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REMARK ON LOCAL EXISTENCE OF $\{e\}$ -STRUCTURE WITH PRESCRIBED STRUCTURAL FUNCTIONS ON A MANIFOLD OF DIMENSION TWO

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This paper is partially connected with my previous paper [1] the definitions and notations of which are freely used here. Manifolds, mappings, functions etc. are always differentiable of class C^∞ .

Let M be a differentiable manifold of dimension 2 , $c^1, c^2, d_1^1, d_1^2, d_2^1, d_2^2$ differentiable functions on M . The results of [1] lead to the following question:

Does there exist an $\{e\}$ -structure on M for which c^1, c^2 are structural functions of first order and d_1^1 , d_1^2 , d_2^1 , d_2^2 structural functions of second order ?

Or, equivalently:

Does there exist vector fields v_1 , v_2 on M such that $v = \{v_1, v_2\}$ is a full parallelism on M and such that:

$$
(1)\qquad
\begin{aligned}
[v_1, v_2] &= c^1 v_1 + c^2 v_2 \ , \\
[v_1, [v_1, v_2]] &= d_1^1 v_1 + d_1^2 v_2 \ , \\
[v_2, [v_1, v_2]] &= d_2^1 v_1 + d_2^2 v_2 \ .
\end{aligned}
$$

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It is easy to see that such an $\{e\}$-structure need not exist for any choice of c^i, $i = 1, 2$; d^i_j, $i, j = 1, 2$. (e.g. if c^1, c^2 are constant, then necessarily $d^1_1 = d^1_2 = d^2_1 = d^2_2 = 0$).

We solve the problem locally in a neighborhood of the point 0 in so called general case - i.e. we assume that the differentials $dc^1(0)$, $dc^2(0)$ generate the cotangent space at 0. Then the functions c^1, c^2 can be taken as a coordinate system (x_1, x_2) in a neighborhood $\mathcal{U}$ of $0 \in \mathbb{R}^2$ and we can reformulate the equations (1) into the form:

$$
\begin{aligned}
(2) \qquad & [v_1, v_2] = x^1 v_1 + x^2 v_2 , \\
& [v_1, [v_1, v_2]] = d^1_1 v_1 + d^2_1 v_2 , \\
& [v_2, [v_1, v_2]] = d^1_2 v_1 + d^2_2 v_2 ,
\end{aligned}
$$

where d^i_j, $i = 1, 2$ are functions of the variables x^1, x^2 on $\mathcal{U}$.

<u>Theorem</u>. There exists an $\{e\}$-structure with the structural functions (1) $d^1_1, d^1_2, d^2_1, d^2_2$ if and only if the following two conditions are satisfied:

(i) d^i_j, $i, j = 1, 2$ satisfy the differential equations:

$$
d^1_1 \frac{\partial d^1_2}{\partial x^1} + d^2_1 \frac{\partial d^1_2}{\partial x^2} - d^1_2 \frac{\partial d^1_1}{\partial x^1} - d^2_2 \frac{\partial d^1_1}{\partial x^2} - x^1 x^2 \frac{\partial d^1_2}{\partial x^1} -
$$

$$
- (x^2)^2 \frac{\partial d^1_2}{\partial x^1} - (x^1)^2 \frac{\partial d^1_1}{\partial x^1} - x^1 x^2 \frac{\partial d^1_1}{\partial x^2} + x^1 (d^1_1 + d^2_2) = 0 ,
$$

$$d_2^1 \frac{\partial d_1^2}{\partial x^1} + d_2^2 \frac{\partial d_1^2}{\partial x^2} - d_1^1 \frac{\partial d_2^2}{\partial x^1} - d_1^2 \frac{\partial d_2^2}{\partial x^2} + (x^1)^2 \frac{\partial d_1^2}{\partial x^1} +$$
$$+ x^1 x^2 \frac{\partial d_2^2}{\partial x^1} + x^1 x^2 \frac{\partial d_1^2}{\partial x^2} +$$
$$+ (x^2)^2 \frac{\partial d_2^2}{\partial x^2} - x^2 (d_1^1 + d_2^2) = 0 ,$$

(ii) $\det((d_j^i)) + x^1x^2(d_1^1 - d_2^2) - (x^1)^2 d_1^2 + (x^2)^2 d_2^2 \neq 0$ on $\mathcal{U}$.

<u>Proof</u>. We shall try to find functions a_j^i, $i, j = 1, 2$ of the variables x^1, x^2 so that the vector fields $v_i = \sum_{j=1}^{2} a_i^j \frac{\partial}{\partial x^j}$, $i = 1, 2$ satisfy (2). Substituting in (2) we get immediately the conditions (i) and (ii).

If (i) and (ii) are satisfied, then the vector fields v_1 , v_2 can be found in the form:

$$v_1 = (d_1^1 - x^1 x^2) \frac{\partial}{\partial x^1} + (d_1^2 - (x^2)^2 \frac{\partial}{\partial x^2} \quad , \tag{3}$$
$$v_2 = (d_2^1 + (x^1)^2) \frac{\partial}{\partial x^1} + (d_2^2 + x^1 x^2) \frac{\partial}{\partial x^2} \quad .$$

<u>Corollary</u>. The necessary and sufficient conditions for the existence of an $\{e\}$-structure on $\mathcal{U}$ with constant structural coefficients d_i^j in (2) are

(iii) $d_1^1 = - d_2^2$,

(iv) $\mathcal{U}$ does not intersect the curve :

$$(x^1)^2 d_1^2 - (x^2)^2 d_2^1 - 2 x^1 x^2 d_1^1 + (d_1^1)^2 + d_1^2 d_2^1 = 0 .$$

R e f e r e n c e

[1] Jarolím BUREŠ: Deformation and equivalence G-structures I., to appear in Czech.Math.J.

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