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Olga Štěpánková

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CONSTRUCTION BY TRANSFINITE INDUCTION

Olga ŠTĚPÁNKOVÁ, Praha

In an attempt to formulate the minimal set theory sufficient to deduce some elementary properties of real numbers we became interested in so called continual theory of sets (CTS), proposed for this purpose by Fraenkel [1] . By CTS we understand a set theory without the power set axiom. An often encountered problem in CTS is an iteration for a Gödelian term without set image property x) (for example when constructing Ker). The iteration called our attention to two properties of Gödelian terms, which we shall name locality and invertibility. The Gödelian term $\mathcal{T}(X)$ is <u>local</u> iff it is provable in CTS, that

$$\forall X(\mathcal{T}(X) = \bigcup_{y \in \mathcal{P}(X)} \mathcal{T}(y)) \quad .$$

The term $\mathcal{T}(X)$ is <u>invertible</u> with inverse term $\mathcal{S}(y)$ iff the next formulae are provable in CTS:

1) $y \subseteq \mathcal{T}(x) \rightarrow \mathcal{S}(y) \subseteq X$,

2) $y \subseteq \mathcal{T}(\mathcal{S}(y))$ & $\mathcal{M}(\mathcal{S}(y))$.

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x) Throughout the text the notation of [2] is used.

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The term of power $\mathcal{P}$ with the sum $\bigcup$ as its inverse term can serve as an example of a local and invertible Gödelian term.

This paper presents the following generalization of the meta-theorems on iteration (see [2, Theorem 2145]):

Metatheorem 1. Let $\mathcal{T}$ be local term, then

$$CTS + D1 \vdash \forall X \exists T [\mathcal{D}(T) = \mathcal{O}n \,\&\, T''\{0\} = X \,\&\, (\forall \alpha \in \mathcal{O}n)$$

$$(T''\{\alpha + 1\} = \mathcal{T}(T''\{\alpha\}) \,\&$$

$$\& \, (\alpha \in \mathcal{O}n_{II} \rightarrow T''\{\alpha\} = T''\alpha))] \, ,$$

where $D1$ is the axiom

$$\forall R [Rel(R) \,\&\, \mathcal{M}(\mathcal{D}(R)) \rightarrow \exists r (\mathcal{M}(r) \,\&\, r \in R \,\&\, \mathcal{D}(r) =$$
$$= \mathcal{D}(R))] \, .$$

Metatheorem 2. If $\mathcal{T}$ is a local and invertible term, then the claim of Metatheorem 1 is provable in CTS (without $D1$). The following property of CTS extended by S -axiom $\forall x (\mathcal{M}(x) \rightarrow x \approx \omega_0)$ and $D3$ -axiom of $V = Ker$ can be proved by the application of the above metatheorem.

Theorem 3. $CTS + S + D3 \vdash E2 \equiv \exists F (\mathcal{F}ce_2(F) \,\&$
$\& \, \mathcal{D}(F) = \mathcal{P}(\omega_0) \,\&\, \mathcal{W}(F) \subseteq \mathcal{O}n)$ where $E2$ is the class axiom of choice and $\mathcal{F}ce_2$ stands for one-one function.

We should like to stress that similar metatheorems in the theory of semisets (TSS) could be demonstrated by simple modification of the above statements. More precisely, the condition of semisets-image property of Gödelian term is not necessary for iteration even in TSS (compare [2, Theorem 4228]).

§ 1. <u>Demonstration of the metatheorem 1</u>

The principle of the proof is based on slight modification of the standard method. When handling the Gödelian **term** $\mathcal{T}(x)$ with class image property it is necessary to cover the iteration process by set relations (not by functions) which have on the ordinals $\alpha + 1$ the property $r''\{\alpha + 1\} \subseteq \mathcal{T}(r\ \{\alpha\})$.

