_j - x_j^0) \right| \leq \end{aligned}$$

$$\leq C_1 \cdot \sum_{i=1}^n \left| \frac{\partial F}{\partial x_i} (\psi(x^1)) \right| \cdot \|x - x^0\|.$$

In a similar way we can estimate

$$\begin{aligned} \left| \frac{\partial F}{\partial x_i} (\psi(x^1)) \right| &= \left| \frac{\partial F}{\partial x_i} (\psi(x^1)) - \frac{\partial F}{\partial x_i} (\psi(x^0)) \right| \leq \\ &\leq C_1 \sum_{t=1}^n \left| \frac{\partial^2 F}{\partial x_i \partial x_t} (\psi(x^2)) \right| \cdot \|x^1 - x^0\| \end{aligned}$$

where $\|x^0 - x^1\| \leq \|x - x^0\|$. Further we can estimate $\frac{\partial^2 F}{\partial x_i \partial x_t} (\psi(x^2))$ etc. After a finite number of steps we obtain our assertion.

Remark 2.3. (Hausdorff measure.) Suppose A is a subset in E_n and ρ is a positive real number. For each $\varepsilon > 0$ define $\mu_{\rho, \varepsilon}(A) = \inf \sum_{i=1}^{\infty} (\text{diam } A_i)^\rho$, the infimum being taken over all countable coverings $\{A_i\}_{i=1}^{\infty}$ of A such that $\text{diam } A_i < \varepsilon$. The number $\mu_\rho(A) = \lim_{\varepsilon \rightarrow 0+} \mu_{\rho, \varepsilon}(A)$ is said to be ρ -Hausdorff measure of A . If $\mu_\rho(A) = 0$, then we say A is ρ -null.

It is easy to see: if A is ρ -null, then A is κ -null for each $\kappa > \rho$. If $\rho = n$, then we obtain Lebesgue measure.

3. Some estimates for functions of the class $C^{k, \lambda}(\Omega)$

Theorem 3.1. Let $f \in C^{k, \lambda}(\Omega)$ be a function. Then there exists a countable system of sets $\{M_t\}_{t=1}^{\infty}$ such that

(4) $Z \setminus \bigcup_{t=1}^{\infty} M_t$ is countable;

(5) for each positive integer t there exists $\epsilon_t > 0$ such that $|f(x_1) - f(x_2)| \leq C_t \|x_1 - x_2\|^{k_t + \alpha}$

for each $x_1, x_2 \in M_t$.

Remark 3.1. A similar assertion is proved in [2], but for $\lambda = 0$ only. A.P. Morse proves it by using induction for $n + k$. Theorem 3.1 can be proved in a similar way. But in this paper, a constructive proof is given. This proof is based on the fact that each set M_t lies in some hyperplane; this hyperplane is characterized by the mapping

$\Phi = \Phi_1 x \dots x \Phi_p$ (on some neighborhood of a point x^0) from Construction 3.1 and Lemma 3.1; the number d_p is the dimension of this hyperplane.

Construction 3.1. Suppose $x^0 \in Z$ is a fixed point. We shall associate a finite number of mappings Φ_1, \dots, Φ_p to this point.

Let k_1 be the smallest entire number such that $D^\beta f(x_0) = 0$ for all $|\beta| \leq k - k_1$. If $k_1 = 0$, then we need not any mapping, that means our hyperplane (see Remark 3.1) has dimension n .

