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Products as reflections

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Situations when a product of topological spaces is a reflection of a subspace are investigated. Consequences and connections: pseudocompact spaces, \( \mathfrak{R} \)-spaces, reflections of products.

All the spaces considered are assumed to be uniformizable Hausdorff.

During the author’s stay in Mathematical Center in Amsterdam, Autumn 1970, the following question was raised in a discussion: Is the complement in \( \mathbb{R}^{\omega_1} \) (real line) of a point homeomorphic to the complement of two points? The answer is easy if one realizes a Corson’s theorem [5] implying that in this case \( \mathbb{R}^{\omega_1} \) is the Hewitt realcompactification of the complements. In general, if we have a product space \( \prod \{ P_i \} \) and two of its non-homeomorphic subsets \( A, B \), we want to know whether \( \prod \{ P_i \} - A \), \( \prod \{ P_i \} - B \) are homeomorphic. The question is answered in the negative if one knows that any homeomorphism between \( \prod \{ P_i \} - A \), \( \prod \{ P_i \} - B \) extends to an autohomeo-
morphism on \( \Pi \{ P_i \} \) and this last property holds in the case that \( \Pi \{ P_i \} \) is a reflection of both \( \Pi \{ P_i \} - A \), \( \Pi \{ P_i \} - B \). Clearly, if \( \Pi \{ P_i \} \) is a reflection of a space \( P \) in a reflective replete subcategory \( \mathcal{X} \) of \( \text{Top}_{\mathcal{W}} \), then \( P_i \in \mathcal{X} \) for all \( i \) since any reflective replete subcategory \( \mathcal{X} \) of a category \( \mathcal{Z} \) is closed under formation of retracts in \( \mathcal{Z} \), so \( \Pi \{ P_i \} \) is a reflection of \( P \) in the smaller full subcategory of \( \mathcal{X} \) generated by all the products of \( P_i \) (such a reflection will be denoted by \( \beta_{\{ P_i \} P} \)). Thus we have got the following (trivial) assertion:

**Proposition.** Let \( X, Y \) be dense subspaces of \( \Pi \{ P_i \} \). If any continuous mapping on \( X \) or \( Y \) into \( P \) can be continuously extended to \( \Pi \{ P_i \} \), then any homeomorphism on \( X \) onto \( Y \) can be extended to an automorphism on \( \Pi \{ P_i \} \) and, hence, \( X \) and \( Y \) are not homeomorphic provided \( \Pi \{ P_i \} - X \) and \( \Pi \{ P_i \} - Y \) are not homeomorphic.

If we omit density of \( X, Y \) we must assume instead of it that the extensions are unique. Usually one meets dense reflections so that we shall investigate only the cases indicated in the assertion.

In the sequel we will be interested in the assumption of Proposition and shall have in mind that it entails extension of homeomorphisms and "non-homeomorphism" of complements.

By \( \psi, \omega, d, u \) we shall denote the following cardinal functions: pseudocharacter, weight, density, uniform character, respectively (see [14] for these and other car-
dinal functions on topological spaces).

A standard way how to obtain mapping-extensions from subspaces of a product is to factorize them via a subproduct where it is easier to extend the factorized mapping (e.g. [5],[10]). One usually assumes that the projection of the subspace is the whole subproduct (e.g. [3],[5]). Since this assumption will be used very often throughout the paper we will formulate it before the statements:

Definition. Let $X$ be a subspace of a product $\Pi \{ P_i \mid i \in I \}$ and $\alpha$ be an infinite cardinal. The subspace $X$ is said to have the property $V(\alpha)$ if $\mu_X[X] = \Pi\{ P_i \mid i \in J \}$ whenever $J \subset I$, $\text{card } J < \alpha$.

As we shall see from the following Lemma the property $V(\alpha)$ entails inductive generation of all the projections $\mu_J : X \to \Pi\{ P_i \mid i \in J \}$, $\text{card } J < \alpha$, so that any factorization of a continuous mapping on $X$ via $\mu_J$ is continuous.

Lemma. A subspace $X$ of $\prod \{ P_i \mid i \in I \}$ has $V(\alpha)$ if and only if $\mu_J[ U \cap X ] = \mu_J[U]$ for any canonical open set $U$ in $\prod \{ P_i \mid i \in I \}$ and any $J \subset I$, $\text{card } J < \alpha$.

Proof. Let $U = \Pi \{ U_i \mid i \in I \}, \quad x \in \mu_J[U], \quad \text{card } J < \alpha$.

In accordance with [3] we denote $R(U) = \{ i \mid U_i \neq P_i \}$; then $\text{card } J' < \alpha$ for $J' = J \cup R(U)$. Choose $x' \in \mu_{J'}[U]$ such that $\mu_{J'}x' = x$. $V(\alpha)$ implies the existence of an $x'' \in X$ such that $\mu_{J''}x'' = x'$. Clearly, $x'' \in U \cap X$ and $\mu_Jx'' = x$.