For the more detailed proof let us define classes R, T

$$r \in \mathrm{R} \equiv \mathrm{Rel}(r) \,\&\, \mathfrak{M}(r) \,\&\, \mathfrak{D}(r) \subseteq On \,\&\, r''\{0\} \subseteq X\,(\forall \alpha \in \dot{\mathfrak{D}}(r))$$

$$[(\alpha \in On_{\mathrm{I}} \rightarrow r''\{\alpha\} \subseteq \mathcal{T}(r''\{\alpha - 1\})) \,\&\, (\alpha \in On_{\mathrm{II}} \rightarrow r''\{\alpha\} - r''\alpha)] \ ,$$

$\mathrm{T} = \cup \mathrm{R}$, i.e. $y \in \mathrm{T} \equiv \exists r\, (r \in \mathrm{R} \,\&\, y \in r)$.

The assumption that T has not the required properties leads to contradiction. Let α be the minimal ordinal number for which T does not satisfy the conditions of Metatheorem 1. The ordinal α can acquire the following 3 types of values:

(i) $\alpha = 0$ which contradicts the definition of T ,

(ii) $\alpha = \beta + 1$. We shall show that $\mathrm{T}''\{\beta + 1\} = \mathcal{T}(\mathrm{T}''\{\beta\})$.

The inclusion $\mathrm{T}''\{\beta + 1\} \subseteq \mathcal{T}(\mathrm{T}''\{\beta\})$ is the consequence of locality of $\mathcal{T}$ and the definition of T .

Conversely, let $x \in \mathcal{T}(\mathrm{T}''\{\beta\})$, then there exists such $y \subseteq \mathrm{T}''\{\beta\}$ that $x \in \mathcal{T}(y)$. Let us seek a relation r which belongs to the system R and satisfies the condition $y \subseteq r''\{\beta\}$. If such a relation exists we can define the relation $s = r \upharpoonright (\beta + 1) \cup \langle x, \beta + 1 \rangle$. According to the definition of s $s \in \mathrm{R} \,\&\, x \in s''\{\beta + 1\}$, then $x \in \mathrm{T}''\{\beta + 1\}$. Relation r can be found thanks to

Axiom $D1$. Consider $Q = \{\langle p, v\rangle : v \in y \ \& \ p \in R \ \& \ v \in p''\{\beta\}\}$. According to $D1$, q can be chosen in such a way that $q \subseteq Q \ \& \ \mathcal{M}(q) \ \& \ \mathcal{D}(q) = \mathcal{D}(Q) = y$.

If we define $r = \bigcup \mathcal{W}(q)$ then $r \in R$ and $y \subseteq r\{\beta\}$. This case leads to contradiction.

(iii) $\alpha \in \mathit{On}_{II}$. The inclusion $T''\{\alpha\} \subseteq T''\alpha$ is obvious. Let $x \in T''\alpha$, i.e. $(\exists \gamma \in \alpha)(x \in T''\gamma)$. Choose such a relation $r_0 \in R$ that $x \in r_0''\{\gamma\}$ and define r as follows:

$$\langle u, \delta\rangle \in r \equiv$$

$$\equiv (\gamma \geq \delta \ \& \ \langle u, \delta\rangle \in r_0) \vee$$

$$\vee (\gamma < \delta \leq \alpha \ \& \ \delta \in \mathit{On}_{II} \ \& \ u \in r''(\gamma + 1)) .$$

It is apparent that $r \in R$, $x \in r''\{\alpha\}$ and then $x \in T''\{\alpha\}$. This is a contradiction.

§ 2. Demonstration of Metatheorem 2

The analysis of the above proof shows that $D1$ is needed only in the search for relation $r \in R$ which for some fixed $\beta \in \mathit{On}$, $y \subseteq T''\{\beta\}$ has the property $y \subseteq r''\{\beta\}$ (see § 1 Case ii). In this paragraph we shall describe the construction which allows to avoid the application of $D1$ in case ii) of the demonstration of Metatheorem 1 for an invertible Gödelian term.