Assume $k_1 > 0$. Then $\frac{\partial}{\partial x_j} D^\beta f(x^0) \neq 0$ for some $j, \beta, 1 \leq j \leq n, |\beta| = k - k_1$. Denote $Z_{k_1} = \{x \in Z; D^\beta f(x) = 0 \text{ for all } |\beta| \leq k - k_1\}$. From the implicit function theorem it follows that there exist the balls

$D(x^0) \subset \Omega$, $D(y^0) \subset E_{d_1}$, ($d_1 < n$) and a mapping $\Phi_1 \in C^{k_1, 2}(D(y^0))$ such that

$$(6) \quad Z_{k_1} \cap D(x^0) \subset \Phi_1(D(y^0)) \subset \Omega, \quad \Phi_1(y^0) = x^0$$

and such that either $d_1 = 1$ or

$$(7) \quad \frac{\partial}{\partial y_j} (D^\beta f * \Phi_1)(y^0) = 0$$

for each $|\beta| = k - k_1$, $j = 1, \dots, d_1$

(see Remark 2.1; we set $F_i = D^{\alpha^i} f$, where α^i , $i = 1, \dots, b$ are all nullindexes such that $|\alpha^i| = k - k_1$,

$$\frac{\partial}{\partial x_j} D^{\alpha^i} f(x^0) \neq 0 \text{ for some } j \text{). Define } D_1 = D(y^0).$$

If $d_1 = 1$, then we set $r = 1$ and we conclude our construction.

Suppose $d_1 > 1$. Let k_2 be the smallest number such that $k_2 < k_1$ and

$$(8) \quad D^\beta (D^{\beta'} f * \Phi_1)(y) = 0$$

for each $|\beta'| = k - k_1$, $|\beta| \leq k_1 - k_2$

for $y = y^0$ (β denotes d_1 -dimensional multiindex in (8)).

If $k_2 = 0$, then we set $r = 1$ and we conclude our construction.

Suppose $k_2 > 0$ and denote

$$Z_{k_1, k_2} = \{x \in Z_{k_1}; x = \Phi_1(y), \text{ (8) is valid}\}.$$

We have $\frac{\partial}{\partial y_j} D^\beta (D^{\beta^1} f * \Phi_1)(y^0) \neq 0$ for some $\beta^1, \beta, j, |\beta^1| = k - k_1, |\beta| = k_1 - k_2, 1 \leq j \leq d_1$.

We can, by using implicit function theorem (analogously as in the case of Φ_1 - see Remark 2.1) construct the balls $D(x^0) \subset \Omega, D_2 \subset E_{d_2}, (d_2 < d_1)$ and a mapping $\Phi_2 \in C^{k_2, \lambda}(D_2)$ such that

$$(6') \quad Z_{k_1, k_2} \cap D(x^0) \subset \Phi_1 * \Phi_2(D_2) \subset \Omega, \Phi_2(v^0) = y^0$$

and such that either $d_2 = 1$ or

$$(7') \quad \frac{\partial}{\partial v_j} (D^\beta (D^{\beta^1} f * \Phi_1) * \Phi_2)(v^0) = 0$$

for each $|\beta^1| = k - k_1, |\beta| = k_1 - k_2, j = 1, \dots, d_2$.

If $d_2 = 1$, then we set $r = 2$ and conclude our construction. Suppose $d_2 > 1$. Analogously as k_2 , we can take the smallest entire number k_3 such that $k_3 < k_2$ and

$$(8') \quad D^\beta (D^{\beta^2} (D^{\beta^1} f * \Phi_1) * \Phi_2)(v) = 0$$

for each $|\beta^1| = k - k_1,$

$|\beta^2| = k_1 - k_2,$

$|\beta| \leq k_2 - k_3,$

and for $v = v^0$ (β^1, β^2, β is m -dimensional, d_1 -dimensional,

d_2 -dimensional multiindex, respectively). If $k_3 = 0$, then we set $r = 2$. Assume $k_3 > 0$. Then we can (analogously as $Z_{k_1, f, z}(\Phi_1, \Phi_2)$) construct the sets Z_{k_1, k_2, k_3} , $Z_{k_1, k_2, k_3, k_4} \dots$ and mappings Φ_3, Φ_4, \dots , respectively. It is easy to see that after a finite number of steps we obtain the following assertion:

Lemma 3.1. To each point $x^0 \in Z$, a finite number of mappings Φ_1, \dots, Φ_r and a ball $D(x^0)$ can be associated such that (we use the notation from Construction 3.1)

$$(9) \quad \Phi_\ell \in C^{k_\ell^2}(D_\ell), D_\ell \text{ is a ball in } E_{d_\ell}, \ell = 1, \dots, r,$$

where $k_r < k_{r-1} < \dots < k_1 \leq k$; $d_r < d_{r-1} < \dots < d_1 < n$;

$$(10) \quad \Phi_\ell(D_\ell) \subset D_{\ell-1}, Z_{k_1, \dots, k_\ell} \cap D(x^0) \subset \Phi_1 * \dots * \Phi_\ell(D_\ell) \subset \Omega, \\ \ell = 1, \dots, r;$$

$$(11) \quad D^\beta(D^{\beta^2}(\dots(D^{\beta^2}(D^{\beta^1}f * \Phi_1) * \Phi_2)\dots) * \Phi_\ell)(v) = 0$$

for $v = v^0, (\Phi_1 * \dots * \Phi_\ell(v^0) = x^0)$,

$$|\beta^1| = k - k_1, |\beta^2| = k_1 - k_2, \dots, |\beta^\ell| = k_{\ell-1} - k_\ell,$$

$$|\beta| \leq k_\ell - k_{\ell+1} \text{ and for } \ell = 1, \dots, r-1;$$

if $d_r > 1$, then this holds for $\ell = r, k_{r+1} = 0$, too.

Let us define $\Phi(v) = \Phi_1 * \dots * \Phi_r(v)$ for $v \in D_r$.

Lemma 3.2. There exists a finite number of sets

Z^1, \dots, Z^q such that $\bigcup_{j=1}^q Z^j = Z$ and each set Z^j contains all points $x \in Z$ of the same type in the following sense:

if $x^1, x^2 \in Z^j$ and if $\Phi_1^1, \dots, \Phi_{r_1}^1$; $\Phi_1^2, \dots, \Phi_{r_2}^2$; respectively, are the corresponding mappings associated to the points x_1, x_2 , respectively, by Lemma 3.1, then $r_1 = r_2$, $\kappa_i^1 = \kappa_i^2$ and the implicit function theorem is used for the same combination of variables in each step of Construction 3.1 (i.e. the domains of Φ_i^1, Φ_i^2 lie in the same subspace of E_m , $i = 1, \dots, r_1 = r_2$).

Proof. The assertion follows from Construction 3.1 and Lemma 3.1.

Remark 3.2. Assume $x^1, x^2 \in Z^j$ (j fixed). Let Φ_i^1, Φ_i^2 , $i = 1, \dots, r$ be the corresponding mappings (see Lemma 3.1, 3.2) with the domains D_i^1, D_i^2 . Then $\Phi_i^1 = \Phi_i^2$ on $D_i^1 \cap D_i^2$. It follows from the construction of these mappings, from the fact that $x^1, x^2 \in Z^j$ for the same j and from the unicity of the implicit function.

Remark 3.3. Assume $x^0 \in Z^j$. Then the condition (11) is fulfilled for each $v \in D_\ell$ such that $\Phi_1 * \dots * \Phi_\ell(v) \in Z^j$. This follows from Remark 3.2 and from the validity (11) for mappings associated to the point $x = \Phi_1 * \dots * \Phi_\ell(v)$.

Remark 3.4. Suppose $x^0 \in Z^j$. Then $D(x^0) \cap Z^j \subset \Phi(D_r)$. This follows from (10), because

$D(x^0) \cap Z^{\dot{z}} \subset Z_{k_1, \dots, k_p}$ for some set Z_{k_1, \dots, k_p}
 (see Construction 3.1 and Remark 3.2).

