Commentationes Mathematicae Universitatis Carolinae

Jindřich Nečas

On the range of nonlinear operators with linear asymptotes which are not invertible

Commentationes Mathematicae Universitatis Carolinae, Vol. 14 (1973), No. 1, 63--72

Persistent URL: http://dml.cz/dmlcz/105470

Commentationes Mathematicae Universitatis Carolinae

14,1 (1973)

ON THE RANGE OF NONLINEAR OPERATORS WITH LINEAR ASYMPTOTES WHICH ARE NOT INVERTIBLE

Jindřich NEČAS, Praha

Abstract: Let A be a linear, bounded, selfadjoint operator from a real Hilbert space to itself with a closed range. Let $0 < \dim \operatorname{Ker} A < \infty$. Let P be a completely continuous operator. If the operator P has weak asymptotes $\ell(w)$ for $w \in \operatorname{Ker} A$, then the condition $(w, h) < (w)$ is sufficient for $h \in \operatorname{Range} (A + P)$. This condition can be also necessary.

Key words: nonlinear operator, completely continuous operator, weak asymptote, fixed point, boundary value problem, closed range, alternative problem

AMS, Primary: 47H10, 47H15 Ref. Ž. 7.956, 7.978.4
Secondary: 35J65

---

§ 1. Introduction. Let A be a linear, bounded, selfadjoint operator from a real Hilbert space H to itself with a closed range. Let $0 < \dim (\operatorname{Ker} A) < \infty$. Let P be a completely continuous operator, in general nonlinear, from H to H , such that for all u from H

(1.1) $$\| Pu \| \leq \alpha < \infty .$$

Let us suppose that the operator P has a "weak asymptote $\ell(w)$ on every halfray with the slope from the $\operatorname{Ker} A$ ": there eixsts a finite $\lim_{t \to \infty} (w, P(u + tw)) = \ell(w)$, uniformly with respect to bounded sets of u and with respect to w from $\mathrm{Ker}\, A$ such that $\|w\| = 1$.

Put $Tu = Au + Pu$, $T(H) = R$, and let us look for the conditions implying $h \in R$.

Results:

If for every $w \in \mathrm{Ker}\, A$, $\|w\| = 1$

(1.2) $\quad (w, h) < \ell(w) \; ((w, h) > \ell(w))$,

then $h \in R$.

If for every $u \in H$ and $w \in \mathrm{Ker}\, A$, $\|w\| = 1$,

(1.3) $\quad (w, Pu) < \ell(w) \; (\leq, >, \geq)$

then (1.2) $(\leq, >, \geq)$ is necessary.

The necessary condition is obvious; for to prove the sufficient condition, we use the Cesari-Lazar type alternative problem, see L. Cesari [1] and Schauder's fixed point theorem.

As an example, we consider a general boundary value problem for one partial differential equation $\sum_{|i|,|j| \leq k} (-1)^{|i|} D^{i} (a_{ij} D^{j} u) + g(u) = f$ and we obtain, as a partial result, the assertion of the paper of S.A. Williams [2], which is a generalization of the paper of E. Landesman, A. Lazar [3]. This paper can be considered as a generalization of the above papers.

In the paper of the author, see J. Nečas [4] or [5], the $\varkappa$-asymptote of a nonlinear operator is introduced. In our case the operator A is the 1-asymptote of the operator T because $\lim\limits_{\|u\| \to \infty} \dfrac{\|Tu - Au\|}{\|u\|} = \lim\limits_{\|u\| \to \infty} \dfrac{\|Pu\|}{\|u\|} = 0$.

§ 2. <u>Abstract results</u>. Let us note $\mathrm{Ker}\, A = H_2, H_1 = H \dot{-} H_2$. Because A is a one-to-one operator from $H_1 \to H_2$, (and $A(H) = H_1$), let S be the inverse of A, restricted to the space H_1. Let $\dim H_2 = k$.

Let L be the Hilbert space defined as $L \times R_k$, of the couples $(u, c) = U$, provided with the scalar product $(U, V) = (u, v) + (c^1, c^2)$. Let P_i be the projections of H to H_i. Let $\{w_i\}_{i=1}^{k}$ be an orthonormal basis of H_2. Let us define a mapping C of L to L, putting $(u, c) \mapsto (u^*, c^*)$ and

$$(2.1)\quad u^* = \sum_{i=1}^{k} c_i w_i + SP_1(h - Pu),\ c_i^* = c_i - (Pu^* - h, w_i),\ \varepsilon > 0 .$$

Clearly C is a completely continuous operator. We obtain immediately

<u>Lemma 2.1</u> (Cezari-Lazar type alternative problem) $Tu = h$ iff (u, c) is a fixed point of C.