Corollary. Let $X$ have $V(\alpha)$ in $\prod \{ P_i \mid i \in I \}$ and $J \subset I$, $\text{card } J < \alpha$. Then the projection $\mu_J : X \to \{ P_i \mid i \in J \}$ is open.
If $X$ has $\mathcal{V}(\alpha)$ in $\prod\{P_i \mid i \in I\}$ and $f : I \to Y$ can be factorized via $\mu_J$ for a $J \subseteq I$, card $J < \alpha$, we can extend $f$ continuously onto $\prod\{P_i \mid i \in I\}$ into $Y$ - e.g. [3] (the whole projection $\mu_J : \prod\{P_i \mid i \in I\} \to \prod\{P_i \mid i \in J\}$ followed by the factorization). There are two general theorems about factorizations of such mappings $f$ - for references and history see [3],[21] (a space $X$ is said to be pseudo-$\alpha$-compact [8],[13] if any locally finite open family in $X$ is of cardinality less than $\alpha$).

**Comfort-Negrepontis [3]:** Let $\alpha$ be a regular uncountable cardinal and $X$ be a subspace of $\prod\{P_i \mid i \in I\}$ with $\mathcal{V}(\alpha)$. If $X$ is a pseudo-$\alpha$-compact, then any $f \in C(X,Y)$, $Y$ metrizable, depends on less than $\alpha$ coordinates.

**Gleason [13]:** Let $X$ be an open subset of $\prod\{P_i \mid i \in I\}$. If all the $P_i$ are separable, then any $f \in C(X,Y)$, $Y$ of countable pseudocharacter, depends on countably many coordinates.

Moreover, W.W. Comfort and S. Negrepontis proved in [3] that under the assumptions stated above ($\alpha$ regular uncountable, $X$ has $\mathcal{V}(\alpha)$), $X$ is pseudo-$\alpha$-compact if and only if any finite subproduct of $\prod\{P_i \mid i \in I\}$ is pseudo-$\alpha$-compact. So this condition on $X$ is in fact a condition on $\{P_i\}$ as in the Gleason's theorem. Only because of shorter expression we shall use the condition formulated for $X$. Next, we shall give slight generalizations and comments to both theorems.

It is seen from the proof of Comfort-Negrepontis' theorem [3] that one needs weaker condition on $Y$ than that of metrizability. Define a cardinal function $\nu$ in the follow-
ing way: \( \nu Y < \alpha \) if there is a system \( \{ \mathcal{U}_\xi \mid \xi < \beta \} \), \( \beta < \alpha \), of reflexive relations on \( Y \) such that

\[
\land \left( \bigcup \mathcal{U}_\xi \right) \mathcal{U}_\xi \mathcal{U}_\xi \mathcal{U}_\xi \mathcal{U}_\xi \mathcal{U}_\xi \mathcal{U}_\xi \mathcal{U}_\xi \mathcal{U}_\xi = \text{the identity and that} \quad \mathcal{U}_\xi [x] \text{ is a neighborhood of } x \text{ for any } x \in X, \xi < \beta.
\]

Evidently, \( \psi Y \leq \psi (\Delta Y, Y \times Y) \leq \nu Y \leq \mu Y \) (even \( \nu Y \leq \mu Y' \) for any coarser space \( Y' \); \( \psi (\Delta Y, Y \times Y) = \psi Y \) provided any neighborhood of \( \Delta Y \) in \( Y \times Y \) is uniformizable, i.e., if any open cover \( \mathcal{A} \) has an open refinement \( \mathcal{B} \) with the property: \( x \in B_1, y \in B_2, B_1 \cap B_2 \neq 0, B_1 \in \mathcal{B} \implies \implies (x, y) \in A \) for an \( A \in \mathcal{A} \) [1],[18]. In particular, \( \psi (\Delta Y, X \times Y) = \psi Y \) if \( Y \) is paracompact or a \( \Sigma \) product of complete separable metric spaces [5] (for further cases see [17]).

**Theorem 1.** Let \( \alpha \) be an uncountable regular cardinal and \( X \subset \prod \{ P_i \mid i \in I \} \) have \( V(\alpha) \). If \( X \) is pseudocompact, then any \( \xi \in C(X, Y) \), \( \nu Y < \alpha \), depends on less than \( \alpha \) coordinates.

We shall now give the Gleason's theorem a form similar to Theorem 1 (for \( \alpha \) isolated and \( X = \prod \{ P_i \mid i \in I \} \) see [14]).

**Theorem 2.** Let \( \alpha \) be an uncountable regular cardinal and \( X \subset \prod \{ P_i \mid i \in I \} \) have \( V(\alpha) \). If \( dP_i < \alpha \) for all \( i \in I \), then any \( \xi \in C(X, Y) \), \( \nu Y < \alpha \) depends on less than \( \alpha \) coordinates.