Proof of Theorem 3.1. An open ball $D(x^0)$ from Lemma 3.1 corresponds to each point $x^0 \in Z^{\dot{z}}$. These balls cover $Z^{\dot{z}}$ and therefore we can select a countable covering $\{D(x^t)\}_{t=1}^{\infty}$ of the set $Z^{\dot{z}}$. We have a finite number of sets $Z^{\dot{z}}$. Hence, it is sufficient to prove: if $x^0 \in Z^{\dot{z}}$ is a fixed point, then there exists a set $M \subset D(x^0) \cap Z^{\dot{z}}$ such that $|f(x^1) - f(x^2)| \leq C \|x^1 - x^2\|^{k_1 + \alpha}$ for each $x^1, x^2 \in M$ and the set $Z^{\dot{z}} \cap D(x^0) \setminus M$ is countable.

Let $x^0 \in Z^{\dot{z}}$ be fixed. We shall use the notation from Construction 3.1 and Lemma 3.1. Denote $A = \{v \in D_p; \Phi(v) \in D(x^0) \cap Z^{\dot{z}}\}$, $M = \Phi(A' \cap A)$, where A' is the set of all limit points of A . By Remark 3.4, we have $D(x^0) \cap Z^{\dot{z}} \subset \Phi(A)$, the set $A \setminus A'$ countable, therefore $D(x^0) \cap Z^{\dot{z}} \setminus M$ is countable. Suppose $x^1, x \in M$, $v^1, v \in A'$, $\Phi(v^1) = x^1$, $\Phi(v) = x$. We have $D^\beta f(x) = 0$ for $|\beta| \leq k - k_1$ (see Construction 3.1 and Lemma 3.2 - we have $x, x^0 \in Z^{\dot{z}}$ for the same \dot{z}). By Remark 2.2 (we put $F = f$, $\psi = \Phi$)

$$\begin{aligned}
 |f(x^1) - f(x)| &\leq C \sum_{|\beta| = k - k_1} |D^\beta f(\Phi(v^2))| \cdot \|v^1 - v\|^{k - k_1} = \\
 (12) \quad &= C \sum_{|\beta| = k - k_1} |(D^\beta f * \Phi_1)(\Phi_2 * \dots * \Phi_p(v^2))| \cdot \|v^1 - v\|^{k - k_1},
 \end{aligned}$$

where $v^2 \in \overline{v^1 v}$, Lemma 3.1 and Remark 3.3 imply

$$\mathcal{D}^\beta (\mathcal{D}^{\beta^1} f * \Phi_1) (\Phi_2 * \dots * \Phi_n (v)) = 0 \text{ for } |\beta^1| = k - k_1, \\ |\beta| \leq k_1 - k_2.$$

From Remark 2.2 we obtain (we put $F = \mathcal{D}^\beta f * \Phi$,

$$\psi = \Phi_2 * \dots * \Phi_n)$$

$$(13) \quad |(\mathcal{D}^{\beta^1} f * \Phi_1) (\Phi_2 * \dots * \Phi_n (v^2))| \leq \\ \leq C \sum_{|\beta^2| = k_1 - k_2} |(\mathcal{D}^{\beta^2} (\mathcal{D}^{\beta^1} f * \Phi_1)) (\Phi_2 * \dots * \Phi_n (v^3))| \cdot \|v^2 - v\|^{k_1 - k_2},$$

$\|v^2 - v\| \leq \|v^1 - v\|$. Analogously, we can proceed: we shall

estimate $\mathcal{D}^{\beta^2} (\mathcal{D}^{\beta^1} f * \Phi_1) * \Phi_2, \mathcal{D}^{\beta^3} (\mathcal{D}^{\beta^2} (\mathcal{D}^{\beta^1} f * \Phi_1) * \Phi_2) * \Phi_3$

etc. After $n - 1$ steps we obtain altogether (from the estimates (12), (13) etc.)

