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A THEOREM ON SUPPORTS IN THE THEORY OF SEMISETS

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The following theorem is proved in the theory of semisets: If there is a total semiset support then each nonempty semiset of ordinal numbers has a least element.

Key-words: theory of semisets, support, complete ultrafilter, complete Boolean algebra

Introductory remark (by Petr Hájek). There are various beautiful results concerning the set theory and the theory of semisets proved by B. Balcar but not published. The result contained in the present paper means a considerable simplification of the discussion on the notion of support (see [1] Chapt. IV Sect. 1 and 2) and was proved by Balcar at the end of 1969. (Cf. [2] 3.7.) It was not possible to include it into [1] but the authors of [1] hoped that Balcar would publish his result elsewhere. Since this hope has remained unsatisfied I have decided to help publish Balcar's result. I simply wrote down the result and its proof as I had learned it from Balcar without trying to generalize or make applications. I am grateful to Balcar that he permitted me to do this and I hope sincerely that this friendly joke will get him to publish more of

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his results in a reasonable time. P.H.

In what follows, familiarity with [1] Chapters I, II and IV (Sect. 1 and 2) is assumed. We freely use denotations introduced there; in particular, we use a, b, x, y etc. to denote sets and σ, φ etc. to denote semisets. (The reader is recommended to use Index of symbols in [1] if necessary.) TSS$'$ denotes the theory of semisets with the regularity axiom (D1). Our aim is to prove the following

<u>Theorem</u> (TSS$'$). If there is a total semiset support then each non-empty semiset of ordinal numbers has a least element. In symbols, (S3) $\rightarrow$ (ST) (the third support axiom implies standardness).

By [1] 4241, we have the following

<u>Corollary</u> (TSS$'$). (S3) iff (S6), i.e. there is a total semiset support iff there is a total semiset support which is a complete ultrafilter on a complete Boolean algebra $\underset{\sim}{b}$.

The theorem is an immediate consequence of Lemma 4 below.

<u>Lemma 1</u> (TSS$'$). A non-empty semiset σ is a support iff there is a set $a \supseteq \sigma$ and a relation $r \subseteq a \times a$ such that the following holds:

(i) r is antireflexive,

(ii) $(\forall c \subseteq a - \sigma)(\exists x \in \sigma)(r''\{x\} \supseteq c)$,

(iii) $(\forall x \in \sigma)(r''\{x\} \subseteq a - \sigma)$.

<u>Remark</u>. Suppose that $\underset{\sim}{b}$ is a complete Boolean algebra and that σ is a complete ultrafilter on $\underset{\sim}{b}$. Put

$a = b - \{0_b\}$ and $r = \{\langle u, v\rangle ; u, v \in a \,\&\, u \wedge v = 0_b\}$ (the relation of disjointness). The reader verifies with ease that (i), (ii), (iii) are satisfied. The notion of disjointness is the motivation of our conditions (i) - (iii).

Proof of Lemma 1. ($\longrightarrow$) Let σ be a support and let $a \supseteq \sigma$. By [1] 4115, $\mathbb{P}(a - \sigma)$ is dependent on σ; so let $\mathbb{P}(a - \sigma) = r_1'' \sigma$. We may suppose w.l.o.g. $\mathbb{D}(r_1) \subseteq a$ and $\mathbb{W}(r_1) \subseteq \mathbb{P}(a)$. Put $\langle y, x\rangle \in r_2 \equiv$ $\equiv (\exists u)(\langle u, x\rangle \in r_1 \& y \in u)$, then $r_2 \subseteq a \times a$ and $r_2''\{x\} =$ $= \bigcup(r_1''\{x\})$ for each $x \in \mathbb{D}(r_2)$. If $x \in \sigma$ and $\langle u, x\rangle \in r_1$ then evidently $x \notin u$, hence we may suppose that $\langle u, x\rangle \in r_1$ implies $x \notin u$. Consequently, r_2 may be supposed antireflexive. If $c \subseteq a - \sigma$ then $\langle c, x\rangle \in r_1$ for some x, which implies $c \subseteq r_2''(x)$. Finally, if $x \in \sigma$ then $r_1''\{x\} \subseteq \mathbb{P}(a - \sigma)$ and hence $r_2''\{x\} \subseteq a - \sigma$.

($\longleftarrow$) Let σ, a, r satisfy (i) - (iii). Let s be a relation such that $\mathbb{D}(s) \subseteq a$ and let $\rho = s''\sigma$. Put $\mathbb{W}(s) = b$. We prove $\mathrm{Dep}(b - \rho, \sigma)$; the result will follow by [1] 1466. Put $r_1 = \{\langle y, x\rangle ; (\mathrm{Cnv}(s))''\{y\} \subseteq r''\{x\}\}$. We prove $r_1''\sigma = b - \rho$. Indeed, suppose $x \in \sigma$ and $\langle y, x\rangle \in r_1$. Then $r''\{x\} \supseteq (\mathrm{Cnv}(s))''\{y\}$; by (iii), $(\mathrm{Cnv}(s))''\{y\} \subseteq a - \sigma$ and consequently $y \in b - \rho$. On the other hand, if $y \in b - \rho$ then $(\mathrm{Cnv}(s))''\{y\} \subseteq a - \sigma$ and, by (ii), there is an $x \in \sigma$ such that $r''\{x\} \supseteq$ $\supseteq (\mathrm{Cnv}(s))''\{y\}$. We have $\langle y, x\rangle \in r_1$ and consequently $y \in r_1''\sigma$.