Metadefinition and metalemma. Let $\mathcal{S}$ be the Gödelian term with set-image property inverse to the local Gödelian term $\mathcal{T}$, R, x constants such that

$$CTS \vdash Rel(R) \ \& \ \mathcal{D}(R) = \omega_0 \ \& \ R''\{0\} =$$

$$= x \ \& \ (\forall n \in \omega_0)(R''\{n+1\} = \mathcal{S}(R''\{n\})) .$$

Define in CTS $\mathcal{S}$-universe of a set x by equation $\mathcal{S}\text{-}univ(x) = \mathrm{R}''\omega_0$. The following claims can be demonstrated:

1) $\mathcal{M}(\mathcal{S}\text{-}univ(x))$.

2) $\mathcal{S}(\mathcal{S}\text{-}univ(x)) \subseteq \mathcal{S}\text{-}univ(x)$.

3) $x \subseteq y \rightarrow \mathcal{S}(x) \subseteq \mathcal{S}(y)$.

Proof. The correctness of 1) is obvious. According to the definition

$$\mathrm{R}''\{n\} \subseteq \mathcal{T}(\mathcal{S}(\mathrm{R}''\{n\})) = \mathcal{T}(\mathrm{R}''\{n+1\}),$$

$$\mathcal{S}\text{-}univ(x) = \bigcup_{n\in\omega_0} \mathrm{R}''\{n\} \subseteq \bigcup_{n\in\omega_0} \mathcal{T}(\mathrm{R}''\{n+1\}) \subseteq \mathcal{T}(\mathcal{S}\text{-}univ(x)).$$

That and the conditions of invertibility prove 2). The claim 3) follows from the inclusion $x \subseteq y \subseteq \mathcal{T}(\mathcal{S}(y))$.

Construction. We use the notation of § 1. β is the last number for which the assumption of induction in § 1 holds, y is an arbitrary fixed subset of $\mathrm{T}''\{\beta\}$. We want to construct a relation $r \in \mathrm{R}$ such that $y \subseteq r''\{\beta\}$. Let us define the relation p, $\mathcal{D}(p) = \beta + 1$,

$$x = p''\{\gamma\} \equiv \gamma \leq \beta \;\&\; [(\gamma = 0 \;\&\; x = \mathcal{S}\text{-}univ(y) \cap X) \vee (\gamma \in \mathcal{O}n_{\mathrm{I}} \;\&\; x = \mathcal{S}\text{-}univ(y) \cap \mathcal{T}(p''\{\gamma - 1\})) \vee (\gamma \in \mathcal{O}n_{\mathrm{II}} \;\&\; p''\{\gamma\} = p''\gamma = x)].$$

Obviously, p is a set and $p \in \mathrm{R}$. We shall show that p is the sought relation, i.e. $y \subseteq p''\{\beta\}$.

By induction on $\gamma \leq \beta$ we show that $p''\{\gamma\} = \mathcal{S}\text{-}univ(y) \cap \mathrm{T}''\{\gamma\}$. If this is true, then $p''\{\beta\} = \mathcal{S}\text{-}univ(y) \cap \mathrm{T}''\{\beta\} \supseteq y$, q.e.d.

The induction assumption holds for $\gamma = 0$ (see the definition). Let it be true for all numbers smaller than γ_0, does it hold also for γ_0 ? For $\gamma_0 \in \mathcal{O}n_{II}$ the answer follows from the equations

$$p\{\gamma_0\} = p''\gamma_0 = \bigcup_{\gamma \in \gamma_0} (\mathcal{S}\text{-}univ(y) \cap T''\{\gamma\}) = \mathcal{S}\text{-}univ(y) \cap T''\gamma_0 .$$

Let $\gamma_0 \in \mathcal{O}n_I$. The left inclusion is obvious. To prove the right one, choose $v \in \mathcal{S}\text{-}univ(y) \cap T''\{\gamma_0\}$, i.e. $\{v\} \subseteq \mathcal{S}\text{-}univ(y) \cap \mathcal{T}(T''\{\gamma_0 - 1\})$.