$$(14) \quad |f(x^1) - f(x)| \leq C \sum_{\beta^1, \dots, \beta^n} |\mathcal{D}^{\beta^n} (\dots (\mathcal{D}^{\beta^2} (\mathcal{D}^{\beta^1} f * \Phi_1) * \Phi_2) \dots \\ \dots) * \Phi_n (v^{n+1})|. \|v^1 - v\|^{k - k_n},$$

the sum is taken over all multiindexes $|\beta^1| = k - k_1, \dots$

$$\dots, |\beta^{n-1}| = k_{n-1} - k_n.$$

If $d_n > 1$, then from Lemma 3.1 and Remark 3.3 it follows

$$\mathcal{D}^\beta (\mathcal{D}^{\beta^n} (\dots (\mathcal{D}^{\beta^2} (\mathcal{D}^{\beta^1} f * \Phi_1) * \Phi_2) * \dots) * \Phi_n) (v) = 0,$$

$$|\beta^1| = k - k_1, \dots, |\beta^n| = k_{n-1} - k_n, |\beta| \leq k_n.$$

Hence, we obtain by using (14) and the mean value theorem

$$\begin{aligned}
& |f(x^1) - f(x)| \leq \\
& \leq C \sum_{\beta^1, \dots, \beta^{n+1}} |D^{\beta^{n+1}} (D^{\beta^n} (\dots (D^{\beta^2} (D^{\beta^1} f * \Phi_1) * \Phi_2) * \dots \\
& \dots) * \Phi_n) (v^{n+2})| \cdot \|v^1 - v\|^{k_n} \leq \\
& \leq C \cdot \|v^{n+2} - v\|^2 \cdot \|v^1 - v\|^{k_n} \leq C \|v^1 - v\|^{k_n+2}
\end{aligned}$$

(the sum being taken over all multiindexes $|\beta^1| = k_n - k_{n-1}, \dots$
 $\dots, |\beta^n| = k_{n-1} - k_n, |\beta^{n+1}| = k_n$), because the functions in
the middle member are λ -Hölderian.

Suppose $d_n = 1$. The functions which are in the right hand side in (14), are the functions of one variable and they are equal to zero on each point from A (see Remark 3.3). But we have $v \in A'$ and from here we see that the derivatives of all orders not exceeding k_n of these functions on v are equal to zero. Hence, we can conclude the proof analogously as in the case $d_n > 1$.

4. Hausdorff measure of the set of critical values

Theorem 4.1. Let f be a function, $f \in C^1(\Omega)$,

$n \geq 1$. Let A be a compact subset of Z and

$$(15) |f(x') - f(x)| \leq C \cdot \|x' - x\|^n$$

for each $x', x \in A$, where $C > 0$. Then $f(A)$ is $\frac{n}{\kappa}$ -null.

Proof. For each positive integer N we shall denote

by $\{I_N^j\}_{j=1}^{k_N}$ a system of all intervals of the type

$\langle k_1 N^{-1}, (k_1 + 1)N^{-1} \rangle \times \dots \times \langle k_n N^{-1}, (k_n + 1)N^{-1} \rangle$

(m -dimensional cubes) which intersect the set A (k_i are entire numbers). Set $J_N^{\sharp} = I_N^{\sharp} \cap A$. We have

$\bigcup_{\sharp} J_N^{\sharp} = A$, therefore $\bigcup_{\sharp} f(J_N^{\sharp}) = f(A)$. From (15) we obtain $\text{diam } f(J_N^{\sharp}) \leq C \cdot N^{-n}$. By the definition of Hausdorff measure we have

$$(16) \quad \mu_{\frac{m}{2}}(f(A)) \leq \lim_{N \rightarrow \infty} \sum_{j=1}^{n_N} [\text{diam } f(J_N^{\sharp})]^{\frac{m}{2}}.$$

Let $\varepsilon > 0$ be arbitrary (but fixed). Let us divide the sets J_N^{\sharp} for each fixed N into two groups:

$$(i) \quad \text{diam } f(J_N^{\sharp}) \leq \varepsilon N^{-n};$$

$$(ii) \quad \text{diam } f(J_N^{\sharp}) > \varepsilon N^{-n}.$$

By $\nu_N^{(1)}$, $\nu_N^{(2)}$ respectively, denote the number of sets which lie in the group (i), (ii). Put $\nu_N = \nu_N^{(1)} + \nu_N^{(2)}$.