Lemma 2 (TSS$'$). A non-empty semiset σ is a support iff there is a set $a \supseteq \sigma$ and a symmetric relation $r \subseteq a \times a$ satisfying (i) - (iii) of Lemma 1.

Proof. Let σ be a support and let r_2 be as in the first part of the proof of Lemma 1. We know that r_2 satisfies (i) - (iii). Put $r_3 = r_2 \cup \mathrm{Cnv}(r_2)$. Then r_3 is symmetric and satisfies (i), (ii). We show that (iii) is also satisfied. Suppose not and let $x \in \sigma$, $r_3''\{x\} \cap \cap \sigma \ni y$. Then $(\mathrm{Cnv}(r_2))''\{x\} \ni y$ and $y \in \sigma$, i.e. $\langle x,y \rangle \in r_2$ and $x, y \in \sigma$. This contradicts the fact that r_2 satisfies (iii).

Lemma 3 (TSS$'$). Let σ be a support and let a, r be as in Lemma 2. Put $x \leq y \equiv r''\{x\} \supseteq r''\{y\}$. Then $\langle a, \leq \rangle$ is a quasiordered set and σ is a complete ultrafilter on $\langle a, \leq \rangle$ in the following sense:

(iv) $(\forall x, y \in a)(y \geq x \ \& \ x \in \sigma \rightarrow y \in \sigma)$,

(v) $(\forall x, y \in \sigma)(\exists z \in \sigma)(z \leq x \ \& \ z \leq y)$,

(vi) if $q \subseteq a$ and $(\forall x \in a)(\exists y \in q)(y \leq x)$ then $q \cap \sigma \neq 0$.

Proof. (iv) Let $y \geq x \ \& \ x \in \sigma$. Then $r''\{y\} \subseteq \subseteq r''\{x\} \subseteq a - \sigma$; suppose $y \notin \sigma$. By (ii), there is a $z \in \sigma$ such that $r''\{z\} \supseteq \{y\} \cup r''\{y\}$. Hence $\langle y,z \rangle \in r$; $\langle z,y \rangle \in r$ by symmetry, i.e. $\langle z, z \rangle \in r$, which contradicts (i).

(v) Let $x, y \in \sigma$; by (iii), $r''\{x\} \cup r''\{y\} \subseteq a - \sigma$ and by (ii) there is a $z \in \sigma$ such that $r''\{z\} \supseteq \supseteq r''\{x\} \cup r''\{y\}$. Hence $z \leq x, y$.

(vi) The condition $(\forall x \in a)(\exists y \in q)(y \leq x)$ is equivalent to $(\forall x \in a)(\exists y \in q)(\varkappa''\{x\} \subseteq \varkappa''\{y\})$. Suppose $q \cap \sigma = 0$, i.e. $q \subseteq a - \sigma$. By (ii), there is an $x \in \sigma$ such that $\varkappa''\{x\} \supseteq q$. For this x we have a $y \in q$ such that $\varkappa''\{y\} \supseteq \varkappa''\{x\} \supseteq q$; this implies $y \in \varkappa''\{y\}$, which contradicts (i).

Lemma 4 (TSS′). If σ is a support and if $\wp$ is a non-empty semiset of ordinal numbers dependent on σ then $\wp$ has a least element.

Proof. Let $\sigma \subseteq a$ and let $\varkappa, \leq$ be as in Lemma 3. Suppose $\wp = s''\sigma$; we can assume $\mathrm{D}(s) = a$ and $\mathrm{W}(s) \subseteq \mathit{On}$. Define a set-function f as follows:

$$f'x = \min \bigcup_{y \geq x} s''\{y\} .$$

By Lemma 3(iv), $f''\sigma \subseteq s''\sigma$ and evidently $(\forall \alpha \in s''\sigma)(\exists \beta \leq \alpha)(\beta \in f''\sigma)$. Hence it suffices to show that $f''\sigma$ has a least element. Put $b = \{x ; (\forall y \leq x)(f'y = f'x)\}$. Since $y \leq x \rightarrow f'y \leq f'x$ evidently holds for each $x, y \in a$, the set b fulfils $(\forall x \in a)(\exists y \in b)(y \leq x)$. By Lemma 3(vi), $b \cap \sigma \neq 0$. For each $x \in b \cap \sigma$, $f'x$ is minimal in $f''\sigma$. Indeed, suppose $y \in \sigma$ and $f'y < f'x$. By Lemma 3 (v), there is a $z \in \sigma$ such that $z \leq x \;\&\; z \leq y$. For this z we have $f'z \leq f'y < f'x$, hence $z \leq x \;\&\; f'z < f'x$, which contradicts $x \in b$.

The proof of Lemma 4 is complete; the theorem is an immediate consequence of the last lemma by the definition of a total support.

R e f e r e n c e s

[1] P. VOPĚNKA and P. HÁJEK: The theory of semisets, Academia Prague and North Holland Publ. Comp.Amsterdam 1972.

[2] P. VOPĚNKA: The theory of semisets, Actes du Congrès international des mathématiciens 1970, 255-260.

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