<u>Theorem 2.1</u>. Let A be a linear, bounded, selfadjoint operator from H to H with a closed range and let $0 < \dim(\mathrm{Ker}\, A) < \infty$. Let P be a completely continuous operator from H to H (nonlinear), satisfying (1.1). Let P have a weak asymptote $\ell(w)$ on every halfray with the slope from the $\mathrm{Ker}\, A$. Then the condition (1.2) is sufficient for h to be in the $\mathrm{Range}\,(A + P)$.

<u>Proof</u>. Let us look for a fixed point of the operator C. Note $|c| = \rho$, $\sum_{i=1}^{k} c_i w_i = w$, $(Pu^* - h, w_i) = t_i$.

We have

$$(Pu^* - h, w) = \rho\,(P(\frac{w}{\rho}\rho + SP_1(h - Pu)) - h, \frac{w}{\rho}) \overset{df}{=} \rho\,\alpha\,(w, \rho)\ .$$

Because $(\frac{w}{\rho}, P(u + t\,\frac{w}{\rho})) \longrightarrow l(\frac{w}{\rho})$ uniformly, $l(\omega)$ for $\|\omega\| = 1$, ω from $Ker\ A$, is continuous and there exists $\rho_1 > 0$ such that for $\rho \geq \frac{\rho_1}{2}$; $\alpha\,(w, \rho) \geq \alpha_0 > 0$. Consider $\rho_1 \geq \rho \geq \frac{\rho_1}{2}$. $(c^*, c^*) = \rho^2 - 2\varepsilon\rho\,\alpha\,(w, \rho) + \varepsilon^2 |t|^2$. $|t|$ is bounded because of the condition (1.1), so we can choose $\varepsilon_0 > 0$ such that for $0 < \varepsilon \leq \varepsilon_0$ and $\frac{\rho_1}{2} \leq \rho \leq \rho_1$:

(2.2) $$|c^*|^2 \leq \rho^2 \leq \rho_1^2\ .$$

If we choose ε small enough, we obtain for $0 \leq \rho \leq \frac{\rho_1}{2}$

(2.3) $$|c^*|^2 \leq \rho_1^2\ .$$

It follows from the condition (1.1) that

(2.4) $$\|u^*\|^2 \leq |c|^2 + M^2\ .$$

Put $D = \{ u \mid \|u\|^2 \leq \rho_1^2 + M^2,\ |c|^2 \leq \rho_1^2 \}$. D is a closed, convex set in the space L . It follows from (2.2), (2.3), (2.4) that the mapping C maps D into itself. Because C is completely continuous, there exists by the Schauder's fixed point theorem a fixed point that in virtue of the lemma 2.1 gives the result.

Remark 2.1. If for some subspace H_3 of H, $H_1 \subset H_3 \subset \subset H$, the above operator $P : H \longrightarrow H_3$, we can restrict our considerations to the subspace H_3. If $H_3 = H_1$, we have $\mathrm{Range}\,(A + P) = H_1$ because of the Fredholm alternative, see J. Nečas [4].

We obtain easily the necessary conditions for $h \in \in \mathrm{Range}(A+P)$; we formulate the situation for the inequality $<$, the reader can do it for $>$, $\leq$, $\geq$.

Proposition 2.1. Let for all $u \in H$ and $w \in \mathrm{Ker}\, A$, $\|w\| = 1$

$$(2.5) \qquad (w, Pu) < \ell(w) .$$

Let the conditions of the theorem 2.1 be satisfied (clearly) without (1.2)). Then if $h \in \mathrm{Range}\,(A + P)$ the inequality

$$(2.6) \qquad (w, h) < \ell(w)$$

is valid.

Clearly: $Au + Pu = h \Longrightarrow (w, Pu) = (w, h) < \ell(w)$.

Remark 2.2. For the proposition 2.1 to hold, the condition (1.1) is not necessary; only the limit $\ell(w)$ must exist, eventually infinite.

§ 3. Application to general boundary value problems.