**Proof.** The assumption on \( Y \) implies the existence of \( J_x \subset I \), \( \text{card } J_x < \alpha \), for any \( x \in X \), such that \( \xi x = \xi y \) whenever \( y \in X \), \( \nu_{J_x} y = \nu_{J_x} x \). Take now an arbitrary nonvoid \( J_0 \subset I \) with \( \text{card } J_0 < \alpha \).

Since density character of \( \prod \{ P_i \mid i \in J_0 \} \) is less
than \( \alpha \) (\( \alpha \) is regular), there is a set \( S'_0 \) of cardinality less than \( \alpha \) and dense in \( \prod \{ P_i \mid i \in J_0 \} \); take \( S_0 \) to be a subset of \( X \) such that \( \nu_{J_0} \) is a bijection on \( S_0 \) onto \( S'_0 \). Now put \( J_1 = J_0 \cup \{ J_x \mid x \in S_0 \} \).

Evidently, \( \text{card } J_1 < \alpha \) and we can construct a set \( S_1 \) for \( J_1 \) in the same way as \( S_0 \) for \( J_0 \). By an inductive procedure we obtain sequences \( \{ J_m \} \) (increasing), \( \{ S_m \} \) such that \( J_m \subset I \), \( S_m \subset X \), \( \text{card } J_m < \alpha \), \( \text{card } S_m < \alpha \) and \( \nu_{J_m} \) maps \( S_m \) injectively onto a dense subset of \( \prod \{ P_i \mid i \in J_m \} \). Put \( J = \bigcup \{ J_m \} \), \( S = \bigcup \{ S_m \} \). First, we shall notice that \( \nu_{J} [S] \) is dense in \( \prod \{ P_i \mid i \in J \} \) and that \( \xi x = \xi y \) whenever \( x, y \in X, \nu_{J} x = \nu_{J} y \in \nu_{J} [S] \).

Indeed, if \( U \) is a canonical open set in \( \prod \{ P_i \mid i \in J \} \), then \( \chi (U) \subset J_m \) for an \( m \) and so there is an \( b \in S_m \) such that \( \nu_{J_m} b \in \nu_{J_m} [U] \); it follows \( \nu_{J} b \in U \).

To prove the second assertion, take \( x, y \in X \) with \( \nu_{J} x = \nu_{J} y \). Indeed, if \( \nu_{J} x = \nu_{J} y \in \nu_{J} [S] \); there is an \( m \) and \( b \in S_m \) such that \( \nu_{J_m} b = \nu_{J_m} x = \nu_{J_m} y \) and, consequently, \( \nu_{J_m} x = \nu_{J_m} y \) and \( \xi b = \xi x = \xi y \) by definition of \( J_m \). Choose now an arbitrary \( x, y \in X \) such that \( \nu_{J} x = \nu_{J} y \). For any canonical neighborhoods \( U, V \) of \( x, y \) respectively there is an \( b' \in \nu_{J} [S] \cap \nu_{J} [U] \cap \nu_{J} [V] \); by Lemma, there are \( u_{U,V} \in U \cap X, v_{U,V} \in V \cap X \) such that \( \nu_{J} u_{U,V} = \nu_{J} v_{U,V} = b' \). The nets \( \{ \xi u_{U,V} \}, \{ \xi v_{U,V} \} \) converge in \( X \) to \( x, y \) respectively and \( \xi u_{U,V} = \xi v_{U,V} \). Hence \( \xi x = \xi y \). The proof is complete.

The condition "\( dP_i < \alpha \) for all \( i \)" in Theorem 2 implies pseudo-\( \alpha \)-compactness of any dense subspace in
and hence, the condition on $X$ in Theorem 1. On the other hand, the condition $\forall Y < \alpha$ in Theorem 1 implies the condition $\exists Y < \alpha$ in Theorem 2. In both cases the converse implications are false (take $\alpha = \omega_1$, $P_i = \omega_\alpha$ for all $i$ and $Y = \omega_\alpha$). But $\omega_\alpha$ is a bad counterexample to the "union" of both Theorems; details and generalizations will appear in a forthcoming paper.

The condition $\forall(\alpha)$ is essentially set-theoretic. We will show now that $\forall(\alpha)$ follows from "nice" topological conditions. For instance, $\forall(\omega_1)$ implies $\exists \varphi$-density of $X$ in $\prod\{P_i \mid i \in I\}$ and the converse is true provided all the $P_i$ are of countable pseudocharacters; similarly for higher cardinals.

If $\mu_\beta \{X\} = \prod\{P_i \mid i \in J\}$, then the complement of $X$ contains a homeomorph of $\prod\{P_i \mid i \in I - J\}$ as a retract. Thus, if $\forall$ is a topological property preserved by retracts and such that no space $\prod\{P_i \mid i \in I - J\}$, $\text{card } J < \alpha$ has $\forall$ but $\prod\{P_i \mid i \in I - X\}$ has, then $\forall(\alpha)$ for $X$ holds.