Let us suppose that we have proved the following assertion:

$$(17) \quad \nu_N = O(N^m), \quad \nu_N^{(2)} = o(N^m).$$

Then

$$\begin{aligned} \sum_{j=1}^{n_N} [\text{diam } f(J_N^{\sharp})]^{\frac{m}{2}} &= \sum_{J_N^{\sharp} \in (i)} [\text{diam } f(J_N^{\sharp})]^{\frac{m}{2}} + \\ &+ \sum_{J_N^{\sharp} \in (ii)} [\text{diam } f(J_N^{\sharp})]^{\frac{m}{2}} \leq \nu_N^{(1)} (\varepsilon N^{-n})^{\frac{m}{2}} + \nu_N^{(2)} (C_1 N^{-n})^{\frac{m}{2}} \leq \\ &\leq \varepsilon^{\frac{m}{2}} \nu_N^{(1)} N^{-n} + C_2 \nu_N^{(2)} N^{-n}. \end{aligned}$$

The second member in the right hand side converges to zero (if $N \rightarrow \infty$) by (17) and the first member can be made arbitrarily small by a convenient choice of ε . From here and from (16) we obtain $f(A)$ is $\frac{\eta}{\kappa}$ -null.

Hence, it is sufficient to prove (17).

Suppose

(18) there exists $\sigma > 0$ (dependent of ε only, independent of N, j) such that $m_n(J_N^j) \leq (1 - \sigma) N^{-n}$

for each $J_N^j \in (ii)$ (where m_n denotes the n -dimensional Lebesgue measure).

Set $A_N = \nu_N N^{-n} - m_n(A)$. We have $A_N \rightarrow 0$, because A is compact. From here $\nu_N = O(N^{-n})$. We have

$$m_n(A) \leq \nu_N^{(1)} N^{-n} + (1 - \sigma) \nu_N^{(2)} N^{-n},$$

hence

$$\nu_N^{(1)} + \nu_N^{(2)} = m_n(A) N^n + \sigma(N^n) \leq \nu_N^{(1)} + (1 - \sigma) \nu_N^{(2)} + \sigma(N^n).$$

From here $\sigma \nu_N^{(2)} = \sigma(N^n)$, i.e. $\nu_N^{(2)} = \sigma(N^n)$, hence

(17) is valid. Hence, it is sufficient to prove (18).

Let J_N^j be an arbitrary set of the group (ii). There exist $a, b \in J_N^j$ such that $\text{diam } f(J_N^j) = f(b) - f(a) > \varepsilon N^{-n}$.

From (15) we obtain

$$(19) \quad |f(b') - f(a')| \geq \frac{1}{2} \varepsilon N^{-n}$$

for each

$$(20) \ a', b' \in I_N^j, \|a' - a\| < \left(\frac{\varepsilon}{4C}\right)^{\frac{1}{k}} N^{-1}, \|b' - b\| < \left(\frac{\varepsilon}{4C}\right)^{\frac{1}{k}} N^{-1}.$$

Consider two points a', b' which fulfil (20) and

$\overline{a'b'} \cap A \neq \emptyset$. Then there exist the open segments

$$S_i, \quad i = 1, 2, \dots \quad \text{such that } \overline{a'b'} \setminus A = \bigcup_{i=1}^{\infty} S_i.$$

Denote the extreme points of these segments by a^i, b^i .