Let $\Omega \subset R_n$ be a bounded domain with Lipschitz boundary. Let $W^{k,2}(\Omega) = W^{k,2}$ be the Sobolev space of real functions u such that u and its derivatives (in the sense of distribution) up to the order k are square-integrable in Ω. $W^{k,2}$ is a Hilbert space with the scalar product

$$(3.1) \qquad (u, v)_k = \int_\Omega \sum_{|\alpha| \le k} D^\alpha u D^\alpha v \, dx .$$

Let $W_0^{k,2}$ be the subspace of $W^{k,2}$ of functions whose derivatives $D^\alpha u = 0$ on $\partial\Omega$ for $|\alpha| < k$. (For details, see for example J. Nečas [6].) Let V be a closed subspace of $W^{k,2}$ such that $W_0^{k,2} \subset V \subset W^{k,2}$, $a_{ij} \in L_\infty(\Omega)$, $|i|, |j| \le k$, $a_{ij} = a_{ji}$ and

$$(3.2) \qquad \sum_{|i|,|j|=k} a_{ij} \xi_i \xi_j \ge c \sum_{|i|=k} \xi_i^2 , \quad c > 0 .$$

Let $A_{ij} \in L_\infty(\partial\Omega)$, $A_{ij} = A_{ji}$, $|i|, |j| < k$. Let $g(s)$ be a real, continuous function on the real line, such that $\lim_{s\to\infty} g(s) = g(\infty)$, $\lim_{s\to-\infty} g(s) = g(-\infty)$, both $g(\infty)$ and $g(-\infty)$ being finite. Put

$$(3.3) \qquad \begin{aligned} A(v, u) &= \int_\Omega \sum_{|i|,|j|\le k} a_{ij} D^i v D^j u \, dx + \\ &+ \int_{\partial\Omega} \sum_{|i|,|j|<k} A_{ij} D^i v D^j u \, dS . \end{aligned}$$

$A(v,u)$ is a symmetric bounded bilinear form on $W^{k,2} \times W^{k,2}$ and define $A: V \to V$ by

$$(3.4) \qquad (Av, u)_k = A(v, u) .$$

Define $(v, Pu)_k = (v, g(u))_0$. Let $f \in L_2(\Omega)$. (We can consider $f \in V'$.) Let us look for the generalized solution u of the boundary value problem with homogeneous boundary data, i.e. we seek u in V such that for all $v \in V$:

(3.5) $\quad (A(v,u) + (v, g(u))_0 = (v, f)_0$.

For details see J. Nečas [6]. Put $(v, f)_0 = (v, h)_k$. So the problem (3.5) can be formulated as the problem to solve

(3.6) $$Au + Pu = h \ .$$

Because of the condition (3.2) and the fact that the imbedding $W^{k,2}(\Omega) \to W^{k-1,2}(\Omega)$ and the imbedding $W^{1,2}(\Omega) \longrightarrow \longrightarrow L_2(\partial\Omega)$ is completely continuous, we obtain easily that $dim\,(Ker\,A) < \infty$. If $Ker\,A = \{\theta\}$, according to the remark 2.1 $Au + Pu$ is onto, so the problem (3.5) has a solution for every $f \in L_2$.

Let $0 < dim\,(Ker\,A)$. Put $Ker\,A = H_2$ and let $V = H$.

Lemma 3.1. For $u \in H$, $w \in H_2$, there exists $\lim_{t\to\infty} (w, P(u+tw))_k$ uniformly with respect to $\|u\|_k \le c_1$, $\|w\|_k = 1$, $w \in H_2$.

Proof. Let $\Omega_+ = \{x \in \Omega \mid w(x) > 0\}$, $\Omega_- = \{x \in \Omega \mid w(x) < 0\}$. We have

(3.7) $$(w, P(u+tw))_k = \int_{\Omega_+} w(x)\, g(u(x) + tw(x))\,dx + \\ + \int_{\Omega_-} w(x)\, g(u(x) + tw(x))\,dx \ .$$

For almost all x from Ω_+ ,

(3.8) $$\lim_{t\to\infty} w(x)\, g(u(x) + tw(x)) = w(x)\, g(\infty)$$

and for almost all x from Ω_- :

$$(3.9) \quad \lim_{t \to \infty} w(x)\, g(u(x) + t w(x)) = w(x)\, g(-\infty) .$$

From the Lebesgue's theorem on the integrable majorants, it follows from (3.8) and (3.9) that

$$(3.10) \quad \ell(w) = g(\infty)\int_{\Omega_+} w(x)\,dx + g(-\infty)\int_{\Omega_-} w(x)\,dx .$$