First take $\forall$ to be a property described by cardinal functions. Let $\varphi$ be a cardinal function being not increased by retracts; if $\varphi(\prod\{P_i \mid i \in I - X\}) = \min\{\varphi(\prod\{P_i \mid i \in I - J\}) \mid \text{card } J < \alpha \}$, then $\forall(\alpha)$ holds. Since $\varphi(\prod\{P_i \mid i \in I - J\}) \geq \sup\{\varphi(P_i \mid i \in I - J) \mid \text{card } J < \alpha \}$ and, in particular, $\varphi(\prod\{P_i \mid i \in I - X\}) < \min\{\sup\{\varphi(P_i \mid i \in I - J) \mid \text{card } J < \alpha \}, \text{card } J < \alpha \}$, then $\forall(\alpha)$ holds. Since $\varphi(\prod\{P_i \mid i \in I - J\}) \geq \sup\{\varphi(P_i \mid i \in I - J) \mid \text{card } J < \alpha \}$ it suffices to require $\varphi(\prod\{P_i \mid i \in I - X\}) < \min\{\sup\{\varphi(P_i \mid i \in I - J) \mid \text{card } J < \alpha \}, \text{card } J < \alpha \}$.

To simplify statements we shall suppose for a moment
that $p_i = p$ for all $i \in I$ and $\text{card } P > 1$. Without loss of generality we shall also suppose $\alpha_0 \leq \alpha \leq \text{card } I$.

The condition $V(\alpha)$ (for any $\alpha \leq \text{card } I$) is then implied by existence of a topological property $Y$ preserved by retracts and such that $P^I$ does not have $V$ and $P^I - X$ has $V$. The above formulas may be given in this case the following form:

$$g(P^I - X) < gP^I \text{ or } g(P^I - X) = gP.$$ 

For instance, $V(\alpha)$ holds if $\text{card } (P^I - X) < 2^{\text{card } I}$ or if $d(P^I - X) = dP$. Let $\alpha \leq \text{card } I$.

**Theorem 3.** Let $P$ be a space and $A$ a subspace of $P^I$ such that $gA < gP^I$ for a cardinal function $g$ being not increased by retracts. If $\alpha \leq \text{max } (dP, \psi Y)$, then $P^I - A$ is $C(P^I - A, Y)$-embedded in $P^I$.

**Proof.** Put $\alpha = (\text{max } (dP, \psi Y))^+$. Then $P^I - A$ has $V(\alpha)$ and, by Theorem 2, any continuous $f : P^I - A \to Y$ can be continuously factorized via $\text{pr}_j$, $\text{card } J < \alpha$ and, hence, has a continuous extension on $P^I$ into $Y$.

**Corollary 1.** Let $P$ be a space and $A$ a subspace of $P^I$ such that $gA < gP^I$ for a cardinal function being not increased by retracts. If $\text{card } I > \text{max } (dP, \psi P)$, then $\beta_p(P^I - A) = P^I$.

By a $\beta$-compact space (in the sense of Herrlich [11]) we mean a space each of its $\mathcal{E}$-ultrafilters with $\beta$-intersection property is fixed.

**Corollary 2.** Let $P$ be not a $\beta$-compact space and let $\text{card } I > \text{max } (dP, \psi P)$. If $A$ is a $\beta$-com-
Proof. Put \( \varphi \) to be the compactness degree in the sense of Herrlich [11], i.e., \( \varphi P \) is the first cardinal \( \alpha \) such that \( P \) is \( \alpha \)-compact.

**Corollary 3.** Let \( P \) be the space \( \mathbb{R} \) of reals, \( \text{card } I > \omega_0 \) and \( A, B \) non-homeomorphic subspaces of \( \mathbb{R}^I \) with cardinalities smaller than \( 2^{\text{card } I} \). Then \( \mathbb{R}^I - A \), \( \mathbb{R}^I - B \) are not homeomorphic.

Putting \( \beta = \omega_0 \) in Corollary 2 we obtain the following generalization of Theorem 1 in [22] (for a more general version see Theorems 4, 6 in the sequel):

Let \( P \) be a realcompact non-compact space and \( A \) be a compact subspace of a product \( P^I \), where \( \text{card } I > d P \). Then \( P^I - A = P^I \).

Choose now other properties for \( V \), e.g. to be normal or a \( \kappa \)-space.

**Theorem 4.** Let \( P \) be not compact, \( \alpha > \omega P \). Then \( \beta_P (P^\alpha - A) = P^\alpha \), for any normal subspace \( A \) of \( P^\alpha \).

**Proof.** If \( V \) is the property "to be normal", then by [19], [15], \( P^\omega \) does not have \( V \) provided \( P \) is not compact and \( \alpha > \omega P \). Clearly \( \alpha > \max (d_P, \varphi P) \).