The validity of the next inclusions is a consequence of Metalemma 1 and of the induction assumption

$$\mathcal{S}(\{v\}) \subseteq \mathcal{S}(\mathcal{S}\text{-}univ(y)) \cap T''\{\gamma_0 - 1\} \subseteq \mathcal{S}\text{-}univ(y) \cap T''\{\gamma_0 - 1\} = p''\{\gamma_0 - 1\}.$$

The above inclusions lead to $\{v\} \subseteq \mathcal{T}(\mathcal{S}(\{v\})) \subseteq$ $\subseteq \mathcal{T}(p''\{\gamma_0 - 1\})$. Because the set v was chosen from $\mathcal{S}\text{-}univ(y)$, it holds $v \in p''\{\gamma_0\}$.

§ 3. Proof of Theorem 3

Metatheorem 2 makes it possible to iterate the term of power $\mathcal{P}$ to construct the relation P with the following properties:

$$(\forall \alpha \in \mathcal{O}n)(P''\{0\} = \mathcal{P}(\omega_0) \; \& \; P''\{\alpha + 1\} =$$
$$= \mathcal{P}(P''\{\alpha\}) \; \& \; (\alpha \in \mathcal{O}n_{II} \rightarrow P''\{\alpha\} = P''\alpha)) ,$$

which correctly defines the class Ker in CTS.

Lemma (CTS + S). Let there be 1-1 mapping F between $\mathcal{P}(\omega_0)$ and $\mathcal{O}n$. Then 1-1 mapping $\tilde{F}$ of $\mathcal{P}(\mathcal{P}(\omega_0))$ and $\mathcal{O}n$ can be constructed.

Corollary (CTS + S). The existence of 1-1 mapping F_2 between $\mathcal{P}(\mathcal{O}n)$ and $\mathcal{O}n$ follows from the existence of 1-1 mapping F_1 of $\mathcal{P}(\omega_0)$ into $\mathcal{O}n$.

Proof of the lemma. It is sufficient to find a regular well ordering of $\mathcal{P}(\mathcal{P}(\omega_0))$, equivalent to the regular well ordering of $\mathcal{P}({}^{\omega_0}2)$. First we shall define 1-1 function of $\mathcal{P}({}^{\omega_0}2)$ into ${}^{\omega_0\times\omega_0}2$. Thanks to such relation the ordering of ${}^{\omega_0\times\omega_0}2$ can be transferred to $\mathcal{P}({}^{\omega_0}2)$.

Let us define the relation R

$$\langle g,x\rangle \in R \equiv x \in \mathcal{P}({}^{\omega_0}2)\ \&\ g \in {}^{\omega_0\times\omega_0}2\ \&\ \exists\varphi\,[\mathcal{F}ce_2(\varphi)\ \&\ \mathcal{D}(\varphi)=\omega_0\ \&\ \mathcal{W}(\varphi)=x\ \&\ (\forall i,j\in\omega_0)(g(\langle i,j\rangle) = \varphi(i)(j))] .$$

Axiom S guarantees $\mathcal{D}(R)=\mathcal{P}({}^{\omega_0}2)$. It is easy to prove that

$$(\forall x_1,x_2\in\mathcal{P}({}^{\omega_0}2))[R''\{x_1\}\equiv {}^{\omega_0\times\omega_0}2\ \&\ (x_1\neq x_2\rightarrow R''\{x_1\}\cap R''\{x_2\}=\emptyset)] .$$

