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A THEOREM ON HAMILTONIAN LINE GRAPHS

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Abstract: In this paper, the following theorem is proved: Let G be a graph with at least five vertices and $\bar{G}$ be the complement of G; then for at least one graph G' of the graphs G and $\bar{G}$, G' is connected and the line graph of G' is hamiltonian.

Key words: hamiltonian graphs; line graphs; the complement of a graph

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In [5] Harary and Nash-Williams raised the problem of characterizing those graphs the line graph of which is hamiltonian. The present paper is a contribution to this topic.

We shall say that a graph G_1 is an LH-subgraph of a graph G_0 if (i) G_1 is a subgraph of G_0, (ii) G_1 is either trivial or eulerian, and (iii) for each edge $x = uv$ of G_0, at least one of the vertices u and v belongs to G_1. (For the terms of the theory of graphs which are not defined here, see Behzad and Chartrand [1].)

Lemma. Let G be a connected graph with at least three edges. Then the line graph $L(G)$ of G is hamiltonian if and only if G contains an LH-subgraph.

This lemma directly follows from Proposition 8 in [5]. (Note that for $G = K(1,2)$ this proposition does not hold.)

The path P_4, which is self-complementary, is the only graph G with four vertices such that (i) G and the complement $\overline{G}$ of G are connected, and (ii) neither $L(G)$ nor $L(\overline{G})$ is hamiltonian.

Theorem. Let G be a graph with $p \geq 5$ vertices. Then for at least one graph G' of the graphs G and $\overline{G}$, G' is connected and $L(G')$ is hamiltonian.

Proof. For $p = 5$, the proof of the statement can be obtained by exhaustion (diagrams of all 34 graphs with 5 vertices can be found in Harary [4]). Assume that $p = n \geq 6$ and that for $p = n - 1$, the statement is proved. The case when $\{G, \overline{G}\} = \{K_p, \overline{K}_p\}$ is obvious. We shall assume that G contains a vertex x such that $1 \leq deg_G\, x \leq p - 2$. Denote $G_0 = G - x$. By the induction hypothesis, for at least one graph G'' of the graphs G_0 and $\overline{G}_0$, G'' is connected and $L(G'')$ is hamiltonian. Without loss of generality we assume that $G'' = G_0$. As G_0 has at least $p - 2 \geq 4$ edges, then G_0 contains an LH-subgraph. Obviously, G is connected. We shall assume that $L(G)$ is not hamiltonian. Let G_1 be an LH-subgraph of G_0 with the maximum number of vertices.

I) Let G_1 be trivial. Then $G_0 = K(1, p-2)$. As $L(G)$ is not hamiltonian, $\overline{G}$ is connected and $L(G)$ is hamiltonian.

II) Let G_1 be nontrivial. By V_0 and V_1 we denote the vertex set of G_0 and G_1, respectively. By E and $\overline{E}$ we denote the edge set of G and $\overline{G}$, respectively. We denote $V_2 = V_0 - V_1$; by m we denote the number of vertices of V_2 . As $L(G)$ is not hamiltonian, there exists $s \in V_2$ such that $rs \in E$. Obviously, the complete graph with the vertex set V_2 is a subgraph of $\overline{G}$. If there exists $w_0 \in V_1$ such that $rw_0, sw_0 \in E$, then G contains an LH-subgraph, which is a contradiction. Thus for each vertex $v \in V_1$, either $rv \in \overline{E}$ or $sv \in \overline{E}$. Let $w_1, w_2 \in V_1$ such that $w_1 w_2 \in E$. As G_1 is an LH-subgraph of G_0 with the maximum number of vertices and G contains no LH-subgraph, we can easily prove that either $rw_1, sw_1 \in \overline{E}$ or $rw_2, sw_2 \in \overline{E}$. As G_1 contains a cycle, there exist distinct vertices $t, u \in V_1$ such that $rt, st, ru, su \in \overline{E}$.