A similar assertion can be formulated for the property "to be a \( \kappa \)-space". First we must prove the following analogon of the Noble's theorem:

**Theorem 5.** Let \( P \) be not compact. Then there is \( \alpha \) such that \( P^\alpha \) is not a \( \kappa \)-space.

**Proof.** Suppose first that \( P \) is not locally compact.

Let \( \alpha_0 \) be a point of \( P \) without compact neighborhood,
\(x_0 \in P - (x_0)\) and let \(\mathcal{C}\) be a system of compact sets in \(P\) containing \(x_0\) and such that any compact set containing \(x_0\) is contained in a member of \(\mathcal{C}\). We will show that \(P^\mathcal{C}\) is not a \(K\)-space. We may suppose that \(C_0 = (x_0) \in \mathcal{C}\).

For finite \(F \subset \mathcal{C}\), denote \(A_F = \{\{\psi_c\} \in P^\mathcal{C} \mid \psi_c = x_0\} \subset \mathcal{C}\) for \(F = (C_0, C_1, \ldots, C_m)\) and \(U_i, i = 0, \ldots, m\), be neighborhoods of \(x_0\) in \(P\); then the point \(\{\psi_c\} \in A_F\) for \(C \in \mathcal{C} - F\), \(\psi_{C_i} = x_0\) for \(i = 1, \ldots, m\), \(\psi_{C_0} \in U_0 - UF\) (\(x_0\) has no compact neighborhood !), belongs to \(A_F \cap \cap \{U_C \mid C \in \mathcal{C}\}\), where \(U_C = X\) for \(C \in \mathcal{C} - F\), \(U_{C_0} = U_i\).

(1) \(x_0 \in A\) : let \(P\) be a finite subset of \(\mathcal{C}\), \(P = (C_0, C_1, \ldots, C_m)\) and \(U_i, i = 0, \ldots, m\), be neighborhoods of \(x_0\) in \(P\); then the point \(\{\psi_c\}\), where \(\psi_c = x_1\) for \(C \in \mathcal{C} - F\), \(\psi_{C_i} = x_0\) for \(i = 1, \ldots, m\), \(\psi_{C_0} \in U_0 - UF\) (\(x_0\) has no compact neighborhood !), belongs to \(A_F \cap \cap \{U_C \mid C \in \mathcal{C}\}\), where \(U_C = X\) for \(C \in \mathcal{C} - F\), \(U_{C_0} = U_i\).

(2) \(K\) compact in \(P^\mathcal{C}\), \(x_0 \in K \implies x_0 \notin \bar{K} \cap A\); there is a \(C_1 \in \mathcal{C}\), \(C_1 \neq C_0\) such that \(C_1 \supset \psi_{C_0} [X]\) and a neighborhood \(U\) of \(x_0\) not containing \(x_1\); assume that \(\{\psi_c\} \in K \cap A \cap \cap \{U_C \mid C \in \mathcal{C}\}\), where \(U_C = X\) for \(C \in \mathcal{C} - (C_0, C_1)\), \(U_{C_0} = U_{C_1} = U\). Then \(\{\psi_c\} \in A_F\) for a finite \(F \subset \mathcal{C}\); clearly \(C_1 \in F\) because \(\psi_{C_1} = x_1\) and, consequently, \(\psi_{C_0} \neq C_1\) but this contradicts to \(\psi_{C_0} \in \psi_{C_0} [X] \subset C_1\).

If \(P\) is locally compact, then \(P^\omega\) is not locally compact and we may accept the preceding construction for \(P^\omega\) instead of \(P\). The proof is complete.

Define \(\gamma P\) to be the least cardinal of a base for the ideal of compact sets in \(P\) and, if \(P\) is not compact, \(\gamma P = \min \{\gamma F \mid F\) closed in \(P^\omega), F\) not locally
compact). Then we have proved:

If \( P \) is not compact (i.e. \( \gamma P \leq \omega \)), then \( P^\omega \)

is not a \( \aleph \)–space provided \( \alpha \leq \gamma P \).

It is easy to show that \( \gamma P \leq \gamma (P^{\omega_0}) \leq \text{card exp}_\aleph \gamma P \)

(the cardinality of the set of all countable subsets of \( \gamma P \),

which is \( 2^{\omega_0} \). \( \gamma P \) if \( \text{cof} \gamma P > \omega_0 \) and \( \gamma P \leq \gamma (P^{\omega_0}) \); thus, mostly, it suffices to assume \( \alpha \leq 2^{\omega_0} \).