It follows from (3.10) that $\ell(w)$ is continuous on the sphere $\| w \|_{\mathcal{R}} = 1$, $w \in \operatorname{Ker} A$. Let us suppose that the limit is not uniform. Then there exist $t_n \to \infty$, $w_n \to w$ in V and almost everywhere in Ω, $u_n \to u$ in L_2 (from the compactness of the imbedding) and almost everywhere in Ω and $\varepsilon > 0$ such that

$$(3.11) \quad | (w_n, g(u_n + t_n w_n))_0 - \ell(w_n) | \geq \varepsilon .$$

It follows from the continuity of $\ell(w)$ that for $n \geq n_0$

$$(3.12) \quad | (w, g(u_n + t_n w_n))_0 - \ell(w) | \geq \frac{\varepsilon}{2} .$$

But $g(u_n(x) + t_n w_n(x)) \longrightarrow g(\infty)$ for almost all $x \in \Omega_+$ and $g(u_n(x) + t_n w_n(x)) \longrightarrow g(-\infty)$ for almost all $x \in \Omega_-$, so once more from the Lebesgue's theorem it follows $\lim_{n \to \infty} (w, g(u_n + t_n w_n))_0 = \ell(w)$, which is contradictory with (3.12).

Theorem 3.1. Let the conditions for the boundary value problem be satisfied. Let for $w \in \operatorname{Ker} A$, $\| w \|_{\mathcal{R}} = 1$

$$(3.13) \quad \int_{\Omega} w(x) f(x)\,dx < g(\infty)\int_{\Omega_+} w(x)\,dx + g(-\infty)\int_{\Omega_-} w(x)\,dx .$$

Then the problem (3.5) has a solution. (The same for $>$ in (3.13).)

Remark 3.1. The set of f satisfying (3.13) is not empty if for example $g(-\infty) < 0 < g(\infty)$. If $\dim(\operatorname{Ker} A) = 1$ it is enough that $g(-\infty) < g(\infty)$.

Theorem 3.2. Let $g(-\infty) < g(s) < g(\infty)$. Then a necessary condition for the boundary value problem (3.5) has a solution, is (3.3). If there is $g(-\infty) \leq g(s) \leq g(\infty)$ (or other clear combinations as for example $g(-\infty) > g(s) \geq \geq g(\infty)$), we obtain the necessary condition in the form

$$(3.14) \quad \int_{\Omega} w(x) f(x) dx \leq g(\infty) \int_{\Omega_+} w(x) dx + g(-\infty) \int_{\Omega_-} w(x) dx$$

$$\left(\int_{\Omega} w(x) f(x) dx \geq g(\infty) \int_{\Omega_+} w(x) dx + g(-\infty) \int_{\Omega_-} w(x) dx \right).$$

Clearly:

$$(w, Pu)_k = \int_{\Omega_+} w(x) g(u(x)) dx + \int_{\Omega_-} w(x) g(u(x)) dx <$$

$$< g(\infty) \int_{\Omega_+} w(x) dx + g(-\infty) \int_{\Omega_-} w(x) dx .$$

Remark 3.2. We can easily modify the theorem 3.1 and 3.2 replacing $(v, g(u))_0$ in (3.5) by $\sum_{|\alpha| < k} (D^\alpha v, g_\alpha(x, D^\alpha u))_0$.

R e f e r e n c e s

[1] L. CESARI: Functional analysis and Galerkin's method, Michigan Math. J. 11(1964), 385-414.

[2] S.A. WILLIAMS: A sharp sufficient condition for solution of a nonlinear elliptic boundary value problem, J.Diff.Eq.8(1970),580-586.

[3] E. LANDESMAN, A. LAZAR: Nonlinear perturbations of linear elliptic boundary value problems at resonance, J.Math.Mech.19(1970),n.7,609-623.

[4] J. NEČAS: Fredholm alternative for nonlinear operators and applications to partial differential equations and integral equations, Časopis pěst.mat. 97(1972),65-71.

[5] J. NEČAS: Remark on the Fredholm alternative for nonlinear operators with application to nonlinear integral equations of generalized Hammerstein type, Comment. Math.Univ.Carolinae 13(1972),109-120.

[6] J. NEČAS: Les méthodes directes en théorie des équations elliptiques, Academia Prague,1967.

Matematicko-fyzikální fakulta

Karlova universita

Sokolovská 83, Praha 8

Československo

(Oblatum 25.1.1973)