It is easy to see that $\overline{G}$ is connected. We shall construct an LH-subgraph of $\overline{G}$. Let F denote the subgraph of $\overline{G}$ induced by V_1 . Let $x = v_1 v_2$ be an edge of F ; by $A(x)$ we denote a set $\{v_1 v_2, v_1 v_1', v_2 v_2'\}$ where (i) $v_1', v_2' \in \{r, s\}$, (ii) $v_1 v_1', v_2 v_2' \in \overline{E}$, and (iii) if there exists $v' \in \{r, s\}$ such that $v_1 v', v_2 v' \in E$, then $v_1' = v_2'$. Consider a maximum matching M in the graph $F - t - u$. By A we denote the set $\bigcup_{x \in M} A(x)$. Let j denote the number of those $x \in M$ that there exists an $r - s$ path of $\overline{G}$ induced by $A(x)$. Let v_0 be a vertex of F and Y be a subset of $\overline{E}$; by $D_{v_0}^{Y}$ we denote the set of those vertices of V_1 which are adjacent to v_0 in F and incident with no edge of Y . If j is even, then by B we denote the set $A \cup \{rt, st, ru, su\}$. Let j be odd. If $D_u^A - \{t\} = \emptyset$, then by B we denote the set $A \cup \{rt, st\}$. If $D_u^A - \{t\} \neq \emptyset$, then by B

we denote a set $A \cup \{rt, st\} \cup A(u\,u')$, where u' is a vertex of $D_u^A - \{t\}$. If $m = 1$, then by Z we denote the set B . If $m \geq 3$, then by Z we denote a set $B \cup Z^*$, where Z^* is the edge set of a cycle with the vertex set V_2 . Let $m = 2$ and s' be the only vertex of V_2 different from s . If each vertex $w \in V_1$ adjacent to s' in $\overline{G}$ is incident with an edge of B , then by Z we denote the set B . Let there exist $w' \in V_1$ such that $s'w' \in \overline{E}$ and w' is incident with no edge of B . If $sw' \in \overline{E}$, then by Z we denote $B \cup \{ss', sw', s'w'\}$. Let $sw' \notin \overline{E}$. Then $rw' \in \overline{E}$. If $D_t^B = \emptyset$, then by Z we denote $(B - \{rt, st\}) \cup \{ss', s'w', rw'\}$; if $w' \in D_t^B$, then by Z we denote $(B - \{st\}) \cup \{ss', s'w', tw'\}$; if $D_t^B \neq \emptyset$ and $w' \notin D_t^B$, then by Z we denote $(B - \{rt, st\}) \cup \{ss', s'w', rw'\} \cup A(t, t')$, where t' is a vertex of D_t^B .

Now, let H denote the subgraph of G induced by Z . It is easy to see that H is an LH-subgraph of $\overline{G}$. Thus $L(\overline{G})$ is hamiltonian and the proof is complete.

Corollary. Let G be a nontrivial graph. Then for at least one graph G' of the graphs G and $\overline{G}$, G' is connected and $L(G')$ contains a hamiltonian path.

Remark. It is possible to ask for connections between the present theorem (and its proof) and sufficient conditions for a graph to be hamiltonian which depend on properties of the degree sequence (as in [4], pp. 66-68 , [1, pp. 131-135] , and the most generally in Chvátal [2]), or on the other quantitative indices (Chvátal and Erdös [3]). The following example gives a partial answer to the problem in question. Let $p \geq 12$ and G be the graph which we obtain from the path P_3 and the complement $\overline{C}_{p-2}$ of the cycle with $p - 2$ vertices in such a way that we identify one vertex of $\overline{C}_{p-2}$ with one end-vertex of P_3. Obviously, $L(G)$ is not hamiltonian. Let u denote the only end-vertex of G; it is easy to see that $L(G - u)$ is hamiltonian. The graph $L(\overline{G})$ has $3p - 7$ vertices, the maximum degree p, the connectivity 5, and the independence number $\{(p-1)/2\}$. The graph $L(\overline{G})$ is, of course, hamiltonian but its degree sequence does not fulfil the condition of the first statement of Theorem 1 in [2], and the relation between its connectivity and its independence number does not fulfil the condition of Theorem 1 in [3].

R e f e r e n c e s

[1] M. BEHZAD, G. CHARTRAND: Introduction to the Theory of Graphs, Allyn and Bacon, Boston (Mass.) 1971.

[2] V. CHVÁTAL: On Hamilton's ideals, J. of Combinatorial Theory 12(B) (1972),163-168.

[3] V. CHVÁTAL, P. ERDÖS: A note on Hamiltonian circuits, Discrete Mathematics 2(1972),111-113.

[4] F. HARARY: Graph Theory, Addison-Wesley, Reading (Mass.) 1969.

[5] F. HARARY, C.ST.J.A.NASH-WILLIAMS: On eulerian and hamiltonian graphs and line graphs, Canadian Math.Bull. 8(1965), 701-709.

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