\( \gamma P \). If the continuum hypothesis does not hold, then it may happen that \( \gamma (P^{\omega_0}) < 2^{\omega_0} \). \( \gamma P \)

even for non-compact spaces (e.g. \( \gamma (T^{\omega_0}) = \gamma T^{\omega_0} = \omega_1 \)); under continuum hypothesis always \( \gamma (P^{\omega_0}) = 2^{\omega_0} \). \( \gamma P \)

whenever \( \text{cof} \gamma P > \omega_0 \) or \( \gamma P = \omega_0 \) since \( \gamma (P^{\omega_0}) > \omega_0 \)

provided \( P \) is not compact. It is shown in [16] that if \( P \)

is not countably compact, then \( P^{\omega_1} \) is not a \( \aleph \)–space.

This assertion does not follow from that of ours. If continuum hypothesis is not true then it may happen that still \( \gamma'N = 2^{\omega_0} \). Indeed, if \( F \) is closed in \( N^N \) and not locally compact, then \( F \)

contains a closed subset homeomorphic to the space \( T = N \times N \cup (\infty) \), where points of \( N \times N \)

are isolated and any neighborhood of \( \infty \) contains all the \( (m) \times N, m \in N \), except finite number. Thus \( \gamma'N = \gamma T \).

Since compact sets in \( T \) are of the form \( (\infty) \cup A \), where \( A \cap ((m) \times N) \)

is finite for every \( m \), the problem of finding \( \gamma T \) reduces to finding small cofinal subsets in the system of all functions \( f: N \rightarrow N \), ordered pointwise. As was communicated to the author by B. Balcar and P. Štěpánek, there is an ultrafilter on \( N \) such that in the corresponding ultraproduct-model any such cofinal part is
of cardinality $2^{\omega_1}$.

It is easy to see that Theorem 5 and remarks following it remain valid in larger classes of topological spaces than completely regular Hausdorff (of course, in dependence on definitions of local compactness and of $\kappa$-spaces).

**Theorem 6.** Let $\mathcal{P}$ be not compact, $\alpha > \max (\mathcal{P}, \mathcal{P}_1, \mathcal{P}_2)$.

Then $\beta_\mathcal{P}(\mathcal{P}^\kappa - \mathcal{A}) = \mathcal{P}^\kappa$ for any $\kappa$-space $\mathcal{A}$ in $\mathcal{P}^\kappa$.

**Corollary.** Let $\alpha$ be an uncountable cardinal and $\mathcal{A}, \mathcal{B}$ non-homeomorphic $\kappa$-spaces or normal spaces in $\mathcal{R}^\kappa$ (in particular, metrizable or compact). Then $\mathcal{R}^\kappa - \mathcal{A}$ is not homeomorphic to $\mathcal{R}^\kappa - \mathcal{B}$.

Similar statements can be given e.g. for the case when $\mathcal{A}, \mathcal{B}$ are zero-dimensional subspaces of $\mathcal{R}^\kappa$, etc. (In general, if $\mathcal{K}$ is a productive and closed-hereditary class of spaces, i.e. epireflective in $\text{Top}_\kappa$, then $\mathcal{P} \in \mathcal{K}, \mathcal{A} \in \mathcal{K}$ entails $\mathcal{P}^\kappa - \mathcal{A}$ has $\mathcal{V}(\alpha)$ for all $\alpha \leq \text{card} \ I$.)

N. Noble proved in [20] that if $X \in \prod \{ \mathcal{P}_i \mid i \in I \}$, any finite subproduct of $\{ \mathcal{P}_i \}$ satisfies countable chain condition and any countable subproduct is perfectly normal realcompact, then $\nu X = \Pi \{ \mathcal{P}_i \mid i \in I \}$ if and only if $X$ is $G_\delta$-dense in $\prod \{ \mathcal{P}_i \mid i \in I \}$; the conditions are satisfied if e.g. all the $\mathcal{P}_i$ are separable metrizable.

We shall prove now a more general

**Theorem 7.** Let $X$ be a subspace of a product $\Pi \{ \mathcal{P}_i \mid i \in I \}$ of realcompact spaces with countable pseudocharacters and let all the finite subproducts of $\{ \mathcal{P}_i \}$ be pseudo-$\omega_1$-compact. Then $\nu X = \Pi \{ \mathcal{P}_i \mid i \in I \}$ if and only if $X$ is $G_\delta$-dense in $\Pi \{ \mathcal{P}_i \mid i \in I \}$.
Proof. The nontrivial part is the "if" part. If \( X \) is \( G_\sigma \)-dense in \( \prod \{ P_i \mid i \in I \} \), then \( X \) has \( V(\omega_i) \) because \( \psi P_i = \omega_0 \) for all \( i \). By Comfort-Negrepontis' theorem we obtain directly that \( X \) is \( C \)-embedded in \( \prod \{ P_i \mid i \in I \} \).

Corollary. Let \( X \) be a subspace of a product \( \prod \{ P_i \mid i \in I \} \) of realcompact separable spaces with countable pseudocharacters. Then \( \nu X = \prod \{ P_i \mid i \in I \} \) if and only if \( X \) is \( G_\sigma \)-dense in \( \prod \{ P_i \mid i \in I \} \).