We obtain

$$\begin{aligned} |f(b') - f(a')| &\leq \sum_{i=1}^{\infty} |f(b^i) - f(a^i)| \leq \\ &\leq C \cdot \sum_{i=1}^{\infty} (\text{diam } S_i)^k \leq C \cdot \left(\sum_{i=1}^{\infty} \text{diam } S_i\right)^k = \\ &= C \cdot [m_1(\overline{a'b'} \setminus A)]^k. \end{aligned}$$

If $m_1(\overline{a'b'} \setminus A) < \left(\frac{\varepsilon}{2C}\right)^{\frac{1}{k}} N^{-1}$, then we obtain

$$|f(b') - f(a')| < \frac{1}{2} \varepsilon N^{-k}. \quad \text{But it is not possible by}$$

(19), (20), hence

$$(21) \text{ if } \|a' - a\| \geq \frac{1}{4} \left(\frac{\varepsilon}{C}\right)^{\frac{1}{k}} N^{-1}, \|b' - b\| \geq \frac{1}{4} \left(\frac{\varepsilon}{C}\right)^{\frac{1}{k}} N^{-1},$$

$$\overline{a'b'} \setminus A \neq \emptyset, \text{ then } m_1(\overline{a'b'} \setminus A) \geq \frac{1}{2} \left(\frac{\varepsilon}{C}\right)^{\frac{1}{k}} N^{-1}.$$

If $\overline{a'b'} \cap A = \emptyset$, then the last inequality holds, too.

It is easy to see there exists $C_4 > 0$ (dependent of the dimension n only, independent of j, N) such that there

exist $a^0, b^0 \in I_N^j$ which fulfil the conditions

$$D(a^0, C_4 \varepsilon^{\frac{1}{k}} N^{-1}) \subset D(a, \frac{1}{4} \left(\frac{\varepsilon}{C}\right)^{\frac{1}{k}} N^{-1}) \cap I_N^j;$$

$$D(\mathcal{B}^0, C_4 \varepsilon^{\frac{1}{2}} N^{-1}) \subset D(\mathcal{A}, \frac{1}{4} (\frac{\varepsilon}{C})^{\frac{1}{2}} N^{-1}) \cap I_N^{\frac{1}{2}}.$$

Let K be a convex closure of the set

$$D(\mathcal{A}^0, C_4 \varepsilon^{\frac{1}{2}} N^{-1}) \cup D(\mathcal{B}^0, C_4 \varepsilon^{\frac{1}{2}} N^{-1}).$$

By using (21) we obtain

$$m_m(K \setminus A) \geq P \frac{1}{2} \left(\frac{\varepsilon}{C}\right)^{\frac{1}{2}} N^{-1},$$

where P is the volume of $(m-1)$ -dimensional ball with

$$\text{diam } P = 2 \cdot C_4 \varepsilon^{\frac{1}{2}} N^{-1}. \text{ It is easy to see from here}$$

$$m_m(K \setminus A) \geq C_5 \varepsilon^{\frac{m}{2}} N^{-m},$$

where C_5 depends on C and m only. Further,

$$m_m(I_N^{\frac{1}{2}} \setminus A) \geq m_m(K \setminus A).$$

It is sufficient to put $\sigma = C_5 \varepsilon^{\frac{m}{2}}$ and the assertion (18) is proved. This completes the proof of Theorem 4.1.

Theorem 4.2. If $f \in C^{k, \lambda}(\Omega)$ is a function, then the set $f(Z)$ is $\frac{n}{k+\lambda}$ -null.

Proof. It is easy to see that we can suppose that the sets M_k from Theorem 3.1 are compact. Our assertion follows from here and from Theorem 4.1.

Remark 4.1. If $\lambda < \frac{n}{k+\lambda}$, then there exists a function from the class $C^{k, \lambda}$ such that $\mu_\lambda(f(Z)) > 0$ (see [1]).

Remark 4.2. If $f \in C^\infty$ (i.e. f has continuous derivatives of all orders), then the set $f(Z)$ is λ -null

for each $\epsilon > 0$. This follows from Theorem 4.2. But the set $f(Z)$ need not be countable. We must demand f is real-analytic to obtain such a strong assertion (see [5]).

R e f e r e n c e s

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