Well ordering of ${}^{\omega_0\times\omega_0}2$ makes it possible to define the mapping $\tilde{R}$ of $\mathcal{P}({}^{\omega_0}2)$ into ${}^{\omega_0\times\omega_0}2$ from R in this way:

$$\langle g,x\rangle \in \tilde{R} = g = \min R''\{x\} .$$

Demonstration of Theorem 3. The implication from left to right is obvious. The proof of inverse statement is based on the construction of the mapping of sets into $\mathcal{O}n$ simultaneously with their construction in Ker, i.e. we find such a relation T that for each $\alpha\in\mathcal{O}n$, $T''\{\alpha\}$ there is 1-1 mapping of $P''\{\alpha\}$ into $\mathcal{O}n$. 1-1 mapping M of Ker into $\mathcal{O}n\times\mathcal{O}n$ can be easily derived from T:

$$\langle\alpha,\beta,x\rangle\in M = x\in P''\{\beta\}\setminus P''\beta\ \&\ \alpha = T''\{\beta\}(x) .$$

Let P, F_1, F_2 be classes as mentioned formerly, N the

1-1 mapping of $\mathcal{O}n \times \mathcal{O}n$ into $\mathcal{O}n$. The required relation T could be constructed by iteration of the terms $\mathcal{T}_1$ on isolated and $\mathcal{T}_2$ on limited ordinals with $T''\{0\} = F_1 \times \{0\}$, where

$$\mathcal{T}_1(X) = \{\langle u, v, \gamma+1\rangle : \gamma = \bigcup(\mathcal{D}(X)) \; \&$$
$$\& \; v \in \mathcal{P}(\mathcal{D}(X''\{\gamma\})) \; \& \; u = F_2((X''\{\gamma\})(v))\} \; ,$$
$$\mathcal{T}_2(X) = \{\langle u, v, \gamma\rangle : \gamma = \mathcal{D}(\mathcal{D}(X)) \; \&$$
$$\& \; (\exists \delta \in \gamma)(v \in P''\{\delta\} \setminus P''\delta \; \& \; u =$$
$$= N(\langle X''\{\delta\}(v), \delta\rangle)\} \; .$$

In other words, the term $\mathcal{T}_1$ changes the class $X \times \{\gamma\}$, when X is a mapping of $\mathcal{D}(X)$ into $\mathcal{O}n$, into composition of $\widetilde{X}$ (conventionally derived mapping of $\mathcal{P}(\mathcal{D}(X))$ into $\mathcal{P}(\mathcal{O}n)$) and F_2 . The term $\mathcal{T}_2$ applied to the ordered system of mappings into $\mathcal{O}n$ transforms the system in a similar way as described in the definition of M from T . Both $\mathcal{T}_1$, $\mathcal{T}_2$ are local and are not invertible in a strict sense and therefore we cannot use them directly for the construction of T (Metatheorem 2). According to the invertibility of the term $\mathcal{P}$, which alone is responsible for the loss of the set-image property, the use of Axiom $D1$ in the iteration can be eliminated. If we follow the method of construction of T from § 2, the proof will differ only in the construction of the relation R for some fixed y such that $R''\{\beta\} \supseteq y$ and has all other properties enumerated in § 2. For the purpose of the iteration of $\mathcal{T}_1$, $\mathcal{T}_2$ the relation R can be found by the iteration of the terms $\widetilde{\mathcal{T}}_1$, $\widetilde{\mathcal{T}}_2$ with

$r''\{0\} = (F_1 \upharpoonright \mathit{Univ}(x)) \times \{0\}$, where

$$\tilde{\mathcal{T}}_1(x) = \mathcal{T}_1(x) \cap V \times \mathit{Univ}(x) \times \mathit{On} \quad ,$$

$$\tilde{\mathcal{T}}_2(x) = \mathcal{T}_2(x) \cap V \times \mathit{Univ}(x) \times \mathit{On} \quad .$$

The terms $\tilde{\mathcal{T}}_1$, $\tilde{\mathcal{T}}_2$ have obviously the set-image property and the relation constructed by their iteration has the predicted features. The rest of the proof is obvious.

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R e f e r e n c e s

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Ústav výpočtové techniky ČVUT
Horská 3, Praha 2
Československo

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