As was remarked earlier, W.W. Comfort and S. Negrepontis proved in [3] that \( X \subset \prod \{ P_i \mid i \in I \} \) with \( V(\omega_i) \) is pseudo-\( \alpha \)-compact, \( \alpha \) regular uncountable, if and only if any finite subproduct is pseudo-\( \alpha \)-compact. The corresponding theorem for \( \alpha = \omega_0 \) is not true [2],[9] in general, but holds in special important cases - see [6],[7] for the case of compact discrete \( P_i \). We give here a generalization of the Efimov-Engelking's theorem.

Theorem 8. Let \( P_i \) be compact for all \( i \in I \) and \( X \subset \prod \{ P_i \mid i \in I \} \) have \( V(\omega_i) \). Then \( X \) is pseudocompact and \( \nu X = \beta X = \prod \{ P_i \mid i \in I \} \).

Proof. It suffices to prove that \( X \) is \( C \)-embedded in \( \prod \{ P_i \mid i \in I \} \) and this assertion follows directly from the Comfort-Negrepontis' theorem.

Corollary. Let \( X \) be a dense subspace of a product of compact metrizable spaces \( P_i \). Then \( X \) is pseudocompact if and only if \( X \) is \( G_\sigma \)-dense in \( \prod \{ P_i \mid i \in I \} \).

Proof. The property "\( X \) is \( G_\sigma \)-dense in \( \prod \{ P_i \mid i \in I \} \)" is equivalent to "\( X \) has \( V(\omega_i) \)" in our case. If \( X \) is pseudocompact, \( J \subset I \), and \( J \subset \omega \), then \( \nu_{\omega} \{ X \} \) is a
dense pseudocompact subspace of the compact metrizable
\( \prod_i P_i \mid i \in I \) and, hence, \( \mu P_i \cap X = \prod_i P_i \mid i \in I \). The
converse implication follows from Theorem 8.

**Corollary 2.** Let \( X \) be a dense pseudocompact subspace
of a product of compact metrizable spaces \( P_i \). Then \( vX =
\beta X = \prod_i P_i \).

The same procedure may be used if we know that the pro­
duct is pseudocompact.

**Theorem 9.** Let the product \( \prod_i P_i \mid i \in I \) be pseudo­
compact and \( X \subseteq \prod_i P_i \mid i \in I \) have \( V(\omega_i) \). Then \( X \)
is pseudocompact.

**Proof.** \( X \) is \( C \)-embedded in \( \prod_i P_i \mid i \in I \).

**Corollary.** If \( X \subseteq T_{\omega_i} \) has \( V(\omega_i) \), then \( X \) is pseu­
docompact.

At the end we shall give an analogon to the Glicksberg's
theorem on \( \beta \prod P_i \) [10] (it should be noted that our re­
sult is analogous to a corollary, not to the main theorem in
[10]). The \( \alpha \)-compactification of \( X \) [11] is denoted by
\( \beta_{\alpha} X \) (i.e., \( X \subseteq \beta_{\alpha} X \subseteq \beta X \), \( \beta_{\alpha} X \) is the set of
all \( X \)-ultrafilters on \( X \) with \( \alpha \)-intersection property).
First a lemma (for two factors and \( \alpha = \omega_i \) see [4]).

**Lemma.** Let \( \prod_i P_i \) be \( C^* \)-embedded in \( \prod_{i=1}^n P_i \).
Then \( \beta_{\alpha} \prod_i P_i \) is \( C^* \)-embedded in \( \prod \beta_{\alpha} P_i \).

**Proof.** It is proved in [12] that \( \alpha \)-compact spaces form
the epireflective hull in \( \text{Top}_\mathcal{U} \) of the spaces \( I^\beta - (\nu), \)
\( \beta < \alpha \), where \( I \) denotes now the real interval \( [0, 1] \)
and \( \nu \) is a point of \( I^\beta \). Thus it suffices to prove that
if \( \prod_i P_i \) is \( C^* \)-embedded in \( \prod_i \beta_{\alpha} P_i \), then any
\[ \varepsilon: \prod \beta_i \to \beta^\alpha - (\eta), \quad \beta < \alpha, \]
can be continuously extended to \( \prod \beta_i \to \beta^\alpha - (\eta) \). Let \( \beta < \alpha \), \( \varepsilon: \prod \beta_i \to \beta^\alpha - (\eta) \). Then each \( \nu_i \circ \varepsilon \) \( (\nu_i: I^\beta - (\mu) \to I) \)
is the \( \varepsilon \)'s projection, \( \varepsilon < \beta \) has a continuous extension \( \tilde{\varepsilon}: \prod \beta_i \to I \). Put \( \tilde{\varepsilon} = \prod (\tilde{\varepsilon}_i) \mid \beta \), i.e. \( \tilde{\varepsilon} = (x \mapsto \{ \tilde{\varepsilon}_i \mid \beta \}): \prod \beta_i \to I^\alpha \).

Suppose that \( A = \varepsilon^{-1}[\eta] \neq \emptyset \). We may assume that the index set of \( \{ P_i \} \) is the well-ordered set \( \{ \eta \mid \eta < \gamma \} \).

Let \( \{ x^\mu \} \in A \) and \( \{ \eta_\alpha \} \subset \{ \eta \mid \eta < \gamma \} \) be the set of all indices \( \eta \) such that \( x^\mu \not\in P_\eta \). We shall define inductively a net \( \{ x^\mu \} \mid \mu \) of points of \( A \) such that \( x^\mu \in P_\eta \) for all \( \eta \leq \eta_\mu \), \( x^\mu = x^\mu_\eta \) for \( \eta = \eta_\mu \) or \( \eta > \eta_\mu \) and such that \( x^{\mu'} = x^{\mu'}_\eta \) for \( \mu' < \mu'' \), \( \eta \leq \eta_\mu \). For \( \mu = 0 \) there is a \( x^0_\eta \in P_\eta \)
such that \( x^0_\eta \in A \), where \( x^0_\eta = x^0_\eta \) for \( \eta = \eta_0 \), for otherwise \( \emptyset \neq \beta_\alpha \beta_\eta \cap A \subset \beta_\alpha \beta_\eta - \beta_\eta \) (here \( \beta_\alpha \beta_\eta \) is the copy of \( \beta_\alpha \beta_\eta \) by embedding \( x \mapsto \{ x \} \mid \eta \)
\( x^0_\eta = x, x^0_\eta = x^0_\eta \) for \( \eta = \eta_0 \) and similarly \( \beta_\eta \)), which is impossible because, in that case, \( \varepsilon/\beta_\eta \to I^\beta - (\eta) \) and has the unique continuous extension \( \tilde{\varepsilon}/\beta_\alpha \beta_\eta \to I^\beta - (\eta) \). Suppose that \( \{ x^\mu \} \) are defined for all \( \mu < \tilde{\mu} \) and put \( t = \{ t_\eta \} = \lim \{ x^\mu \} \mid \mu < \tilde{\mu} \) (the net \( \{ x^\mu \} \mid \mu < \tilde{\mu} \) is coordinate-wise almost constant). Then \( t \in A \) and we can construct \( \{ x^{\tilde{\mu}} \} \) from \( t \), \( \eta_{\tilde{\mu}} \) in the same way as \( \{ x^0 \} \)
from \( \{ x^\mu \} \), \( \eta_0 \), \( \{ x^0 \} \) has the required properties.

The limit \( \lim \{ x^\mu \} \mid \mu \) is a point of \( -797 - \)
\begin{align*}
A \cap \prod \{ P_i \mid i \in I \} & \quad \text{a contradiction.}
\end{align*}

**Theorem 10.** Let \( \prod \{ P_i \mid i \in I \} \) be pseudo-\( \omega_1 \)-compact. Then \( \beta^\alpha \prod \{ P_i \mid i \in I \} = \prod \{ \beta^\alpha P_i \mid i \in I \} \) if and only if \( \beta^\alpha \prod \{ P_i \mid i \in J \} = \prod \{ \beta^\alpha P_i \mid i \in J \} \) for all \( J \subseteq I \), \( \text{card } J < \omega_1 \).

**Proof.** One implication is clear. To prove the other, by the preceding Lemma, it suffices to show that \( \prod \{ P_i \mid i \in I \} \) is \( C^* \)-embedded in \( \prod \{ \beta^\alpha P_i \mid i \in I \} \). If \( f : \prod \{ P_i \mid i \in I \} \to [0,1] \) then, by the Comfort-Negrepontis' theorem, \( f = f' \circ \mu_i \) for a \( J \subseteq I \), \( \text{card } J < \omega_0 \); by our condition \( f \) can be extended on \( \prod \{ \beta^\alpha P_i \mid i \in \hat{J} \} \) and, hence, \( f \) can be extended to \( \prod \{ \beta^\alpha P_i \mid i \in I \} \).

It may be interesting that under pseudo-\( \omega_1 \)-compactness of the product, the commutation of \( \prod \) with \( \beta^\alpha \) depends on countable subproducts for any \( \alpha \). In the Glickberg's theorem, i.e. \( \alpha = \omega_0 \), there is no loss of generality in assumption that the product is pseudo-\( \omega_1 \)-compact but, clearly, this is not the case for \( \alpha > \omega_0 \). The only generalization of Theorem 10 we know is to put any uncountable regular \( \alpha \) instead of \( \omega_1 \).

**References**


COMFORT W.W. and S. NEGREPONTIS: Extending continuous functions on $X \times Y$ to subsets of $\beta X \times \beta Y$, Fund. Math. 59 (1966), 1-12.


- 799 -


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